

Eco529: Macro, Money, and Finance

Lecture 10: Money

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Outline

1 Money Model

- Model Setup
- Frictionless Benchmark
- Adding Financial Frictions
- Adding Monetary Frictions
- Separating Money $\mathcal{M}$ and Gov. Bonds $\mathcal{B}$

2 Monetary Policy

- Interest Rate Policy
- Quantitative Easing
- Inflation–Fiscal Link
- Sargent and Wallace

3 Price Level Determination

- Fiscal Backing and the Fiscal Theory of the Price Level
- Bubble Theories and (In-)Determinacy
- "Pure" Unit of Account Theory

The 3 Roles of Money

■ Store of value

- fundamental cash flows due to backing
e.g., commodity standard, exchange rate regime, fiscal backing
- non-cash-flow benefits: helps overcome *intertemporal* financial frictions
e.g., OLG (Samuelson), spatial separation (Townsend), uninsured idiosyncratic risk (Bewley)
- store of value role not exclusive to money: non-monetary assets are substitute stores of values

■ Medium of exchange

- helps overcome monetary frictions = frictions in *intratemporal* exchange
- key monetary friction: double coincidence of wants problem
- makes money special relative to assets, which cannot serve as substitute media of exchange

■ Unit of account

- contractual values denominated in monetary unit
- e.g., nominal goods prices (+ commitment to sell at quoted price), nominal debt contracts, nominal labor contracts

The Value of Money and the Price Level

- Core question of monetary economics: what determines the general level of nominal goods prices?
- Equivalently: what are the determinants of the value of money?
 - price level $\mathcal{P}_t$: price of real goods basket in units of money
 - real value of a single unit of money: $1/\mathcal{P}_t$
- Two aspects:
 - 1 which economic considerations justify the value of money in a given equilibrium?
 - 2 determinacy question:
 - does the model have a unique prediction for the value of money / price level? ($\sim$ equilibrium uniqueness)
 - more broadly, which economic forces lead to coordination on a specific monetary equilibrium?

Classification of Monetary Theories

1 Backing theories: value of money derives from fundamental cash flows that back it

- store of value role (money is just another asset)
- example: Fiscal Theory of the Price Level (FTPL)

2 Bubble theories: money valued because it can be passed on to others can be rational expectation if trading money overcomes market frictions:

- 1** intertemporal financial frictions (e.g., incomplete markets)
 - emphasizes store of value role
 - examples: Samuelson, Townsend, Bewley, Brunnermeier-Merkel-Sannikov
- 2** intratemporal monetary frictions (e.g., cash-in-advance constraint)
 - emphasizes medium of exchange role
 - example: (New) Monetarism

3 Money as a pure unit of account

- value of money derives from role of money as a unit of account
- not from the value of any monetary assets
- example: New Keynesianism

Monetary Assets: Credit, Deposits, Cash, Reserves, Gov-Debt

In first two classes of theories, different assets may play the role of “money”:

- **Credit** can substitute for
 - store of value assets (credit balances to keep track of resource distribution)
 - media of exchange (exchange goods against credit balance)imperfect credit prerequisite for bubble theories
- **Bank deposits, cash, and central bank reserves** all play a role in the payment system as media of exchange
- Government-provided **outside money** vs. **inside money**
 - outside money: positive net supply, backed by government fiscal capacity
 - inside money: zero net supply, backed by bank assets
- **Cash & reserves** (narrow outside money) vs. **nominal government liabilities** (broad outside money)
 - (primarily) narrow money provides medium of exchange services
 - but all nominal government liabilities
 - compete for the same backing real resources
 - serve as a store of value
 - are affected symmetrically by changes in the price level

A Unified Model of Money

- Next: develop and solve a simple model that illustrates several monetary theories
 - fiscal theory of the price level (backing theory)
 - money (gov. debt) as a safe asset (store of value bubble theory)
 - money providing transaction services (medium of exchange bubble theory)
- For now, we disregard the determinacy question
 - we always select a specific equilibrium: the monetary steady state
 - will return to the determinacy question in the end

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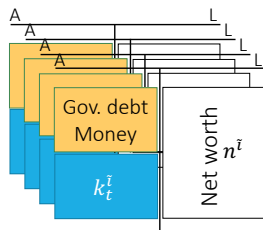
Model with Money

- Continuum of (heterogeneous) agents $\tilde{i} \in [0, 1]$: choose $\tilde{c}_t^{\tilde{i}}, \theta_t^{\tilde{i}}, \iota_t^{\tilde{i}}$ to maximize

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\log c_t^{\tilde{i}} + f(\mathcal{G}_t K_t) \right) dt \right]$$

$$s.t. \quad \frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} dt + dr_t^{\mathcal{MB}} + (1 - \theta_t^{\tilde{i}})(dr_t^{K, \tilde{i}}(\iota_t^{\tilde{i}}) - dr_t^{\mathcal{MB}}) \quad \& \quad n_t^{\tilde{i}} \geq 0$$

- Each agent operates physical capital $k_t^{\tilde{i}}$
 - output (net of investment & transaction cost):
 $y_t^{\tilde{i}} dt = (a k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}} - \mathfrak{T}(\nu_t^{\tilde{i}}) k_t^{\tilde{i}}) dt$
 - $\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} + d\Delta_t^{k, \tilde{i}},$
 $(d\tilde{Z}_t^{\tilde{i}} \text{ idiosyncratic Brownian risk})$
 - output tax $\tau_t a k_t^{\tilde{i}} dt$



- No aggregate risk dZ_t
- Government budget constraint:
$$\underbrace{(\mu_t^{\mathcal{MB}} - i_t^{\mathcal{MB}})}_{=: \check{\mu}_t^{\mathcal{MB}}} \mathcal{MB}_t + \mathcal{P}_t K_t \underbrace{(\tau_t a - \mathcal{G})}_{=: s_t} = 0$$

Frictions

Let us build up model step by step adding one friction at a time:

1 Frictionless benchmark: $\tilde{\sigma} = 0$, $\mathfrak{T} \equiv 0$

- only tax backing present

2 Financial friction (intertemporal)

- $\tilde{\sigma} > 0$ and incomplete markets friction: $d\tilde{Z}_t^i$ -shocks uninsurable
- money serves as a safe asset

3 Monetary friction (intratemporal)

- transaction costs $\mathfrak{T}_t(\nu_t^i)$ increasing in velocity ν_t^i
- reduced-form device for medium of exchange role
- interpretation: transaction costs incurred in unmodeled supply chain

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Assets, Aggregate Resource Constraint, and Markets

■ Assets: capital and money

- q_t^K capital price
- $q_t^{\mathcal{MB}} := \frac{\mathcal{MB}_t}{P_t K_t}$ value of money per unit of capital
- $q_t := q_t^K + q_t^{\mathcal{MB}} = N_t/K_t$ wealth per unit of capital
- $\vartheta_t := \frac{\mathcal{MB}_t/P_t}{q_t^K K_t + \mathcal{MB}_t/P_t} = \frac{q_t^{\mathcal{MB}}}{q_t^K + q_t^{\mathcal{MB}}}$ share of nominal wealth
- Postulate Ito price processes
 $dq_t^K/q_t^K = \mu_t^{q,K} dt, \quad dq_t^{\mathcal{MB}}/q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}} dt, \quad d\vartheta_t/\vartheta_t = \mu_t^\vartheta dt$
- SDF for each agent $\tilde{i}$: $d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt$

■ Aggregate resource constraints:

- Output: $C_t + \iota_t K_t + \mathcal{G} K_t = a K_t$
- Capital: $\int k_t^{\tilde{i}} d\Delta k_t^{k,\tilde{i}} d\tilde{i} = 0$

■ Markets: Walrasian goods, money, and capital markets

Return Processes

■ Return on capital

$$\begin{aligned} dr_t^{K, \tilde{i}}(\iota) &= \left(\frac{a(1 - \tau_t) - \iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q, K} \right) dt \\ &= \left(\frac{a - \mathcal{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K} \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q, K} \right) dt \end{aligned}$$

- second line uses government budget constraint

$$\check{\mu}_t^{\mathcal{MB}} \mathcal{MB}_t + \mathcal{P}_t K_t (\tau_t a - \mathcal{G}) = 0 \Leftrightarrow \tau_t a - \mathcal{G} = -\check{\mu}_t^{\mathcal{MB}} q_t^{\mathcal{MB}}$$

■ Return on money

$$\begin{aligned} dr_t^{\mathcal{MB}} &= i_t^{\mathcal{MB}} dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} &= i_t^{\mathcal{MB}} dt + \frac{d(q_t^{\mathcal{MB}} K_t / \mathcal{MB}_t)}{q_t^{\mathcal{MB}} K_t / \mathcal{MB}_t} \\ &= \left(i_t^{\mathcal{MB}} + \mu_t^{q, \mathcal{MB}} + \mu_t^K - \mu_t^{\mathcal{MB}} \right) dt &= \left(\mu_t^{q, \mathcal{MB}} + \mu_t^K - \check{\mu}_t^{\mathcal{MB}} \right) dt \end{aligned}$$

Optimal Investment and Goods Market Clearing

- Optimal investment: Tobin's Q condition

$$q_t^K = \frac{1}{\Phi'(\tilde{l}_t)} = 1 + \phi \tilde{l}_t$$

- in particular, all agents choose same investment rate: $\tilde{l}_t = l_t$
- Goods market clearing

$$\rho q_t^K + l_t + \mathcal{G} = a$$

- Solve for q_t , use $q_t^K = (1 - \vartheta_t)q_t$, plug into optimal investment condition:

$$(1 - \vartheta_t) \frac{a - l_t - \mathcal{G}}{\rho} = (1 - \vartheta_t)q_t = 1 + \phi l_t \quad \Rightarrow \quad l_t = \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho}$$

- where $\check{a} = a - \mathcal{G}$

Interim Conclusion: Equilibrium Asset Prices in Terms of ϑ_t

- Plugging ι_t expression back into $q_t = (\check{a} - \iota_t)/\rho$:

$$q_t = \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho}$$

- q_t , q_t^K , $q_t^{\mathcal{MB}}$, ι_t only depend on the nominal wealth share ϑ_t :

$$\begin{aligned} q_t &= \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} & \iota_t &= \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho} \\ q_t^K &= (1 - \vartheta_t) \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} & q_t^{\mathcal{MB}} &= \vartheta_t \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} \end{aligned}$$

- Hence, ϑ_t is the key variable in this model!
 - in equilibrium, $\vartheta_t = \theta_t$ (asset market clearing)
 - so ϑ_t should be determined by portfolio choice

Characterizing ϑ_t : Portfolio Choice Conditions

■ Portfolio choice conditions

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = r_t = \frac{\mathbb{E}_t[dr_t^{K, \tilde{i}}]}{dt}$$

■ Substitute in return expressions

$$-\check{\mu}_t^{\mathcal{MB}} + \cancel{\mu}_t^K + \mu_t^{q, \mathcal{MB}} = \frac{a - \iota_t - \mathcal{G} + q_t^{\mathcal{MB}} \check{\mu}_t^{\mathcal{MB}}}{q_t^K} + \cancel{\Phi(\iota_t) - \delta} + \mu_t^{q, K}$$

■ Use

- $a - \iota_t - \mathcal{G} = \rho q_t$ (goods market clearing)
- $q_t^K = (1 - \vartheta_t)q_t$, $q_t^{\mathcal{MB}} = \vartheta_t q_t$ (def. of ϑ_t)
- $\mu_t^\vartheta = (1 - \vartheta_t)(\mu_t^{q, \mathcal{MB}} - \mu_t^{q, K})$ (Itô)

and solve for μ_t^ϑ :

$$\mu_t^\vartheta = \rho + \check{\mu}_t^{\mathcal{MB}}$$

Characterizing ϑ_t : The Money Valuation Equation

- By definition, $\mu_t^\vartheta = \mathbb{E}_t[d\vartheta_t]/\vartheta_t$, so last equation can be written as

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}}) \vartheta_t dt$$

- this version is preferable because it remains valid for $\vartheta_t = 0$
- but our derivation assumed $\vartheta_t > 0$ (otherwise $dr_t^{\mathcal{MB}}$ is not well-defined)
- This is the **money valuation equation**, characterizes portfolio demand for money
- For interpretation, integrate forward in time

(this would be a BSDE in a stochastic setting, i.e., a forward-looking choice condition, so we need to integrate forward in time)

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} (-\check{\mu}_{t'}^{\mathcal{MB}}) \vartheta_{t'} dt' \right] = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \frac{S_{t'}}{q_{t'}} dt' \right]$$

- portfolio demand for money arises from expected future primary surpluses ($s_t K_t$)
- these represent real payouts/cash flows made to holders of money

Frictionless Benchmark: Steady State Equilibria

- Consider steady state equilibrium with constant ϑ (and hence q , q^K , $q^{\mathcal{MB}}$, ι) (this is really a balanced growth path because K_t grows at constant rate $g = \Phi(\iota) - \delta$)

- Imposing steady state implies:

- by the money valuation equation ($\mu_t^{\vartheta} = 0$)

$$\check{\mu}^{\mathcal{MB}} = -\rho$$

- by the government budget constraint

$$s = -\check{\mu}^{\mathcal{MB}} q^{\mathcal{MB}} = \rho q^{\mathcal{MB}} \Rightarrow q^{\mathcal{MB}} = \frac{s}{\rho}$$

- Some of the remaining equilibrium quantities are then:

$$\vartheta_t = \frac{s(1 + \phi\rho)}{s + \rho(1 + \phi\check{\alpha})} \quad q^K = \frac{1 + \phi(\check{\alpha} - s)}{1 + \phi\rho} \quad \iota_t = \frac{\check{\alpha} - s - \rho}{1 + \phi\rho}$$

Remark: we need $0 < s < \check{\alpha} + \frac{1}{\phi}$ for both assets to have positive value.

Some Observations from Frictionless Benchmark

- 1 The value of money depends on **fiscal backing**
 - need positive **primary surpluses** ($s > 0$) for money to be valued
 - higher s results in higher $q^{\mathcal{M}^B}$ and ϑ
- 2 The following are inversely related to the value of money
 - the value of capital assets (q^K)
 - the investment rate (ι)
 - the growth rate of the economy ($g = \Phi(\iota) - \delta$)
- 3 The nominal interest rate paid on money does not matter for the real allocation
 - raising $i^{\mathcal{M}^B}$ while maintaining $\check{\mu}^{\mathcal{M}^B} = -\rho$ raises $\mu^{\mathcal{M}^B} = i^{\mathcal{M}^B} - \rho$ one for one
 - this affects the inflation rate ($\pi := \mu^{\mathcal{P}} = \mu^{\mathcal{M}^B} - g$) but no real variables

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- SDF for each agent $\tilde{i}$: $d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt - \tilde{\zeta}_t d\tilde{Z}_t^{\tilde{i}}$

■ Aggregate resource constraints:

- Output: $C_t + \iota_t K_t + \mathcal{G} K_t = a K_t$
- Capital: $\int k_t^{\tilde{i}} d\Delta k_t^{k,\tilde{i}} d\tilde{i} = 0$

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■ Return on capital

$$\begin{aligned}dr_t^{K,\tilde{i}}(\iota) &= \left(\frac{a(1 - \tau_t) - \iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} \\&= \left(\frac{a - \mathcal{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K} \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}}\end{aligned}$$

■ Return on money

$$\begin{aligned}dr_t^{\mathcal{MB}} &= i_t^{\mathcal{MB}} dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \\&= \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \check{\mu}_t^{\mathcal{MB}} \right) dt\end{aligned}$$

Optimal Investment and Goods Market Clearing

Exactly as in previous model:

- Optimal investment

$$q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi \iota_t$$

- Combining with market clearing implies

$$\iota_t = \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho}$$

- Implied asset prices

$$q_t = \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} \quad q_t^K = (1 - \vartheta_t) \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} \quad q_t^{\mathcal{M}\mathcal{B}} = \vartheta_t \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho}$$

- Hence, the key variable to determine is the nominal wealth share ϑ_t

Characterizing ϑ_t : Portfolio Choice Conditions

■ Portfolio choice conditions

$$\begin{aligned} -\check{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q, \mathcal{MB}} &= \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = r_t \\ \frac{a - \mathcal{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K} \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q, K} &= \frac{\mathbb{E}_t[dr_t^K]}{dt} = r_t + \tilde{\zeta}_t \tilde{\sigma} \end{aligned}$$

■ New element: idiosyncratic risk premium $\tilde{\zeta}_t \tilde{\sigma}$ on capital

- due to log utility $\tilde{\zeta}_t = \tilde{\sigma}_t^\eta = (1 - \theta_t) \tilde{\sigma}$
- by asset market clearing then $\tilde{\zeta}_t = (1 - \vartheta_t) \tilde{\sigma}$

■ After same steps as before, we obtain

$$\mu_t^\vartheta = \rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2$$

The Money Valuation Equation with Idiosyncratic Risk

- The money valuation equation is now

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2) \vartheta_t dt$$

- In integral form

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} (-\check{\mu}_{t'}^{\mathcal{MB}} + (1 - \vartheta_{t'})^2 \tilde{\sigma}^2) \vartheta_{t'} dt' \right]$$

- Money may be valued for two reasons:

- because of cash flows from fiscal backing $(-\check{\mu}_t^{\mathcal{MB}})$
- because it is a safe asset that facilitates idiosyncratic risk sharing $((1 - \vartheta_t)^2 \tilde{\sigma}^2)$
(we will explore the safe asset features in more detail in the next lecture)

Idiosyncratic Risk Model: Steady State Equilibria

- Possible values for $\check{\mu}^{\mathcal{MB}}$ consistent with $\mu_t^{\vartheta} = 0$

$$-\rho \leq \check{\mu}^{\mathcal{MB}} < -\rho + \tilde{\sigma}^2$$

- For any such value, there are two possible steady-state equilibria:

	Non-Monetary	Monetary (Store of Value)
ϑ	$\vartheta = 0$	$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}$
$q^{\mathcal{MB}}$	$q^{\mathcal{MB}} = 0$	$q^{\mathcal{MB}} = \frac{(\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}})(1 + \phi\check{\alpha})}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\rho\tilde{\sigma}}}$
q^K	$q^K = \frac{1 + \phi\check{\alpha}}{1 + \phi\rho}$	$q^K = \frac{\sqrt{\rho + \check{\mu}^{\mathcal{MB}}}(1 + \phi\check{\alpha})}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\rho\tilde{\sigma}}}$
ι	$\iota = \frac{\check{\alpha} - \rho}{1 + \phi\rho}$	$\iota = \frac{\check{\alpha}\sqrt{\rho + \check{\mu}^{\mathcal{MB}}} - \tilde{\sigma}\rho}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\rho\tilde{\sigma}}}$
s	$s = 0$	$s = -\check{\mu}^{\mathcal{MB}} \frac{(\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}})(1 + \phi\check{\alpha})}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\rho\tilde{\sigma}}}$

Bubbles and Seigniorage

- If $\tilde{\sigma}^2 > \rho$, $\check{\mu}^{\mathcal{MB}} = 0$ is a possible choice in previous solution
 - then, in the monetary steady state, money is still valued
 - ... but there is no fiscal backing ($s = -\check{\mu}^{\mathcal{MB}} q^{\mathcal{MB}} = 0$)
 - money is a (rational) **bubble**: intrinsic value is zero but market value is positive
- We can push this idea further and even pick $\check{\mu}^{\mathcal{MB}} > 0$
 - feasible so long as $\check{\mu}^{\mathcal{MB}} < \tilde{\sigma}^2 - \rho$
 - then the government runs permanent primary deficits, $s < 0$
- Permanent deficits are possible because the government can generate seigniorage by “mining the bubble”
 - print new money that dilutes claims of existing money holders to aggregate bubble
 - bubble mining here acts as a tax on risk sharing (lowers ϑ_t , raises risk exposures)

Observations from Frictionless Benchmark Revisited

- 1 The value of money depends on **fiscal backing** and **idiosyncratic risk**
 - **do not necessarily** need positive **primary surpluses** ($s > 0$) for money to be valued
 - higher s or higher $\tilde{\sigma}$ result in higher $q^{\mathcal{MB}}$ and ϑ
- 2 The following are inversely related to the value of money
 - the value of capital assets (q^K)
 - the investment rate (ι)
 - the growth rate of the economy ($g = \Phi(\iota) - \delta$)

these observations remain correct (only depend on goods market clearing)

- 3 The nominal interest rate paid on money does not matter for the real allocation
 - raising $i^{\mathcal{MB}}$ while maintaining $\check{\mu}^{\mathcal{MB}} = -\rho$ raises $\mu^{\mathcal{MB}} = i^{\mathcal{MB}} - \rho$ one for one
 - this affects the inflation rate ($\pi := \mu^{\mathcal{P}} = \mu^{\mathcal{MB}} - g$) but no real variables

these observations remain also correct

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Adding Monetary Friction: Transaction Costs

- Recall: output produced by $\tilde{i}$ net of investment and **transaction costs**

$$y_t^{\tilde{i}} dt = (ak_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}}) k_t^{\tilde{i}}) dt$$

- We now add the left-out details:

- $\nu_t^{\tilde{i}}$ denotes output velocity of $\tilde{i}$'s money holdings:

$$\nu_t^{\tilde{i}} := \frac{\mathcal{P}_t a k_t^{\tilde{i}}}{m_t^{\tilde{i}}} = \frac{1 - \theta_t^{\tilde{i}}}{\theta_t^{\tilde{i}}} \frac{a}{q_t^K}$$

where $m_t^{\tilde{i}}$ denotes the money holdings of individual $\tilde{i}$

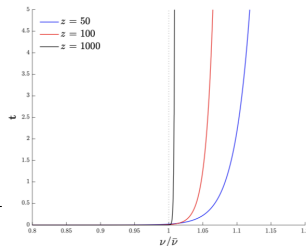
- transaction costs are given by

$$\mathfrak{T}_t(\nu) = \frac{a}{(\mathfrak{z} - 1) \bar{\nu}} \left[\left(\frac{\nu}{\bar{\nu}} \right)^{\mathfrak{z}-1} - \left(\frac{\nu_t^{\text{eq}}}{\bar{\nu}} \right)^{\mathfrak{z}-1} \right]$$

- ν_t^{eq} is velocity of everyone else in equilibrium

- Limit case $\mathfrak{z} \rightarrow \infty$: **cash-in-advance constraint**

$$\nu_t^{\tilde{i}} \leq \bar{\nu} \Leftrightarrow \mathcal{P}_t a k_t^{\tilde{i}} \leq \bar{\nu} m_t^{\tilde{i}}$$



Return Processes

■ Return on capital

$$\begin{aligned} dr_t^{K,\tilde{i}}(\iota, \nu) &= \left(\frac{a(1 - \tau_t) - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} \\ &= \left(\frac{a - \mathcal{G} - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \frac{q_t^M}{q_t^K} \check{\mu}_t^{\mathcal{M}} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} \end{aligned}$$

■ Return on money

$$\begin{aligned} dr_t^{\mathcal{MB}} &= i_t^{\mathcal{MB}} dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \\ &= \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \check{\mu}_t^{\mathcal{MB}} \right) dt \end{aligned}$$

Optimal Investment and Goods Market Clearing

Exactly as in previous model:

- Optimal investment

$$\iota_t = \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho}$$

- Implied asset prices

$$q_t = \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} \quad q_t^K = (1 - \vartheta_t) \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho} \quad q_t^{\mathcal{MB}} = \vartheta_t \frac{1 + \phi\check{a}}{1 - \vartheta_t + \phi\rho}$$

Portfolio Choice

Note: portfolio choice is nonstandard because θ_t enters net worth return nonlinearly via velocity. Therefore, we solve this explicitly using the stochastic maximum principle.

$$H_t = e^{-\rho t} \log c_t - \xi_t c_t + \xi_t n_t \left((1 - \theta_t) \frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} + \theta_t \frac{\mathbb{E}_t[dr_t^{\mathcal{M}\mathcal{B}}]}{dt} \right) - \tilde{\zeta}_t \xi_t n_t (1 - \theta_t) \tilde{\sigma}$$

Maximize H_t with respect to θ_t, ν_t subject to the constraint

$$\theta_t \nu_t = (1 - \theta_t) \frac{a}{q_t^K}$$

Denoting the Lagrange multiplier by $\lambda_t^{\mathcal{M}\mathcal{B}} \xi_t n_t$, the first-order conditions are:

$$\theta_t : \quad \frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{M}\mathcal{B}}]}{dt} = \tilde{\zeta}_t \tilde{\sigma} + \lambda_t^{\mathcal{M}\mathcal{B}} \left(\nu_t + \frac{a}{q_t^K} \right)$$

$$\nu_t : \quad (1 - \theta_t) \frac{\partial \mathbb{E}[dr_t^K(\iota_t, \nu_t)] / dt}{\partial \nu_t} + \lambda_t^{\mathcal{M}\mathcal{B}} \theta_t = 0$$

θ -FOC and Money Valuation Equation

$$\begin{aligned}\frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} &= \overbrace{\frac{a - \mathcal{G} - \iota_t - \mathfrak{T}_t(\nu_t)}{q_t^K}}^{=\rho/(1-\vartheta_t)} + \overbrace{\frac{q_t^{\mathcal{MB}}}{q_t^K}}^{=\vartheta_t/(1-\vartheta_t)} \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,K} \\ \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} &= -\check{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,\mathcal{MB}}\end{aligned}$$

Take the difference:

$$\frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \frac{\rho}{1-\vartheta_t} + \frac{\check{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t}$$

Plug into FOC:

$$\frac{\rho}{1-\vartheta_t} + \frac{\check{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \tilde{\zeta}_t \tilde{\sigma} + \lambda_t^{\mathcal{MB}} \left(\nu_t + \frac{a}{q_t^K} \right) = (1-\vartheta_t) \tilde{\sigma}^2 + \frac{\lambda_t^{\mathcal{MB}} \nu_t}{1-\vartheta_t}$$

Define $\Delta i_t := i_t - i_t^{\mathcal{MB}} = \lambda_t^{\mathcal{MB}} \nu_t$. Intuitively, Δi_t represents a liquidity premium - the spread between a frictionless nominal interest rate and the return on money. Solve for $\mathbb{E}_t[d\vartheta_t]$:

(i_t : shadow nominal rate i_t on nominal asset without transaction services)

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2 \tilde{\sigma}^2 - \Delta i_t) \vartheta_t dt$$

ν -FOC and Quantity Equation

- From capital return and functional form $\mathfrak{T}_t(\nu) = \frac{a}{(\mathfrak{z}-1)\bar{\nu}} \left[\left(\frac{\nu}{\bar{\nu}} \right)^{\mathfrak{z}-1} - \left(\frac{\nu_t^{\text{eq}}}{\bar{\nu}} \right)^{\mathfrak{z}-1} \right]$,

$$\frac{\partial \mathbb{E}[dr_t^K(\nu_t, \nu_t)]/dt}{\partial \nu_t} = -\frac{a}{q_t^K} \frac{1}{\bar{\nu}^2} \left(\frac{\nu_t}{\bar{\nu}} \right)^{\mathfrak{z}-2} = -\frac{\vartheta_t}{1 - \vartheta_t} \frac{1}{\bar{\nu}} \left(\frac{\nu_t}{\bar{\nu}} \right)^{\mathfrak{z}-1}$$

Plug this expression (and $\theta_t = \vartheta_t$) into ν_t -FOC:

$$\lambda_t^{\mathcal{MB}} = \frac{1}{\bar{\nu}} \left(\frac{\nu_t}{\bar{\nu}} \right)^{\mathfrak{z}-1} \Rightarrow \boxed{\Delta i_t = \lambda_t^{\mathcal{MB}} \nu_t = \left(\frac{\nu_t}{\bar{\nu}} \right)^{\mathfrak{z}}}$$

- Solving for ν_t , plugging into definition of ν_t , and aggregating yields the quantity equation

$$\mathcal{P}_t Y_t = \underbrace{\left(\Delta i_t \right)^{\frac{1}{\mathfrak{z}}}}_{\nu_t} \bar{\nu} \mathcal{MB}_t$$

- Remark:* in the CIA limit, $\mathfrak{z} \rightarrow \infty$, two possible cases

$$\begin{cases} \nu_t < \bar{\nu} & \Delta i_t = 0 \\ \nu_t = \bar{\nu} & \Delta i_t \geq 0 \end{cases}$$

Steady State Equilibrium

- Assume $\check{\mu}_t^{\mathcal{MB}} = \check{\mu}^{\mathcal{MB}}$ is constant and consider steady state ($\mu_t^{\vartheta} = 0$)

1 Money Valuation Equation

$$\rho + \check{\mu}^{\mathcal{MB}} = (1 - \vartheta)^2 \tilde{\sigma}^2 + \Delta i$$

2 Quantity Equation

$$\Delta i = \left(\frac{\nu}{\bar{\nu}} \right)^3 = \left(\frac{1}{\bar{\nu}} \frac{1 - \vartheta + \phi \rho}{\vartheta} \frac{a}{1 + \phi \check{a}} \right)^3$$

Remark: last equality follows from equations derived previously

$$\nu_t = \frac{1 - \vartheta_t}{\vartheta_t} \frac{a}{q_t^K}, \quad q_t^K = (1 - \vartheta_t) \frac{1 + \phi \check{a}}{1 - \vartheta_t + \phi \rho}$$

- Combining the two equations yields nonlinear equation for steady-state ϑ
- No closed-form solution except in special cases, e.g.
 - no transaction costs ($\bar{\nu} \rightarrow \infty$) (as analyzed previously)
 - cash-in-advance limit ($\check{a} \rightarrow \infty$) (will consider this one next)

Special Case: Cash in advance constraint ($\beta \rightarrow \infty$)

Two cases:

- 1 $\Delta i = 0, \nu < \bar{\nu}$: valuation equation (store of value role) determines ϑ ,

$$\rho + \check{\mu}^{\mathcal{MB}} = (1 - \vartheta)^2 \tilde{\sigma}^2$$

- 2 $\Delta i > 0, \nu = \bar{\nu}$: quantity equation (medium of exchange role) determines ϑ ,

$$\frac{1}{\bar{\nu}} \frac{1 - \vartheta + \phi \rho}{\vartheta} \frac{a}{1 + \phi \check{\alpha}} = 1$$

	Medium of Exchange	Store of Value
ϑ	$\vartheta = \frac{(1+\phi\rho)a}{a+(1+\phi\check{\alpha})\bar{\nu}}$	$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}$
Δi	$\Delta i = \rho + \check{\mu}^{\mathcal{MB}} - \left(\frac{\bar{\nu} + \phi(\check{\alpha}\bar{\nu} - a\rho)}{a + (1+\phi\check{\alpha})\bar{\nu}} \right)^2 \tilde{\sigma}^2$	$\Delta i = 0$
$q^{\mathcal{MB}}$	$q^{\mathcal{MB}} = \frac{a}{\bar{\nu}}$	$q^{\mathcal{MB}} = \frac{(\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}})(1 + \phi\check{\alpha})}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\tilde{\sigma}\rho}}$
q^K	$q^K = \frac{1 + \phi(\check{\alpha} - a\rho/\bar{\nu})}{1 + \phi\rho}$	$q^K = \frac{\sqrt{\rho + \check{\mu}^{\mathcal{MB}}}(1 + \phi\check{\alpha})}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\tilde{\sigma}\rho}}$
ι	$\iota = \frac{\check{\alpha} - \rho(1 + a/\bar{\nu})}{1 + \phi\rho}$	$\iota = \frac{\check{\alpha} \sqrt{\rho + \check{\mu}^{\mathcal{MB}}} - \tilde{\sigma}\rho}{\sqrt{\rho + \check{\mu}^{\mathcal{MB}} + \phi\tilde{\sigma}\rho}}$

Government Liabilities Valuation Equation(s)

- ϑ_t = fraction of wealth which is invested in the safe assets
- The gov. liabilities money valuation equation (backward equation) is now

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2 - \vartheta_t^{\mathcal{M}} \Delta i_t) \vartheta_t dt$$

- In integral form

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \left(\underbrace{-\check{\mu}_{t'}^{\mathcal{MB}}}_{\text{backing}} + \underbrace{(1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2}_{\text{store of value}} + \underbrace{\vartheta_{t'}^{\mathcal{M}} \Delta i_{t'}}_{\text{medium of exchange}} \right) \vartheta_{t'} dt' \right] \quad (1)$$

$$\Delta i_t = \lambda_t^{\mathcal{M}} \nu \quad (2)$$

- Money may be valued for three reasons:

- 1 cash flows from fiscal backing
- 2 risk sharing benefits from money as a safe asset (store of value)
- 3 transaction benefits from money as a medium of exchange

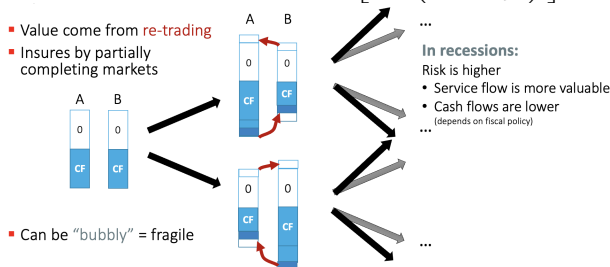
- Interaction: Δi_t affects ϑ_t , which affects Δi_t

Aside: Safe Asset Retrading/Store of Value Service Flow

$$\vartheta_t = \frac{\mathcal{MB}_t}{\mathcal{P}_t} / N_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \left(\underbrace{-\check{\mu}_{t'}^{\mathcal{MB}}}_{\text{backing}} + \underbrace{(1 - \vartheta_{t'})^2 \tilde{\sigma}_{t'}^2}_{\text{store of value}} + \underbrace{\vartheta_{t'}^{\mathcal{M}} \Delta i_{t'}}_{\text{medium of exchange}} \right) \vartheta_{t'} dt' \right].$$

- Fiscal backing may actually be negative ($\check{\mu}^{\mathcal{MB}} = -\hat{\tau}_t > 0$)
 - then money may still be valued if other benefits are sufficiently strong
 - the government then extracts seigniorage revenue from issuing more money
 - money is then a (rational) bubble
- Generalized FTPL-Equation (with individual agent SDF)

$$\frac{\mathcal{MB}_t}{\mathcal{P}_t} = \mathbb{E}_t[\text{PV}(\text{primary surplus})] + \mathbb{E}_t \left[\text{PV} \left(\Delta i_{t'} \frac{\mathcal{M}_{t'}}{\mathcal{P}_{t'}} \right) \right] + \text{Bubble-term}$$



Money and Growth: Tobin Effect

- Observation from all three variants of the model: investment & growth depend negatively on money portfolio demand (ϑ_t)
- Intuition: money crowds out real investment
 - consumption demand depends on total wealth ($C_t = \rho(q^K + q^{\mathcal{M}^B})K_t$)
 - but money is unproductive: higher $q^{\mathcal{M}^B}$ increases wealth
without raising output ($Y_t = aK_t$)
 - since output is fixed, investment must fall to meet increased consumption demand, reducing future capital and thus future output
- Formalizes argument by Tobin (1965) that portfolio choice between monetary and capital assets is a key determinant of real investment
- Aside: Tobin effect distinguishes outside money from bank-created inside money (compare Merkel, 2020)

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2 Monetary Policy

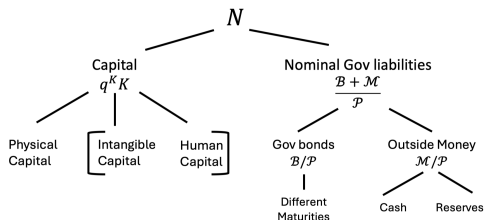
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Money and Nominal Government Debt

- Previous model: money is the only government liability
- More realistic: government issues money $\mathcal{M}_t$ and nominal bonds $\mathcal{B}_t$
 - both serve as a store of value
 - but only $\mathcal{M}_t$ -component of govt. liabilities is medium of exchange
- Model analysis is the same as in



the baseline model, except that we need to reinterpret some variables:

- we need to reinterpret some variables:
 - $q_t^{\mathcal{MB}} \rightarrow q_t^{\mathcal{M}} + q_t^{\mathcal{B}}$ (value of all government liabilities)
 - $\vartheta_t \rightarrow \frac{q_t^{\mathcal{M}} + q_t^{\mathcal{B}}}{q_t^{\mathcal{M}} + q_t^{\mathcal{B}} + q_t^K}$ (nominal wealth share)
 - $\check{\mu}_t^{\mathcal{MB}} \rightarrow \frac{\mathcal{M}_t \check{\mu}_t^{\mathcal{M}} + \mathcal{B}_t \check{\mu}_t^{\mathcal{B}}}{\mathcal{M}_t + \mathcal{B}_t}$ (average dilution rate of nom. liabilities)

- we need to allow for time-varying transaction benefits:

$$\bar{\nu}_t \text{ [money only model]} = \left(\frac{\mathcal{M}_t}{\mathcal{B}_t + \mathcal{M}_t} \right)^{1-1/3} \bar{\nu} \text{ [bond and money model]}$$

- we need to derive new valuation equations:

$$\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2 - \vartheta_t^{\mathcal{M}} \Delta i_t \text{ (Govt. Liability Valuation Equation)}$$

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \left[\int_0^T e^{-r^f t} s_t K_t dt \right] + \mathbb{E}_0 \left[\int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} dt \right] + \mathbb{E}_0 \left[e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} \right] \text{ (FTPL)}$$

Derivation for Govt. Liab. Valuation Equation and FTPL

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i-Interest Rate Policy

- What interest rate? $i_t := \vartheta_t^{\mathcal{M}} i_t^{\mathcal{M}} + (1 - \vartheta_t^{\mathcal{M}}) i_t^{\mathcal{B}}, \quad \Delta i_t := i_t^{\mathcal{B}} - i_t^{\mathcal{M}}.$

i-Interest Rate Policy

- What interest rate? $i_t := \vartheta_t^{\mathcal{M}} i_t^{\mathcal{M}} + (1 - \vartheta_t^{\mathcal{M}}) i_t^{\mathcal{B}}$, $\Delta i_t := i_t^{\mathcal{B}} - i_t^{\mathcal{M}}$.

Proposition: i_t -increase funded by **Debt Expansion**

(no “fiscal surplus implications”, $\check{\mu}_t^{\mathcal{MB}}$ remains unchanged)

- Super-neutrality: No real implications
- **Neo-Fisherian**: increase in i translates one-to-one in higher π

Proposition: i_t -increase funded by **Fiscal Tightening**

(with “fiscal surplus implications”, $\check{\mu}_t^{\mathcal{MB}}$ changes)

- Real effects: Taxes $\tau'_t > \tau_t$ and the nominal wealth share $\vartheta'_t > \vartheta_t$ are higher, and investment $\iota'_t < \iota_t$ are lower: for all t
- **Initial price drop**: $\mathcal{P}'_t < \mathcal{P}_t$ for small t

Interest Rate Policy: Long Term Bond Reevaluation Effect

- We can further distinguish money and bonds by lengthening *bond duration*
- In previous extension, bonds have infinitesimal duration
⇒ nominal bond price = 1
- With long-duration bonds, the nominal bond price can differ from 1
- Unexpected persistent i -increase at $t = 0$ **funded with debt expansion**
⇒ nominal value $\mathcal{B}_t^{(m>0)}$ of any long-term bond declines
- Turns out to not matter a lot:
the maturity composition of government bonds is irrelevant for
 - the real allocation
 - the equilibrium path of ϑ_t
 - ... but it does matter for nominal quantities, the price level, and inflation

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Proposition

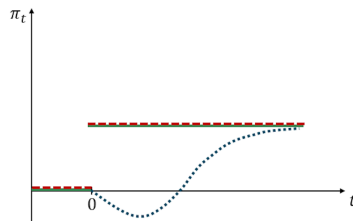
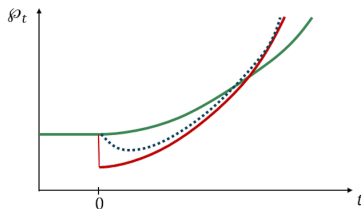
- **Irrelevance of debt maturity** composition **for real** allocation and ϑ_t -path as long as maturity does not impact market incompleteness.
- Modigliani-Miller intuition: the underlying “taxes & service flows” backing bonds independent of maturity structure.

Interest Rate Policy: Long Term Bond Reevaluation Effect

- Unexpected persistent i -increase at $t = 0$ **funded with debt expansion**
 $\Rightarrow$ nominal value $\mathcal{B}_t^{(m>0)}$ of any long-term bond declines

Proposition

- **Irrelevance of debt maturity composition for real allocation and ϑ_t -path** as long as maturity does not impact market incompleteness.
- **Inflation: Sims' Stepping on a Rake:**
 - 1 Interest rate hike devalues nominal debt $\Rightarrow$ price level drop $\mathcal{P}_0$.
 - 2 Subsequently, higher interest rate increases inflation due to Neo-Fisherian effect.
 - 3 **High/Low for too Long:** Interest rate moves extended beyond debt maturity, Neo-Fisherian force dominates.



Δi_t -Spread Interest Rate Policy

- Let nominal interest rate i_t unaffected (no direct fiscal implications)
- Change present value of medium of exchange services

Proposition: Δi_t -spread increase holding i_t constant

- Real effects: higher nominal wealth share ($\vartheta_t \uparrow$), and investment ($\iota_t \downarrow$) for all t ,
- Initial price level drop ($\mathcal{P}_t \downarrow$) for small t
- Implementation: money share of nominal liabilities $\vartheta_t^{\mathcal{M}}$ must fall (and reserve rate $i_t^{\mathcal{M}}$ to keep nominal interest rate i_t from rising)

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Quantitative Easing (QE)

- Assume $\mu_t^{\mathcal{M}} = \mu_t^{\mathcal{B}}$ for all t
- At $t = 0$ QE in form of an unexpected swap of $\mathcal{B}^{(0)}$ -bonds (T-Bill) for money $\mathcal{M}$

T-Bill QE Proposition

T-Bill QE leads to positive price level jump.

Suppose $\mathcal{P}_t$ reacts less, so that real balances $\frac{\mathcal{M}_t}{\mathcal{P}_t}$ expand

⇒ Relaxes CIA constraint and

⇒ permanently lowers Δi (if CIA was binding beforehand)

⇒ lowers “money seigniorage”

⇒ upward jump in the price level (inflation) by

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} s_s K_s ds + \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds + \mathbb{E}_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$$

The quantity equation (with fixed velocity) $\frac{\mathcal{M}_t}{\mathcal{P}_t} = \frac{C_t}{\nu}$ would also lead to upward jump of the price level.

Long-term Bond QE/QT: Wallace Neutrality

Wallace QE/QT “Neutrality”

- QE/QT of long-term assets are **not** fundamentally different from short-term bond QE/QT.
- ... since bond trading is frictionless
- also works through Δi_t
- What breaks the result?
 - SDF along long-term bonds risks are not equalized across agents (see later in the model with banks)

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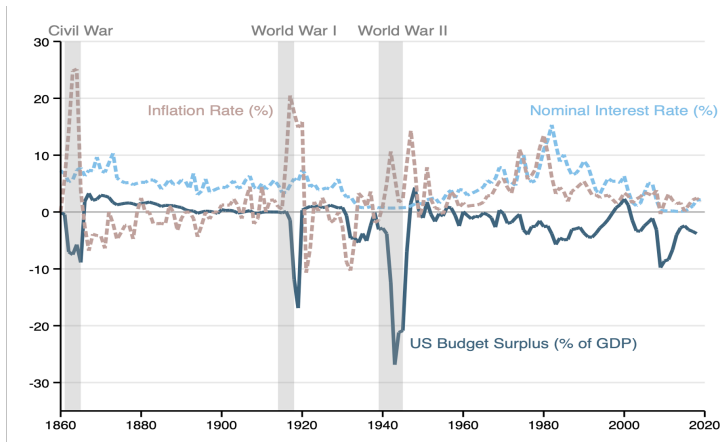
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Inflation–Fiscal Link

- Friedman (1961): “Inflation is always and everywhere a **monetary phenomenon**”
- Sims (1994): “In a fiat-money economy, inflation is a **fiscal phenomenon**, even more fundamentally than it is a monetary phenomenon”.



Source: FRED, MeasuringWorth.com, Mitchell (1908)

Remark: Two Inflation-Fiscal Connection

■ FTPL Channel

Issue additional bonds to finance new economic stimulus

+ don't change future primary surpluses $s_t K_t$

⇒ dilutes value of existing bonds (as # of bonds is higher)

⇒ Inflation

■ Short-run Aggregate Demand Channel

Issue additional bonds to finance new economic stimulus

+ Commit to increase $s_t K_t$, so that bond value is not diluted

(⇒ FTPL Channel is switched off)

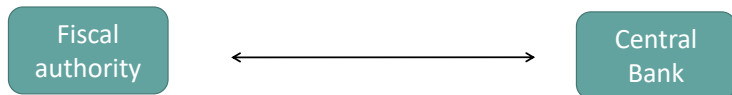
(extra bonds are financed by extra future $s_t K_t$)

If economic model is:

■ Ricardian ⇒ stimulus is neutralized by future taxes

■ Non-Ricardian ⇒ stimulus can boost demand/output
(if there is a negative output gap e.g. in NK models)

Fiscal and Monetary Interaction



■ Monetary dominance

- Monetary tightening leads fiscal authority to reduce fiscal deficit

■ Fiscal dominance

- Interest rate increase does not reduce primary fiscal deficit
- ... only lead to higher inflation

Game of chicken



See [YouTube video 4](#), minute 4:15

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Sargent and Wallace's Unpleasant Monetary Arithmetic

- With medium of exchange role of $\mathcal{M} \rightarrow$ but $\tilde{\sigma} = 0$ to avoid possibility of bubble mining.
- Sargent and Wallace (SW) point out that “*even in an economy that satisfies monetarist assumptions [...] monetary policy cannot permanent control [...] inflation*”
 - They consider an economy in which $\mathcal{P}_t$ is fully determined by money demand $\nu \mathcal{M}_t = \mathcal{P}_t Y_t$
 - But the fiscal authority is “dominant”: sets *deficits* independently of monetary policy actions
- SW emphasize seigniorage from money creation
 - Fiscal needs determine the total present value of *seigniorage*.
 - If monetary authority provides less, lower seigniorage today raises future government debt.
 - Required fiscal backing remains and the shortfall must be made up later via money printing.
 - **Tight money now means higher inflation eventually (Unpleasant Arithmetic).**
- Controlling inflation is not always within the central bank's hands. Even when money demand determines the price level, fiscal policy can dominate in the long run.

Sargent and Wallace (1981)

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The Determinacy Question

- So far: analysis of value of money restricting attention to monetary steady states
 - but this might not be the only equilibrium
 - in fact, for constant $\check{\mu}^{\mathcal{MB}}$ -policies: a second, non-monetary steady state exists
- Important question in monetary economics:
under which conditions is the equilibrium unique?
- Why does this matter?
 - want to use model to analyze comparative statics, policy actions, transmission mechanisms, etc.
 - but this is difficult if there are multiple equilibria
 - which equilibria should we compare?
 - “intrinsic” effects of policy actions vs. effects of changing coordination

Notions of Uniqueness

- Strong notion: unique rational expectations (RE) equilibrium
- Various weaker notions in monetary literature:
 - **locally unique RE equilibrium**: no other equilibrium remains always nearby
 - requires non-negligible change in private-sector beliefs to coordinate on different one
 - **unique Markov-perfect / minimum state variable equilibrium**: no other equilibrium as function of minimal state space
 - without aggregate risk and time trends: steady state uniqueness
 - **unique asymptotically monetary equilibrium**: for all other RE equilibria, value of money vanishes in the long run
 - only equilibrium consistent with expectation that value of money will remain bounded away from zero
- Here: let's focus on strong notion and third weak notion

Remark: Government Policy Paths versus Rules

- Determinacy may depend on government policy
- For many questions, it is sufficient to specify policy along the equilibrium path
- However, for determinacy, this is insufficient:
 - we need to contemplate what the government would do if markets coordinated on different outcomes
 - to do so, we need a full government policy rule (or strategy) that specifies how the government would act at off-equilibrium nodes of the game tree
- Once we specify policy rules, we have to be careful that they are feasible also off-equilibrium, e.g.:
 - the government cannot violate its flow budget constraint at off-equilibrium prices
 - the government cannot commit to fund a primary deficit (negative taxes) in states in which money is worthless

Outline for Determinacy Analysis

- In the following: analyze determinacy in the money model
- To simplify matters:
 - assume no physical investment $\iota = 0$, no government expenditure $\mathcal{G} = 0$, $\phi \rightarrow \infty$, then wealth per unit of capital is constant:

$$q_t = q = \frac{a}{\rho}$$

- keep only one motive for holding money active at a time (backing, safety, transactions)
- Recall that money valuation equation

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2 - \Delta i_t) \vartheta_t dt$$

must hold in any RE equilibrium

- in addition, any solution with $\vartheta_t \in [0, 1] \forall t \geq 0$ corresponds to a valid equilibrium
- $\vartheta_t < 0$ and $\vartheta_t > 1$ inconsistent with free disposal of money or capital

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- Model Setup
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- Adding Financial Frictions
- Adding Monetary Frictions
- Separating Money $\mathcal{M}$ and Gov. Bonds $\mathcal{B}$

2 Monetary Policy

- Interest Rate Policy
- Quantitative Easing
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- Fiscal Backing and the Fiscal Theory of the Price Level
- Bubble Theories and (In-)Determinacy
- "Pure" Unit of Account Theory

Fiscal Theory: Determinacy with Fiscal Backing

- Return to frictionless benchmark, $\tilde{\sigma} = 0$, $\mathfrak{T} \equiv 0$
- Suppose the fiscal authority follows the following policy rule:
 - set constant taxes $\tau > 0$ after any history
 - implies that also primary surplus-capital ratio $s_t = \tau a$ is constant and positive
- Money valuation equation simplifies to

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}_t^{\mathcal{MB}}) \vartheta_t dt = \left(\rho \vartheta_t - \frac{s_t}{q} \right) dt = \rho (\vartheta_t - \tau) dt$$

- This has a unique solution contained in $[0, 1]$:

$$\vartheta_t = \vartheta^{ss} := \tau$$

- if $\vartheta_t > \vartheta^{ss}$, $\mathbb{E}_t[d\vartheta_t] > 0 \rightarrow$ solution eventually > 1
 - if $\vartheta_t < \vartheta^{ss}$, $\mathbb{E}_t[d\vartheta_t] < 0 \rightarrow$ solution eventually > 1
- Conclusion (*Fiscal Theory of the Price Level*): fiscal backing can generate a determinate value of money

FTPL: The Role of Fiscal Policy

- The previous logic generalizes if we replace constant s by any path of *positive* s_t
 - positive is essential: the government must expend real resources to provide backing
 - strictly speaking, $s_t > 0$ for all t not needed, positive present value is sufficient

- But the nature of the fiscal rule matters

- A rule that fixes $\check{\mu}^{\mathcal{B}} \leq -\rho$ instead of s is consistent with continuum of RE equilibria:

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \check{\mu}^{\mathcal{B}}) \vartheta_t dt \Leftrightarrow \vartheta_t = \vartheta_0 e^{(\rho + \check{\mu}^{\mathcal{B}})t}$$

- A rule that adjusts taxes to “keep debt sustainable”, e.g., $\tau_t = \tau^0 + \alpha(\vartheta_t - \tau^0)$ ($\alpha > 1$), leads to indeterminacy:

$$\begin{aligned} \mathbb{E}_t[d\vartheta_t] &= \rho(\vartheta_t - \tau_t) dt = \rho(1 - \alpha)(\vartheta_t - \tau^0) dt \\ \Leftrightarrow \quad \vartheta_t &= \tau^0 + e^{-\rho(\alpha-1)t}(\vartheta_0 - \tau^0) \end{aligned}$$

- Latter case is the baseline assumption in NK literature
→ neutralizes effect on fiscal backing on determinacy

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Bubble Theory: Global Indeterminacy in Models

- Suppose $s = \check{\mu}^{\mathcal{M}^B} = 0$ and either of the following
 - (a) there is idiosyncratic risk $\tilde{\sigma} > \sqrt{\rho}$
 - (b) there are transaction costs $\mathfrak{T}_t(\nu) > 0$
- We focus on case (a) for concreteness, case (b) is similar
(with some complications, see lecture notes)
- The money valuation equation is then

$$\mathbb{E}_t[d\vartheta_t] = \underbrace{(\rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2)}_{\text{strictly increasing in } \vartheta_t} \vartheta_t dt$$

- This has a continuum of solutions contained in $[0, 1]$
 - the non-monetary steady state, $\vartheta_t = 0$
 - the monetary steady state, $\vartheta_t = \vartheta^{ss} := \frac{\tilde{\sigma} - \sqrt{\rho}}{\tilde{\sigma}}$
 - a nonstationary equilibrium for each $\vartheta_0 \in (0, \vartheta^{ss})$ that features $\vartheta_t > 0$ for all t but $\vartheta_t \rightarrow 0$ as $t \rightarrow 0$

Global Indeterminacy: Intuition

- Conclusion from last slide: RE equilibrium is not unique → indeterminacy
 - This is because money does not provide intrinsic value
 - Instead, it generates services from trading it:
 - as safe asset: provides risk sharing because it is *sold* to smooth idiosyncratic shocks
 - as medium of exchange: provides transaction services because it is used to *pay* for goods
 - Value for individual therefore depends on resale value in exchange
 - but resale value depends on value for buyer
 - which in turn depends on resale value in next transaction
 - ⋮
- In bubble theories, value of money depends on *social coordination*: infinite chain of beliefs how others will value it in future transactions

Bubble Theories and Weak Determinacy

- Despite this indeterminacy, there is a good reason to select $\vartheta_t = \vartheta^{ss}$
 - it is the only equilibrium with asymptotically valued money, $\lim_{t \rightarrow \infty} \vartheta_t > 0$
 - to sustain any other equilibrium, agents must believe there is eventual (hyper-)inflation that erodes the value of money
- Aside, $\vartheta_t = \vartheta^{ss}$ has also other properties that sets it apart:
 - it is locally unique
 - it is a minimum state variable equilibrium & the only one in which money has value
 - it is the only equilibrium that survives if there is a positive probability of some (arbitrarily small) fiscal backing in the future

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A Model without Money as an Asset

- Take the frictionless benchmark and set $\mathcal{MB}_t = 0$ (which implies $\tau = s = 0$)
- Then $\vartheta = 0$ and all remaining model equations remain valid
- The real side of this model is trivial:
 - capital grows at a constant rate g
 - agents consume $C_t = aK_t$ (there is no idiosyncratic risk)
 - the real interest rate is $r = \rho + g$
- We can still add money as a unit of account by adding a *zero net supply* nominal bond
 - nominal interest rate i_t controlled by the central bank
 - portfolio choice leads to a Fisher equation (without risk)

$$i_t = r + \pi_t, \quad \pi_t := \mu_t^{\mathcal{P}}$$

- Question: is there a unique equilibrium price level path $\mathcal{P}_t$?
 - answer: it depends on i -policy (and the notion of uniqueness)

Indeterminacy under Exogenous Interest Rates

- Suppose the central bank sets an exogenous time path for i_t
- Then by the Fisher equation

$$\pi_t = i_t - r = i_t - \rho - g$$

is determined

- But the initial price level $\mathcal{P}_0$ is not
- In addition, even π_t is only determined among all perfect foresight equilibria
 - there are additional sunspot RE equilibria with different inflation (and price volatility)

(Local) Determinacy with Wicksellian Feedback Rules

- Let's instead assume the central bank follows a price level feedback rule

$$i_t = i_t^0 + \phi_{\mathcal{P}} \log \mathcal{P}_t, \quad \phi_{\mathcal{P}} > 0$$

- i_t^0 is an exogenous (bounded) intercept path
- $\phi_{\mathcal{P}} \log \mathcal{P}_t$ incorporates feedback from observed price levels to i_t
- This is called a *Wicksellian interest rate rule* (Wicksell 1898)
- Combining this rule with $d\mathcal{P}_t = \pi_t \mathcal{P}_t dt$ and the Fisher equation yields

$$\begin{aligned} d \log \mathcal{P}_t &= d\mathcal{P}_t / \mathcal{P}_t = (i_t^0 - r + \phi_{\mathcal{P}} \log \mathcal{P}_t) dt \\ \Rightarrow \quad \log \mathcal{P}_t &= e^{\phi_{\mathcal{P}} t} (\log \mathcal{P}_0 - \log \mathcal{P}_0^*) - \int_t^\infty e^{-\phi_{\mathcal{P}}(s-t)} (i_s^0 - r) ds, \quad \log \mathcal{P}_0^* := - \int_0^\infty e^{-\phi_{\mathcal{P}} t} (i_t^0 - r) dt \end{aligned}$$

- All but one solutions (the one with $\mathcal{P}_0 = \mathcal{P}_0^*$) lead to unbounded $\mathcal{P}_t$ & π_t
 - there is nothing wrong with these unbounded solutions economically
 - but if we add as an additional selection rule that we seek bounded solutions, then there is a unique $\mathcal{P}_t$ solution
 - in addition, that one is the only locally unique one

(Local) Determinacy with Taylor Rules

- Contemporary literature: inflation instead of price level feedback (Taylor 1993)

$$i_t = i_t^0 + \phi_\pi \pi_t, \quad \phi_\pi > 1$$

- These do *not* work in continuous time without additional inertia, e.g.
 - interest rate smoothing
 - long-term nominal bonds
 - sticky prices
- With such inertia, such a rule can determine the path of inflation in the same way as a Wicksellian rule
 - i.e., we need to add the selection criteria “bounded inflation”
- But it will still not determine the price *level* unless prices are sticky

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Appendix: Derivation for Govt. Liab. and FTPL Equation

$$\begin{aligned}
 H_t = & e^{-\rho t} \log c_t - \xi_t c_t \\
 & + \xi_t n_t \left\{ (1 - \theta_t) \frac{\mathbb{E}_t[dr_t^{K, \tilde{i}}(\iota_t, \nu_t)]}{dt} + \underbrace{\theta_t \left[(1 - \theta_t^{\mathcal{M}}) \frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} + \theta_t^{\mathcal{M}} \frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt} \right]}_{\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} :=} \right\} \\
 & - \xi_t n_t \tilde{\zeta}_t (1 - \theta_t) \tilde{\sigma} \\
 & + \lambda_t^{\mathcal{M}} \xi_t n_t \left[\theta_t \theta_t^{\mathcal{M}} \nu_t - (1 - \theta_t) \frac{a}{q_t^K} \right]
 \end{aligned}$$

First order conditions w.r.t:

$$\begin{aligned}
 \theta_t^{\tilde{i}} : \quad & \frac{\mathbb{E}_t[dr_t^{K, \tilde{i}}(\iota_t, \nu_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \tilde{\zeta} \tilde{\sigma} + \lambda_t^{\mathcal{M}} \left(\nu_t \theta_t^{\mathcal{M}} + \frac{a}{q_t^K} \right) \\
 \theta_t^{\mathcal{M} \tilde{i}} : \quad & \frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt} = \lambda_t^{\mathcal{M}} \nu_t \\
 \nu_t^{\tilde{i}} : \quad & (1 - \theta_t) \frac{\partial \mathbb{E}[dr_t^{K, \tilde{i}}(\iota_t, \nu_t)] / dt}{\partial \nu_t} + \lambda_t^{\mathcal{M}} \theta_t \theta_t^{\mathcal{M}} = 0
 \end{aligned}$$

Recall Return Equation and Take Differences

$$\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}(\iota_t, \nu_t)]}{dt} = \frac{a - \mathcal{G} - \iota_t^{\tilde{i}} - t(\nu_t^{\tilde{i}})}{q_t^K} + \frac{q_t^{\mathcal{M}} \check{\mu}_t^{\mathcal{M}} + q_t^{\mathcal{B}} \check{\mu}_t^{\mathcal{B}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} \quad (1)$$

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} = \check{\mu}_t^{\mathcal{B}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{B}}} = i_t^{\mathcal{B}} - \pi_t \quad (2)$$

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt} = \check{\mu}_t^{\mathcal{M}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{M}}} = i_t^{\mathcal{M}} - \pi_t \quad (3)$$

- Take difference (2) and (3): $\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt} = \Delta i_t$
- Take weighted sum of (2) and (3):

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \underbrace{\vartheta_t^{\mathcal{B}} \check{\mu}_t^{\mathcal{B}} + \vartheta_t^{\mathcal{M}} \check{\mu}_t^{\mathcal{M}}}_{\check{\mu}_t^{\mathcal{MB}}} + \vartheta_t^{\mathcal{B}} \check{\mu}_t^{q^{\mathcal{B}}} + \vartheta_t^{\mathcal{M}} \check{\mu}_t^{q^{\mathcal{M}}} + \Phi(\iota_t^{\tilde{i}}) - \delta \quad (4)$$

- Take difference of (1) and (4)

$$\frac{a - \mathcal{G} - \iota_t^{\tilde{i}} - t(\nu_t^{\tilde{i}})}{q_t^K} + \frac{1}{1 - \vartheta_t} \check{\mu}_t^{\mathcal{MB}} + \underbrace{\mu_t^{q^K} - \vartheta_t^{\mathcal{B}} \mu_t^{q^{\mathcal{B}}} - \vartheta_t^{\mathcal{M}} \mu_t^{q^{\mathcal{M}}}}_{= -\mu_t^{\vartheta} / (1 - \vartheta_t)}$$

Government Liability Valuation Equation

- Plug into FOC w.r.t. θ_t :

$$\underbrace{\frac{a - \mathcal{G} - \tilde{l}_t^j - t(\nu_t^j)}{q_t^K}}_{\substack{= \rho / (1 - \vartheta_t) \\ \text{by goods-mkt clearing}}} + \frac{1}{1 - \vartheta_t} \check{\mu}_t^{\mathcal{MB}} - \frac{\mu_t^\vartheta}{1 - \vartheta_t} = \underbrace{\tilde{\zeta}_t \tilde{\sigma}}_{\substack{= (1 - \vartheta_t) \tilde{\sigma}^2 \\ \text{by log utility}}} + \lambda_t^{\mathcal{M}} \underbrace{\left(\theta_t^{\mathcal{M}} \nu_t + \frac{a}{q_t^K} \right)}_{\substack{= \frac{\vartheta_t^{\mathcal{M}}}{1 - \vartheta_t} \nu_t \\ \text{by volatility def}}}$$

- Plug into FOC w.r.t. $\vartheta_t^{\mathcal{M}}$: $\Delta i_t = \lambda_t^{\mathcal{M}} \nu_t$

Government Liability Valuation Equation:

$$\mu_t^\vartheta = \rho - (1 - \vartheta_t)^2 \tilde{\sigma}^2 + \check{\mu}_t^{\mathcal{MB}} - \vartheta_t^{\mathcal{M}} \Delta i_t$$

FTPL-Equation with $\mathcal{B}$ and $\mathcal{M}$

- Money valuation equation for log utility $\gamma = 1$:

$$\vartheta_t \mu_t^\vartheta = \vartheta_t \underbrace{\left(\rho + \overbrace{g}^{\Phi(\iota) - \delta} - g - (1 - \vartheta_t)^2 \tilde{\sigma}^2 \right)}_{= r^f - g} + \check{\mu}_t^{\mathcal{M}\mathcal{B}} - \vartheta_t^{\mathcal{M}} \Delta i_t$$

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \vartheta_t N_t$$

$$\Rightarrow d \left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} \right) = \left(r^f - \cancel{g} + \check{\mu}_t^{\mathcal{M}\mathcal{B}} - \vartheta_t^{\mathcal{M}} \Delta i + \underbrace{\frac{dN_t}{N_t} = g dt}_{\downarrow \cancel{g}} \right) \left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} \right) dt$$

- Integrate forward:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^T e^{-r^f t} \underbrace{(-\check{\mu}_t^{\mathcal{M}\mathcal{B}} + \vartheta_t^{\mathcal{M}} \Delta i)}_{= sK_t + \frac{\mathcal{M}_t}{\mathcal{P}_t} \Delta i} \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} dt + e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} \right]$$

FTPL Equation:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \left[\int_0^T e^{-r^f t} s_t K_t dt \right] + \mathbb{E}_0 \left[\int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} dt \right] + \mathbb{E}_0 \left[e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} \right]$$

FTPL-Equations with $\mathcal{B}$ and $\mathcal{M}$: Joint and Separately

- Two ways to write FTPL equation

$$\begin{aligned}\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} &= \mathbb{E}_0 \int_0^T e^{-r^f t} s_t K_t dt + \mathbb{E}_0 \int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} dt + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} \\ \frac{\mathcal{B}_0}{\mathcal{P}_0} &= \mathbb{E}_0 \int_0^T e^{-r^f t} s_t K_t dt + \mathbb{E}_0 \int_0^T e^{-r^f t} \mu_t^{\mathcal{M}} \frac{\mathcal{M}_t}{\mathcal{P}_t} dt + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{B}_T}{\mathcal{P}_T}\end{aligned}$$

- Take difference:

$$\frac{\mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} (\Delta i_t - \mu_t^{\mathcal{M}}) \frac{\mathcal{M}_t}{\mathcal{P}_t} dt + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{M}_T}{\mathcal{P}_T}$$

(may contain bubble term when take $T \rightarrow \infty$)

Sargent and Wallace (1981)

- Assume that in equilibrium
 - 1 the payment constraint is always binding
 - 2 surpluses satisfy $s_t = \underline{s}$, $\underline{s} \leq 0$ (constant deficit-GDP ratio)
 - 3 $\nu > \rho$ (given log-utility)
- Then nominal wealth shares must satisfy:

$$\vartheta_t \vartheta_t^{\mathcal{M}} = \rho / \nu \quad (\text{from goods market clearing condition})$$

$$\vartheta_t \vartheta_t^{\mathcal{B}} = \int_t^\infty \rho e^{-\rho(t'-t)} (s_{t'} + \mathcal{J}_{t'}) dt' = \underbrace{\underline{s}}_{<0} + \int_t^\infty \rho e^{-\rho(t'-t)} \mathcal{J}_{t'} dt'$$

- Suppose after time $T < \infty$ the fiscal authority can take control of $\mu_t^{\mathcal{M}}$.
- Fiscal authority chooses seigniorage to keep debt-GDP ratio constant, i.e.

$$\mathcal{J}_t = \hat{\mathcal{J}}(\vartheta_T^{\mathcal{B}}) := -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}}, \quad t \geq T$$

(there are limites on feasible seigniorage but let's ignore this for simplicity)

- For $t \leq T$, the monetary authority chooses (constant) $\mu^{\mathcal{M}}$ independently
 - Also $\mathcal{J}_t = \mu^{\mathcal{M}} q_t^{\mathcal{M}} = \mu^{\mathcal{M}}(a - \mathcal{Q})/\nu =: \mathcal{J}$ is controlled by the monetary authority
- **“Unpleasant Arithmetic” Proposition:**
Tight money now means higher inflation eventually.
 - The (constant) inflation rate over $[T, \infty)$ is strictly decreasing in $\mu^{\mathcal{M}}$ over $[0, T]$

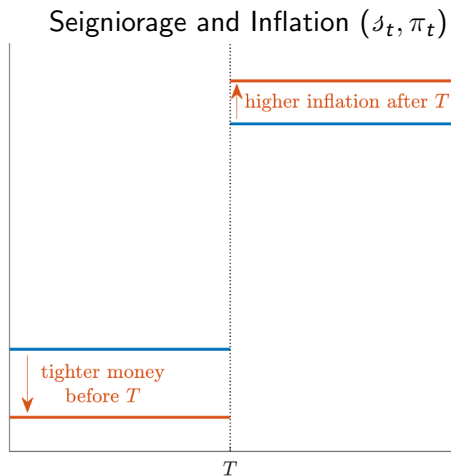
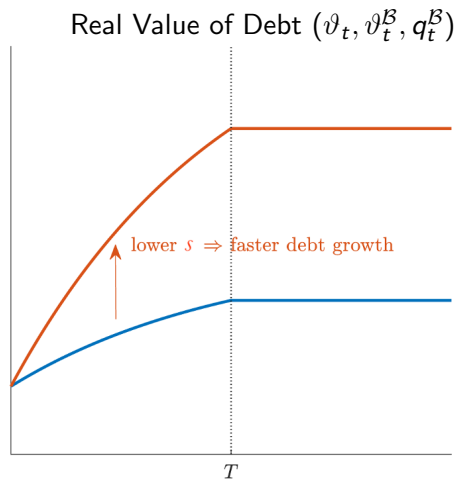
Why Does the Sargent-Wallace Proposition Hold?

- Iterating government liabilities valuation equation forward in time:

$$\vartheta_T \vartheta_T^{\mathcal{B}} = \vartheta_0 \vartheta_0^{\mathcal{B}} - \int_0^T \rho e^{-\rho t} (\underline{s} + \mathcal{J}) dt$$

- Lower money $\mu_t^{\mathcal{M}}$ over $[0, T] \Rightarrow$ lower seigniorage transfers $\mathcal{J} = \mu^{\mathcal{M}}(a - \mathcal{Q})/\nu \Rightarrow$ debt grows faster
- Higher debt at T : need larger seigniorage thereafter to cover interest payments:
 - recall $\hat{\mathcal{J}}(\vartheta_T^{\mathcal{B}}) = -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}}$ is increasing in $\vartheta_T^{\mathcal{B}}$

Illustration of Unpleasant Arithmetic



Monetary Dominance

- Suppose $T = \infty$: monetary authority is always in control of the money supply
- Is there an equilibrium? (suppose also $\mathcal{J} \neq \vartheta_0 \vartheta_0^B - \underline{s}$)
 - not with constant deficit/ K_t -ratio $s_t = \underline{s}$
 - but: a constant deficit is not necessarily feasible policy
- Two cases
 - 1 if $\mathcal{J} > \vartheta_t \vartheta_t^B - \underline{s}$, $s_t = \underline{s} < 0$ remains feasible
 - but fiscal authority will absorb money over time, effective money supply is smaller than $\mathcal{M}_t$
 - fiscal authority controls inflation
(e.g. if real debt to K_t ratio is kept constant, outcomes as if $\mathcal{J} = \vartheta_0 \vartheta_0^B - \underline{s}$)
 - 2 if $\mathcal{J} < \vartheta_t \vartheta_t^B - \underline{s}$, s_t has to rise to avoid default on nominal bonds
 - fiscal authority effectively faces an “intertemporal budget constraint”
 - e.g. smallest constant primary surpluses (per K_t is $s = \vartheta_0 \vartheta_0^B - \mathcal{J}$)