Lectures on Macro, Money, and Finance

A Heterogeneous-Agent Continuous-Time Approach[1]

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List of Symbols

i	Sector type i (experts, households...)
$\tilde{i}$	Agents of sector i with idiosyncratic risk, $\tilde{i} \in [0,1]$
c_t	Consumption
y_t	Output goods
ρ	Discount rate
a	Productivity
$\mathcal{A}(\cdot,\cdot)$	Aggregate productivity
κ	Capital share
χ	Equity share
η	Wealth share
ψ	Share of capital wealth
ϑ	Share of wealth that agents hold in money (fraction of nominal wealth)
ι	Investment rate
ω	Investment opportunities
ϕ	Adjustment costs factor in investment function
$\Phi(\cdot)$	Investment function
$\mathcal{P}$	Money price of goods
q^K	Price of capital
q^M	Rescaled price of money, $q^M = \frac{\mathcal{M}}{\mathcal{P}\cdot K}$
$\mathcal{M}$	positive net supply of money
ζ	Consumption-wealth ratio
ξ	Stochastic discount factor
V	Value function
μ_t	Drift
σ_t	Volatility
ς_t	Price of risk
$\mathrm{d}Z_t$	Brownian motions

Part I

Introduction and Solution Methods

Chapter 1

Introduction

1.1 The Three Watershed Moments in Macroeconomics

Why financial crises are so important? Figure 1.1 depicts the striking resilience of the economy of the United States. The US GDP bounced back after most recessions and returned to the previous trend growth, i.e., the economy made up previous output losses. There are two exceptions: the Great Depression in the 1930s for which the recovery took almost a decade and the global financial crisis (GFC) after 2008. In other words, Figure 1.1 suggests that regular business cycles come and go, but the economy is less resilient after financial crises.

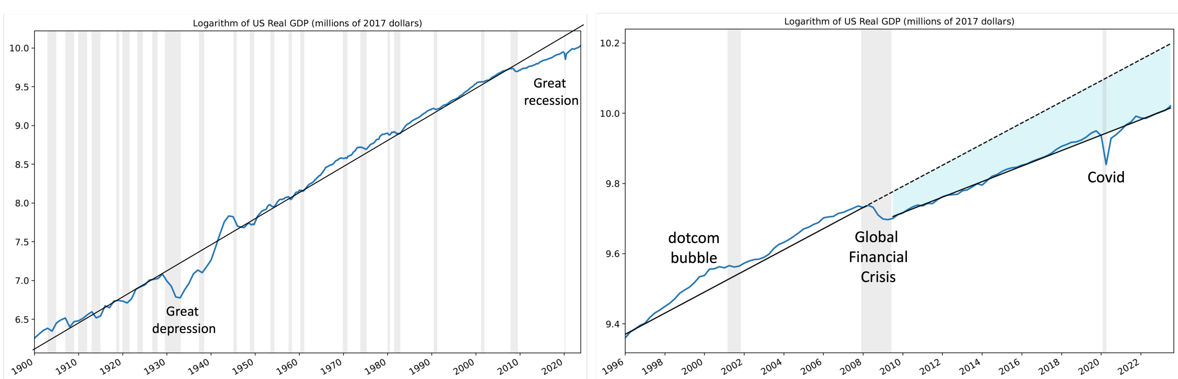

Figure 1.1: Panel A depicts the log-level of US GDP, while Panel B zooms in log GDP-level from 1996 onwards. The shaded areas show recession periods.

After the great depression in the 1930s, the economy did not bounce back for a long time, and arguably it required the fiscal spending associated with WWII to return to the previous GDP trendline. The Global Financial Crisis in 2008 led to the Great Recession, also because - as some observer argued - the fiscal spending was not aggressive enough. Panel B zooms in to stress that after 2008 both the level of GDP and the subsequent growth rate are depressed. So far, there has been no bounce back. Remarkably, the US economy was resilient to the Covid19 pandemic shock: US GDP returned to its post-GFC trend.

1.2 What is Macrofinance?

Macrofinance studies examine how finance impacts the macroeconomy. This field reached new prominence after the global financial crisis, but its roots go far back in history of economic thought. In fact, arguably all eminent economists throughout history have been concerned with the relationship between the macroeconomy and finance.

Macrofinance deals with big issues in macroeconomics, growth and efficiency, as well as with stability of the financial sector as well as the whole macroeconomy. It encompasses many strands of models and empirical analysis. All models involve a dynamic general equilibrium analysis. In most macrofinance models with financial frictions and heterogeneous agents the distribution of wealth matters. Indeed, wealth shares are important state variables. Hence, inequality is also an important policy concern for macrofinance besides growth and stability.

Macrofinance is a "broad church" that touches on most subfields of economics and finance. In finance, it is tightly connected to asset pricing, intermediary finance, corporate, household, and behavioral finance. In economics, the overlap with monetary economics and public finance is arguably the largest.

1.3 Continuous-Time Modeling

Continuous-Time macro-finance models will be the main workhorse of this class. Continuous-time modeling has several attractive features. First, there is a sharp distinction between stocks and flows. Rate changes only affect the stock over a time interval of strictly positive length. Importantly, there is no distinction between beginning-of-period and end-of-period stocks. For example, wealth is equal in the beginning and end of the period, so that consumption is the time-preference rate times the end of period wealth (for log utility). Second, when solving optimal stopping problems, one does not face integer issues.

Third, it helps overcoming time aggregation of data. Different data come in different frequencies. For example, GDP is quarterly, whereas financial data is of much higher frequency. Continuous time models provide an easy framework to deal with these discrepancies.

Fourth, continuous time modeling sometimes squares better with reality. We live in a continuous-time world and do not consume only at the end of the quarter, even though the aggregate macroeconomic data commonly come in quarterly. In discrete time modeling, consumption within periods is simply summed up. This creates an artificial distinction between consumption substitution within and across quarters. Implicitly, the elasticity of intertemporal substitution in a quarterly discrete time model is infinity within quarters, while across quarters it is given by the curvature of the per-period utility function. In contrast, in continuous-time models the elasticity of intertemporal substitution is the same within and across periods.

Besides, continuous-time modeling is often more tractable and allows for a tighter characterization of economic models. For example, in the models discussed in these notes, we can fully characterize the whole dynamical system including the volatility dynamics instead of simply studying a log-linearized representation around the steady state. The well-known Kolmogorov Forward Equation (Fokker Planck Equation) reveals state variables' distribution evolution path starting from any initial distribution, while the impulse response functions in discrete time capture only the expected path after a shock that starts at the steady state.[1] Also, the stationary distribution can be bimodal and exhibit large swings, unlike stable normal distributions that log-linearized models imply.

[1] We will introduce its continuous-time version: "distributional impulse response" in Chapter 3.

Moreover, continuous time allows one to derive more analytical steps and "more" closed form characterizations of the equilibrium before resorting to a numerical analysis. For example, the evolutions of capital, price, net worth are captured by the closed form stochastic differential equations. One can also derive explicit closed-form expressions for amplification terms, as only the slope of the price function is necessary to characterize amplification. In contrast, in discrete-time settings the whole price function is needed, as the jump size may vary. On the other hand, the numerical procedure is more straightforward and faster[2] than discrete time as one no longer needs to search the grid when looking for policy functions.

In terms of tractability, one of the largest benefits of continuous-time models for macrofinance arises in the context of portfolio choice problems. To appreciate this, let us briefly describe a fundamental difficulty of analytically handling portfolio choice in a discrete-time environment. Suppose there are J assets, $j = 1, ..., J$, with (gross) returns R_t^j. In the cross section, returns aggregate additively to portfolio returns. The portfolio assigning weight θ_t^j on asset j has a return of $\sum_{j=1}^{J} \theta_t^j R_t^j$. To retain tractability in the cross-section, we typically assume that R_t^j is normally distributed, so that the portfolio return payoff also follows a normal distribution. However, in the time dimension, returns aggregate multiplicatively. The return of asset j over time is $R_t^j \times R_{t+1}^j \times \cdots$. The returns can only be aggregated over time within the same family of distribution if they are log-normally distributed. To tackle this problem in discrete-time models, it is common place to log-linearize the first-order conditions around the steady state. However, a first-order approximation ignores all volatility terms and makes all assets equivalent. A second-order approximation around steady state is only a partial resolution: it eliminates time-varying volatility, making it impossible to study volatility dynamics. Often one resorts to a log-linearization approximation beyond the steady state à la Campbell and Shiller, (1988a, 1988b), that mimics the continuous time portfolio choice problem and is precise in the continuous time limit.

Admittedly, some of these features are not due to continuous time per se but due to the continuous nature of a particular class of stochastic processes that is typically

[2] Numerical implementations will be studied in Chapter 3. The equilibrium is characterized by partial different equations and solved numerically.

assumed in continuous-time modeling, so-called Itô Processes. An Itô process is a process whose changes over infinitesimally small time intervals are normally distributed. If X_t is the value of a stochastic process at time t, we denote by $\mathrm{d}X_t$ its *time differential*, which is to be interpreted as "$X_{t+dt} - X_t$" for a small (infinitesimal) time increment dt. An Itô process X_t satisfies

$$\mathrm{d}X_t = \mu_{X,t}\mathrm{d}t + \sigma_{X,t}\mathrm{d}Z_t.$$

Here, $\mathrm{d}Z_t$ is the time differential of a *Brownian motion*. Intuitively, one can think of $\mathrm{d}Z_t$ as the continuous-time analog of normal white noise, i.e., i.i.d. standard normal shocks, in discrete time time series models. We discuss Brownian motion in the next subsection. The coefficients $\mu_{X,t}$ and $\sigma_{X,t}$ are called the (arithmetic) *drift* and *volatility* of the Itô process at time t. The volatility dynamics are fully loaded on the Brownian shocks $\mathrm{d}Z_t$, instead of some probabilistic states as is commonly the case in discrete-time models (e.g., a Lucas tree). Importantly, even if the time differential $\mathrm{d}X_t$ of the Itô process is normal at any given time, the time increments $X_t - X_s$ of the process over any positive-length time interval $[s, t]$ can nevertheless be non-normal and exhibit, for example, skewness when the drift and volatility are time-varying.

An important property of Itô processes is that their paths are continuous. In economic models, this means information arrives smoothly in a continuous manner. Implicitly, it also assumes that agents can react continuously to a continuous information flow, so there are no jumps of any variables. On the one hand, continuous paths can greatly simplify analysis and numerical computations. For example, the discrete-time collateral constraint $b_t R_{t,t+1} \leq \min\{q_{t+1}\}k_t$ becomes $b_t \leq (q_t + \mathrm{d}q_t)k_t$ in continuous time. Since continuous paths rule out jumps, $\mathrm{d}q_t$ is infinitesimal compared to q_t, hence the condition simplifies to $b_t \leq q_t k_t$. On the other hand, continuous paths can be restrictive. For example, in an environment with only Itô processes, investors can delever continuously to avoid default, which emboldens investors ex ante and makes it impossible to study default in equilibrium. For more general purposes, one might use Lévy processes which allow for jumps.

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Chapter 2

Portfolio and Consumption Choice in Partial Equilibrium

To start with, we introduce the basic terminology and results we will follow in this course and apply them to solve Merton's portfolio choice problem. The main objective of this lecture is to introduce the basic techniques to solve continuous-time macrofinance models.

2.1 Itô Calculus

2.1.1 Brownian Motion

As described, a Brownian motion is a process Z_t whose time differentials $\mathrm{d}Z_t$ play the role of normally distributed white noise. This can be motivated from a discrete-time approximation as a binomial tree over shrinking time periods Δt, with shrinking steps $h_n = \sigma\sqrt{\Delta t/n}$. Figure 2.1 shows processes for different n. Even though the individual time steps in the discrete tree are not normally distributed (they have a Bernoulli distribution), the change in the tree over a fixed number of time units sums over more and more time steps as we refine the tree (increase n) and, by the central limit theorem, are normally distributed in the limit $n \to \infty$. In Section 2.1, we introduce a formal definition of a Brownian motion on a filtered probability space.

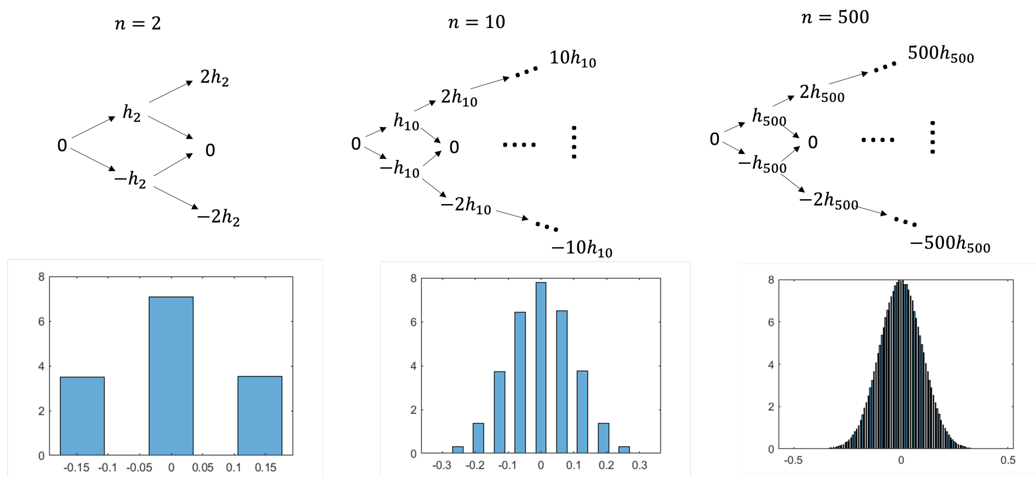

Figure 2.1: Binomial trees for different n.

This section will more formally introduce the basics of Itô calculus, which will be extensively used during the course.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and let X_t be a $\mathcal{F}_t$-measurable process, and let us start by defining Brownian motions and Itô processes.

Definition 2.1. *A Brownian motion Z_t on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies*

1. $Z_0 = 0$
2. *Z_t is almost surely continuous*
3. *Z_t has independent increments*
4. *$Z_t - Z_s \sim \mathcal{N}(0, t-s)$ for $s \in [0, t]$*
5. *Z_t is $\mathcal{F}_t$-measurable*

Definition 2.2. *An Itô process X_t is defined as*

$$X_t = X_0 + \int_0^t \mu_{X,s} ds + \int_0^t \sigma_{X,s} dZ_s$$

with $(\mu_{X,t})_{t\geq 0}$ a predictable Lebesgue integrable process and $(\sigma_{X,t})_{t\geq 0}$ a predictable Z_t-integrable process. That is, $\int_0^t (\sigma_{X,s}^2 + |\mu_{X,s}|) ds < \infty$ for all t. Differentiation with respect to time yields the representation

$$dX_t = \mu_{X,t} dt + \sigma_{X,t} dZ_t,$$

where $\mu_{X,t}$ and $\sigma_{X,t}$ are the arithmetic drift and volatility of X_t.

The previous definition introduced an arithmetic Itô process. However, in this course, we will work mainly with a geometric representation of Itô processes,

$$\frac{\mathrm{d}X_t}{X_t} = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t,$$

where the geometric drift and volatility are defined as $\mu_t^X \equiv \mu_{X,t}/X_t$ and $\sigma_t^X \equiv \sigma_{X,t}/X_t$. This representation is well-defined if the process is always positive or always negative. Throughout, we use the notation convention that subscripts refer to arithmetic and superscripts refer to geometric drift and volatility.

2.1.2 Itô's Lemma

Let us now introduce three important formulas: Itô's lemma, Itô's product rule, and Itô's quotient rule. The latter two are corollaries of Itô's lemma in two dimensions.

Lemma 2.1. *Itô's lemma. For any twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$, $f(X_t)$ is also an Itô process with*

$$\mathrm{d}f(X_t) = \left[f'(X_t)(\mu_t^X X_t) + \frac{1}{2}f''(X_t)(\sigma_t^X X_t)^2\right]\mathrm{d}t + f'(X_t)(\sigma_t^X X_t)\mathrm{d}Z_t$$

.

Moreover, if the function is such that $f(t, X_t)$ depends explicitly on time, then the drift of $f(t, X_t)$ includes $\partial_t f(t, X_t)$.

Corollary 2.1. *Itô's product and quotient rule. For any geometric Itô processes X_t, Y_t, $X_t Y_t$ and X_t/Y_t are Itô processes with*

$$\frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)\mathrm{d}t + (\sigma_t^X + \sigma_t^Y)\mathrm{d}Z_t.$$
$$\frac{\mathrm{d}(X_t/Y_t)}{X_t/Y_t} = \left[\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right]\mathrm{d}t + (\sigma_t^X - \sigma_t^Y)\mathrm{d}Z_t.$$

2.2 A Simple Portfolio Choice Problem

In this section we describe a simple version of the Merton portfolio choice problem with a single risky asset and a single Brownian shock process. Both assumptions are for ease of exposition only and can be generalized without additional conceptual or practical difficulties.

Preferences. Consider an agent who chooses a lifetime stream of consumption $\{c_t\}_{t=0}^{\infty}$ and portfolio weights $\{\theta_t\}_{t=0}^{\infty}$ to maximize

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) \mathrm{d}t.$$

where $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$.

Net worth evolution. The agent's net worth at $t = 0$ is given by n_0 and thereafter it evolves as follows

$$\mathrm{d}n_t = -c_t \mathrm{d}t + n_t \left[\theta_t r_t \mathrm{d}t + (1-\theta_t)\mathrm{d}r_t^a\right]$$

for all $t > 0$ subject to a solvency constraint $n_t \geq 0$. r_t is the risk-free rate and $\mathrm{d}r_t^a = (r_t + \delta_t^a)\mathrm{d}t + \sigma_t^a \mathrm{d}Z_t$ is the risky asset return process with risk premium δ_t^a.

State space. Suppose that returns are a function of an external state variable η_t so that $r_t = r(\eta_t), \delta_t^a = \delta^a(\eta_t)$ and $\sigma_t^a = \sigma^a(\eta_t)$ where η evolves according to a diffusion process

$$\mathrm{d}\eta_t = \mu_t^{\eta}(\eta_t)\eta_t \mathrm{d}t + \sigma_t^{\eta}(\eta_t)\eta_t \mathrm{d}Z_t$$

with η_0 given.

Hence, the decision problem has two states variables (n_t, η_t) where n_t is a controlled state and η_t is an external state.

2.3 Solving Stochastic Control Problems: Hamilton-Jacobi-Bellman Equation, Pontryagin Stochastic Maximum Principle, and Martingale Method

We will now present three methods to solve this optimization in continuous time: the Hamilton-Jacobi-Bellman (HJB) Equation, Pontryagin's Stochastic Maximum Principle, and the Martingale Method.

2.3.1 Hamilton-Jacobi-Bellman (HJB) Equation

The HJB equation is the continuous-time analogue to the Bellman equation. To derive the HJB equation, let $\mathcal{A}(n,\eta)$ be the set of admissible choices $\{c_t,\theta_t\}_{t=0}^{\infty}$ given the initial condition $n_0 = n, \eta_0 = \eta$ and $\mathcal{A}_T(n,\eta)$ be the set of policies $\{c_t,\theta_t\}_{t=0}^{T}$ over $[0,T]$ that have admissible extensions to $[0,\infty)$, $\{c_t,\theta_t\}_{t=0}^{\infty} \subset \mathcal{A}(n,\eta)$. The value function $V(n,\eta)$ of the decision problem is defined to be

$$V(n,\eta) := \max_{\{\theta_t,c_t\}_{t=0}^{\infty}\in\mathcal{A}(n,\eta)} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho t}u(c_t)\mathrm{d}t\right].$$

Notice that V satisfies the Bellman principle of optimality: for all $T > 0$ so that

$$V(n,\eta) = \max_{\{\theta_t,c_t\}_{t=0}^{T}\subset\mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^{T} e^{-\rho t}u(c_t)\mathrm{d}t + e^{-\rho T}V(n_T,\eta_T)\right],$$

where n_T depends on the choice $\{\theta_t,c_t\}_{t=0}^{T}$ over $[0,T]$.

With $V_t := V(n_t,\eta_t)$, we can write the principle of optimality as:

$$0 = \max_{\{\theta_t,c_t\}_{t=0}^{T}\subset\mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^{T} e^{-\rho t}u(c_t)\mathrm{d}t + e^{-\rho T}V_T - V_0\right].$$

Integrating $e^{-\rho T}V_T - V_0$ by parts yields

$$e^{-\rho T}V_T - V_0 = -\rho\int_0^T e^{-\rho t}V_t \mathrm{d}t + \int_0^T e^{-\rho t}\mathrm{d}V_t.$$

Thus,

$$0 = \max_{\{\theta_t, c_t\}_{t=0}^T \subset \mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^T e^{-\rho t}(u(c_t) - \rho V_t)\mathrm{d}t + \int_0^T e^{-\rho t}\mathrm{d}V_t\right].$$

We can now divide by T and take the limit as $T \downarrow 0$ to obtain the stochastic version of the HJB

$$\boxed{\rho V_t \mathrm{d}t = \max_{c_t,\theta_t}\{u(c_t)\mathrm{d}t + \mathbb{E}[\mathrm{d}V_t]\}} \tag{2.1}$$

Now we will transform the previous equation into a non-stochastic differential equation. To do so, recall that $V_t = V(n_t, \eta_t)$ and by Itô's lemma

$$\mathbb{E}[\mathrm{d}V_t] = \left[\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn}V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta}V(n_t,\eta_t)\sigma_{\eta,t}^2\right) + \partial_{n\eta}V(n_t,\eta_t)\sigma_{n,t}\sigma_{\eta,t}\right]\mathrm{d}t,$$

where $\mu_{n,t} = -c_t + n_t\left[r(\eta_t) + (1-\theta_t)\delta^a(\eta_t)\right]$ and $\sigma_{n,t} = n_t(1-\theta_t)\sigma^a(\eta_t)$.

Hence, Equation 2.1 can be written as

$$\begin{aligned}\rho V(n,\eta) = &\max_c\left\{u(c) - \partial_n V(n,\eta)c\right\} + \max_\theta\Big\{\partial_n V(n,\eta)n\left[r(\eta) + (1-\theta)\delta^a(\eta)\right] \\ &+ \left(\frac{1}{2}\partial_{nn}V(n,\eta)n(1-\theta)\sigma^a(\eta) + \partial_{n\eta}V(n,\eta)\sigma_\eta(\eta)\right)n(1-\theta)\sigma^a(\eta)\Big\} \\ &+ \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta}V(n,\eta)\left(\sigma_\eta(\eta)\right)^2\end{aligned}$$

which is a nonlinear partial differential equation in $V(n,\eta)$.

Special case: constant returns

Assume that returns are constant so that $r_t = r, \delta_t^a = \delta^a, \sigma_t^a = \sigma^a$. Then, we can drop η from the problem and write the HJB as

$$\rho V(n) = \max_c \left\{u(c) - V'(n)c\right\} + \max_\theta \left\{V'(n)n\left[r + (1-\theta)\delta^a\right] + \frac{1}{2}V''(n)n^2\left((1-\theta)\sigma^a\right)^2\right\}$$

The first-order conditions yield the optimal consumption and portfolio choices

$$\begin{aligned} u'(c) &= V'(n) \\ 1-\theta &= \left(-\frac{V''(n)n}{V'(n)}\right)^{-1}\frac{\delta^a}{(\sigma^a)^2} \end{aligned}$$

where $-V''(n)n/V'(n)$ is the relative risk aversion coefficient.

To solve this problem, we will guess a functional form for the value function $V(n) = u(\omega n)/\rho$ for some constant $\omega > 0$. Plugging this guess into the HJB equation yields

$$\begin{cases} \log\omega + \log n = \log\rho + \log n - 1 + \frac{1}{\rho}\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right) & \text{if } \gamma = 1 \text{(log utility)} \\ \rho\frac{(\omega n)^{1-\gamma}}{\rho} = \gamma\rho^{1/\gamma}\omega^{1-1/\gamma}\frac{(\omega n)^{1-\gamma}}{\rho} + (1-\gamma)\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)\frac{(\omega n)^{1-\gamma}}{\rho} & \text{if } \gamma \neq 1 \end{cases}$$

Notice that in both cases, n cancels out, thus verifying our guess, and we can then solve for ω. The full solution is given by

$$\begin{aligned} V(n) &= \frac{u(\omega n)}{\rho} \\ c(n) &= \rho^{1/\gamma}\omega^{1-1/\gamma}n \\ 1-\theta(n) &= \frac{1}{\gamma}\frac{\delta^a}{(\sigma^a)^2} \\ \omega &= \rho\left(1 + \frac{\gamma-1}{\gamma}\frac{1}{\rho}\left(r - \rho + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)\right)^{\frac{\gamma}{\gamma-1}} \end{aligned}$$

Let us analyze the optimal consumption choice. We can denote the consumption net worth ratio by $\check{\rho}$ and write it as

$$\check{\rho} := c_t/n_t = \rho^{1/\gamma}\omega^{1-1/\gamma}$$

where ω can be referred to as "investment opportunities". The reaction of c/n to investment opportunities depends on the elasticity of intertemporal substitution $\psi := 1/\gamma$. In particular

i. if $\psi < 1$, then better investment opportunities lead to an increase in consumption and a decrease in savings

ii. if $\psi > 1$, then better investment opportunities lead to a decrease in consumption and an increase in savings

iii. if $\psi = 1$, then the consumption to wealth ratio is independent of investment opportunities.

Behind these results are income and substitution effects. On the one hand, better investment opportunities make the agent wealthier, and she responds by increasing consumption in all periods. On the other hand, these better investment opportunities make saving more attractive, and to benefit from it, the agent needs to postpone consumption today to get more consumption in the future. Whenever $\psi < 1$, the substitution effect, a desire to smooth consumption, is weak and the income effect dominates. However, if $\psi > 1$, the investor is less averse to variation in consumption and the substitution effect dominates.

Special case: time-varying returns

When returns are time-varying, we can use the same approach as before where now we guess $V(n,\eta) = u(\omega(\eta)n)/\rho$. This yields the following optimal consumption and portfolio choices

$$c(n,\eta) = \underbrace{\rho^{1/\gamma}(\omega(\eta))^{1-1/\gamma}}_{\check{\rho}:=} n$$

$$1-\theta(n,\eta) = \underbrace{\frac{1}{\gamma}\frac{\delta^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\frac{\omega'(\eta)}{\omega(\eta)}\sigma^\omega(\eta)\sigma^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{hedging demand}}$$

where now investment opportunities $\omega(\eta)$ are state-dependent. Notice that there is an additional hedging demand term that depends on the covariance $\sigma^\omega\sigma^a$ of investment opportunities with asset returns. To obtain a solution for $\omega(\eta)$, we need to plug in the optimal choices into the HJB equation which will yield an ODE for $\omega(\eta)$ that can be solved numerically.

What is the role of a hedging demand? The variation in future investment opportunities is relevant for portfolio choice for two opposing motives. First, if investment opportunities are good, it is valuable to have available resources. Then, it is reasonable to invest in assets that pay off in states in which investment opportunities are good. However, if investment opportunities are bad, that is a bad time for the investor and additional wealth is valuable. Then it makes sense to invest in assets that pay off in states in which investment opportunities are bad. Which of the two dominates depends on γ. If $\gamma < 1$, the investor is not very risk averse and prefers to have resources available when it is profitable to invest. If $\gamma > 1$, the investor is sufficiently risk averse to want to hedge against bad times. When $\gamma = 1$, the two forces cancel out and the investor acts myopically. Notice that a very conservative investor ($\gamma \to \infty$) cares only about the hedging component.

2.3.2 Pontryagin's Stochastic Maximum Principle

Pontryagin's maximum principle is a method to solve optimal control problems, which is complementary to dynamic programming.

Stochastic maximum principle. Consider a control problem

$$dX_t = \mu(X_t, A_t)dt + \sigma(X_t, A_t)dZ_t,$$

where A_t are the controls and X_t are the states. The stochastic maximum principle is formulated for finite-horizon problems with the objective function of the form

$$\mathbb{E}_0\left[\int_0^T g\left(t, X_t, A_t\right) \mathrm{d}t + G\left(X_T\right)\right],$$

where the payoff flow $g\left(t, X_t, A_t\right)$ depends on t and hence can accommodate discounting.

To solve the optimization problem, one can work with the special adjoint process p_t, which is the dynamic Lagrange multiplier on the state variable X_t. We label p_t and its volatility q_t as *costates* of the system, and then optimize the Hamiltonian

$$H = g(t, X, A) + \langle p, \mu(X, A)\rangle + \operatorname{tr}\left[q^T \sigma(X, A)\right]. \tag{2.2}$$

Under necessary convexity conditions[a], the *stochastic maximum principle* says that p_t must satisfy the BSDE

$$\mathrm{d}p_t = -H_X\left(t, X_t, A_t, p_t, q_t\right) dt + q_t \mathrm{d}Z_t \tag{2.3}$$

with terminal condition $p_T = G'(X_T)$.

[a] See the convexity conditions in Yuliy's "Overview of Stochastic Calculus."

To solve Merton's portfolio choice problem, let ξ_t be the costate and $-\varsigma_t\xi_t$ be its volatility. Then, the Hamiltonian is given by

$$\begin{aligned}
H_t &= e^{-\rho t}\frac{c_t^{1-\gamma}-1}{1-\gamma} + \xi_t n_t \mu_t^n - \varsigma_t \xi_t n_t \sigma_t^n \\
&= e^{-\rho t}\frac{c_t^{1-\gamma}-1}{1-\gamma} + \xi_t\left[-c_t + n_t(1-\theta_t)(r_t+\delta_t^a) + n_t\theta_t r_t - \varsigma_t n_t(1-\theta_t)\sigma_t^a\right]
\end{aligned}$$

The first-order conditions with respect to $\{c_t, \theta_t\}$ yield

$$\begin{aligned}
e^{-\rho t} c_t^{-\gamma} &= \xi_t \\
\delta_t^a &= \varsigma_t \sigma_t^a
\end{aligned}$$

The costate equation is given by

$$\begin{aligned} d\xi_t &= -\partial_n H dt - \varsigma_t \xi_t dZ_t \\ &= -\xi_t \left[r_t + (1-\theta_t)(\delta_t^a - \varsigma_t \sigma_t^a) \right] dt - \varsigma_t \xi_t dZ_t \\ &= -r_t \xi_t dt - \varsigma_t \xi_t dZ_t \end{aligned}$$

where the last line uses the first-order condition with respect to portfolio holdings.

Under the assumption $dc_t = \mu_t^c c_t \, dt + \sigma_t^c c_t \, dZ_t$, the first-order condition with respect to consumption implies by Itô's lemma

$$d\xi_t = -\left[\rho + \gamma \mu_t^c - \frac{1}{2}\gamma(1+\gamma)(\sigma_t^c)^2 \right] \xi_t dt - \gamma \sigma_t^c \xi_t dZ_t$$

Consider the special case of constant returns and log-utility and assume $c_t = a n_t$ for some constant $a > 0$. Then $\mu_t^c = \mu_t^n = -a + r + (1-\theta_t)\delta^a$ and $\sigma_t^c = \sigma_t^n = (1-\theta_t)\sigma^a$.

The costate equation then implies

$$\begin{aligned} r &= \rho - a + r + (1-\theta)\delta^a - (1-\theta)^2(\sigma^a)^2 \\ \varsigma &= (1-\theta)\sigma^a \end{aligned}$$

Combining the last equation with the first order condition with respect to portfolio holdings yields $1 - \theta = \frac{\delta^a}{(\sigma^a)^2}$. Putting this result back into the drift of ξ_t implies that $a = \rho$, which confirms our guess. Therefore, we obtain the same solution as with the HJB equation.

2.3.3 Martingale Method

Now we introduce the martingale approach, a powerful tool for many macro-finance models.

Martingale approach in discrete time.

Consider a standard dynamic portfolio choice problem in discrete time:

$$\max_{\{\boldsymbol{\theta}_\tau, c_\tau\}_{\tau=t}^{\infty}} \quad \mathbb{E}_t \left[\sum_{\tau=t}^{\infty} \frac{1}{(1+\rho)^{\tau-t}} u(c_\tau) \right]$$
$$\text{s.t.} \qquad \boldsymbol{\theta}_t \boldsymbol{p}_t = \boldsymbol{\theta}_{t-1}(\boldsymbol{p}_t + \boldsymbol{d}_t) - c_t \qquad \forall t,$$

where $\{\boldsymbol{\theta}_t, \boldsymbol{p}_t, \boldsymbol{d}_t\}$ are the vectors of holdings, prices and dividends of different assets. WLOG, we focus on an environment with one asset. The FOC w.r.t. θ_t is

$$\xi_t p_t = \mathbb{E}_t \left[\xi_{t+1}(p_{t+1} + d_{t+1}) \right],$$

where $\xi_t = \frac{1}{(1+\rho)^t} \frac{u'(c_t)}{u'(c_0)}$ is the (multi-period) SDF. Consider a self-financing trading strategy A where one reinvests dividend d_t in every period. The price of the strategy p_t^A satisfies

$$\xi_t p_t^A = \mathbb{E}_t \left[\xi_{t+1} p_{t+1}^A \right],$$

i.e., the process $\xi_t p_t^A$ is a martingale.

Martingale approach in continuous time.

Consider a similar portfolio choice problem in continuous time:

$$\max_{\{\iota_t^e, \boldsymbol{\theta}_t^e, c_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) \mathrm{d}t \right]$$
$$\text{s.t.} \qquad \frac{\mathrm{d}n_t}{n_t} = -\frac{c_t}{n_t}\mathrm{d}t + \sum_j \theta_t^j \mathrm{d}r_t^j + \text{ labor income/endowment/taxes}$$
$$n_0 \text{ given.}$$

Here n_t is the net worth of the agent. r_t^j denotes the return of asset j. Let x_t^A be the value of a self-financing trading strategy A where one reinvests all dividends. Again, define the SDF as $\xi_t^i = e^{-\rho t} u'(c_t^i)$. Then it must be that $\xi_t x_t^A$ follows a martingale. (For proof, see Brunnermeier and Sannikov (2016, pg. 19) or separate notes prepared by Sebastian.) Let

$$\frac{\mathrm{d}x_t^A}{x_t^A} = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t.$$

Assume that the SDF follows

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -r_t \mathrm{d}t - \varsigma_t \mathrm{d}Z_t.$$

Using Itô's product rule,

$$\frac{\mathrm{d}(\xi_t x_t^A)}{\xi_t x_t^A} = \left[-r_t + \mu_t^A - \varsigma_t \sigma_t^A\right] \mathrm{d}t + \text{volatility terms}.$$

Since $\xi_t x_t^A$ follows a martingale, its drift equals zero, i.e.,

$$\mu_t^A = r_t + \varsigma_t \sigma_t^A.$$

Example 1. For risk-free asset, $\sigma_t^A = 0$. Hence, $r_t^F = r_t$.

Example 2. For any two assets A, B, we have $\mu_t^A - \mu_t^B = \varsigma_t(\sigma_t^A - \sigma_t^B)$.

Risky Asset as Asset A and Bonds as Asset B. The bond follows the return process

$$\mathrm{d}r_t^B = r_t \mathrm{d}t,$$

and the risky asset follows the return process

$$\mathrm{d}r_t^A = (r_t + \delta_t^a)\mathrm{d}t + \sigma_t^a \mathrm{d}Z_t.$$

Then, the martingale asset pricing condition is

$$\begin{aligned} &\mathbb{E}_t \left[\mathrm{d}r_t^A - \mathrm{d}r_t^B\right] / \mathrm{d}t = \varsigma_t(\sigma_t^a - 0) \\ \iff &(r_t + \delta_t^a) - r_t = \varsigma_t \sigma_t^a \end{aligned}$$

which yields

$$\delta_t^a = \varsigma_t \sigma_t^a. \tag{2.4}$$

We can then recover ς_t by Itô's lemma. Indeed, ξ_t is $e^{-\rho t}u'(c_t) = e^{-\rho t}c_t^{-\gamma}$. [Note: $dc_t = \mu_t^c c_t \mathrm{d}t + \sigma_t^c c_t \mathrm{d}Z_t$]. And, note $u'' = -\gamma c^{-\gamma-1}, u''' = \gamma(\gamma+1)c^{-\gamma-2}$,

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -\underbrace{\left(\rho + \gamma\mu_t^c - \frac{1}{2}\gamma(\gamma+1)(\sigma_t^c)^2\right)}_{r_t^f}\mathrm{d}t - \underbrace{\gamma\sigma_t^c}_{\varsigma_t}\mathrm{d}Z_t.$$

Consider the special case of constant returns and log-utility and assume $c_t = an_t$ for some constant $a > 0$. Then $\mu_t^c = \mu_t^n = -a + r + (1-\theta_t)\delta^a$ and $\sigma_t^c = \sigma_t^n = (1-\theta_t)\sigma^a$. Then, by (2.4) and the previous equation,

$$\frac{\delta^a}{\sigma^a} = \varsigma_t = \sigma_t^c = (1-\theta_t)\sigma^a$$

which reduces to the same solution as before

$$1 - \theta = \frac{\delta^a}{(\sigma^a)^2}.$$

Net Worth as Asset A and Bonds as Asset B. We can use a self-financing strategy that reinvests consisting of an agent's net worth with consumption reinvested. The return on this strategy is

$$\begin{aligned}
\mathrm{d}r_t^n = \frac{\mathrm{d}n_t + c_t\mathrm{d}t}{n_t} &= \theta_t r_t \mathrm{d}t + (1-\theta_t)\mathrm{d}r_t^a \\
&= \theta_t r_t \mathrm{d}t + (1-\theta_t)(r_t + \delta_t^a)\mathrm{d}t + (1-\theta_t)\sigma_t^a \mathrm{d}Z_t \\
&= (r_t + (1-\theta_t)\delta_t^a)\mathrm{d}t + (1-\theta_t)\sigma_t^a \mathrm{d}Z_t,
\end{aligned}$$

The martingale asset pricing condition is therefore

$$\begin{aligned}
\mathbb{E}_t\left[dr_t^n - dr_t\right]/\mathrm{d}t &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff (r_t + (1-\theta_t)\delta_t^a) - r_t &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff (1-\theta_t)\delta_t^a &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff \delta_t^a &= \varsigma_t\sigma_t^a
\end{aligned}$$

which is the same as (2.4).

2.4 Exercises

2.4.1 Solving Differential Equations

1. Read Section 1 of Sebastian's notes on differential equations.

2. Solve the following ODEs

$$y' = y^{-19} \tag{2.5}$$

$$y' = x\cos(x^2)y^2 \tag{2.6}$$

$$y'' = -y \tag{2.7}$$

   on the interval $[0, 10]$ with the initial condition $y(0) = 1$ for all three equations and an additional initial condition $y(0) = 0$ for equation 2.7 using the following three methods:

   (a) explicit Euler method (Section 1.2.1);

   (b) implicit Euler method (Section 1.2.2), using a built-in root-finder of your numerical software;

   (c) a built-in ODE solver of your numerical software.

   Compare the accuracy of explicit and implicit methods across different grid sizes $(N = 11, 51, 501, 10001)$. For each of the three equations, find the grid size that you like the most and plot the results from the three approximation methods together with the respective true solution. These are given by:

$$y(x) = (20x + 1)^{1/20} \qquad y(x) = \frac{1}{1 - \sin x^2/2} \qquad y(x) = \cos x$$

3. Now consider a variation of the implicit method: instead of using a built-in root-finder, perform <u>one</u> step of Newton's method. It is an iterative root-finding algorithm that solves $F(y) = 0$ starting from an initial guess y^0 and updating via

$$y^{n+1} = y^n - (J^n)^{-1}F(y^n)$$

where J^n is the Jacobian of $F(y^n)$, so that $J_{ij}^n = \partial F_i(y^n)/\partial y_j^n$ for the multivariate case. The idea is to compute the tangent of $F(\cdot)$ at y^n and find the point y^{n+1} where this tangent intersects zero. Instead of iterating the algorithm until convergence, we can make a single step and hopefully save some time without losing a lot of accuracy. For our purposes, define $F_i(\cdot)$ at every grid point as follows:

$$F_i(y) \equiv \frac{y - y_{i-1}}{x_i - x_{i-1}} - g(x_i, y)$$

where $g(x_i, y)$ is the RHS of the explicitly written ODE (see equation (2) in Sebastian's notes). In each step, use y_{i-1} as the initial guess and compute y_i via one step of the Newton's method. Compare the results with the "fully-fledged" implicit method. When does the variation work well and when does it fail?

Bibliography

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Part II

Real Models with Financial Frictions

Chapter 3

Simple Macrofinance Models

In the last chapter, we studied the Merton portfolio choice problem as an introduction to continuous-time modeling. The main objective of this lecture is to illustrate some main building blocks of a large body of macro-finance models that employs continuous-time methods. First, we develop a general model with a leverage constraint and risk. Then we cover three specific examples of this model: a complete markets benchmark with no risk, the Basak-Cuoco model (Basak and Cuoco, 1998) which features risk but no leverage constraint, and the Kiyotaki-Moore (Kiyotaki and Moore, 1997) model which features a leverage constraint but no risk.

3.1 Setup with Two-Type of Agents

3.1.1 Model Setup

Environment. Time is continuous. There is no labor and hence capital is the only factor used in production. The economy consists of two types of agents – experts and households. We denote the two types by $i = \{e, h\}$.[1] There is a continuum (with mass one) of both types, $\tilde{i} \in [0, 1]$.[2] Aggregate capital stock evolves exogenously according

[1] In general, i can denote different types/sectors, or different subgroups within the same sector.
[2] Individual-specific analysis will only be needed in environments with idiosyncratic risk.

to

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t,$$

with $g > 0$ and $\sigma > 0$ model parameters. Notice there is no investment in capital.

Notation. We denote the net worth of each sector as N_t^i while N_t is the net worth of the economy. The single endogenous state variable is the wealth share of experts, denoted by $\eta_t \equiv N_t^e/N_t$. The share of the aggregate capital stock held by the expert sector is denoted as κ_t. The portfolio share on capital held by each sector is $\theta_t^{K,i}$. We denote r_t as the risk free rate and $r_t^{K,i}$ as the rate of return on capital for each sector. In general, the sector capital stock is obtained by

$$K_t^i \equiv \int_0^1 k_t^{i,\tilde{i}} \mathrm{d}\tilde{i}.$$

Since in this model there is only aggregate risk, $K_t^i = k_t^{i,\tilde{i}}$ for all $\tilde{i}$ in each sector. The same is true for consumption

$$C_t^i \equiv \int_0^1 c_t^{i,\tilde{i}} \mathrm{d}\tilde{i},$$

where $C_t^i = c_t^{i,\tilde{i}}$ for all $\tilde{i}$ in each sector. The price of capital is denoted by q_t and sector capital holdings are denoted by $K_t^e = \kappa_t K_t$ and $K_t^h = (1-\kappa_t)K_t$. The net worth of the economy is given by the value of the aggregate capital stock so that $N_t = q_t K_t$.

Financial Frictions. No equity issuance is allowed. There is debt issuance D_t from experts to households, but with the collateral constraint $D_t^e \leq \ell \kappa_t^e q_t K_t$. This can be rearranged to $\frac{D_t^e}{N_t^e} \leq \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1-\theta_t^{K,e}) \leq \ell \theta_t^{K,e} \Leftrightarrow (1-\ell)\theta_t^{K,e} \leq 1$. ℓ is an exogenous parameter measuring the degree of the collateral constraint. Finally, each sector is subject to a non-negativity constraint on capital holdings.

Experts' Problem. Experts have a CRS production function $y_t^e = a^e k_t^e$. Denote experts' consumption by c_t^e.[3] Since experts invest $\theta_t^{K,e}$ of their net worth into capital and

[3] Again, we will suppress the superscript $\tilde{i}$ throughout this chapter.

consume at rate c_t^e their net worth process follows

$$\frac{\mathrm{d}n_t^e}{\mathrm{d}t} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t) \right) \right].$$

Experts have a log-utility function and solve the following optimization problem

$$\begin{aligned}
&\max_{c_t^e, \theta_t^{K,e}} \int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t, \quad \text{s.t.} \\
&\quad (1-\ell)\theta_t^{K,e} \leq 1, \\
&\quad \theta_t^{K,e} \geq 0, \\
&\frac{\mathrm{d}n_t^e}{\mathrm{d}t} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t) \right) \right].
\end{aligned}$$

Households' Problem. Households' production function $a^h(1-\kappa_t)k_t^h$ is a function of (aggregate) κ_t^h. Productivity $a^h(1-\kappa_t) \leq a^e$, with equality for $\kappa = 1$ and is strictly decreasing in $1-\kappa$. This means output is given by $y_t^h = a^h(1-\kappa_t)k_t^h = a^h(\cdot)(1-\kappa_t)K_t$. Households consume c_t^h. Since households invest $\theta_t^{K,h}$ of their net worth into capital, their net worth process follows

$$\frac{\mathrm{d}n_t^h}{\mathrm{d}t} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t) \right) \right].$$

Households have a log-utility function and solve the following optimization problem

$$\begin{aligned}
&\max_{c_t^h, \theta_t^{K,h}} \int_s^\infty h^{-\rho^h t} u(c_t^h) \mathrm{d}t, \quad \text{s.t.} \\
&\quad \theta_t^{K,h} \geq 0, \\
&\frac{\mathrm{d}n_t^h}{\mathrm{d}t} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t) \right) \right].
\end{aligned}$$

Market Clearing. We have that the final goods market clears, following

$$C_t^e + C_t^h = a^e K_t^e + a^h(1-\kappa_t)K_t^h.$$

The capital market clears, such that

$$K_t^e + K_t^h = K_t.$$

3.2 A Complete Markets Benchmark

In this subsection, we study a frictionless case in which there is no risk, i.e. $\sigma = 0$, and the leverage constraint ℓ is not active. These assumptions imply that markets are complete (risk-free assets span all possible payoffs without risk and, without a constraint, they can be traded frictionlessly).

3.2.1 Solution Method

Step 1 Goods Market Clearing

The final goods market must clear at each t, giving the following condition

$$C_t^e + C_t^h = a^e K_t^e + a^h(1-\kappa_t)K_t^h$$

For the case of log-utility, we know from Chapter 2 that the consumption-net worth ratio is equal to the discount rate so that $C_t^e = \rho^e N_t^e = \rho^e \eta_t q_t K_t$. Analogously, $C_t^h = \rho^h(1-\eta_t)q_t K_t$. Plugging this in and manipulating yields

$$q_t = \frac{a^e \kappa_t + a^h(1-\kappa_t)(1-\kappa_t)}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

Step 2 Portfolio choice

Let us first obtain the return processes for capital for each sector. Each return process is composed of a dividend yield as well as a capital gain term so that

$$r_t^{K,e} = \frac{a^e}{q_t} + g + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t}$$

$$r_t^{K,h} = \frac{a^h(1-\kappa_t)}{q_t} + g + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t}$$

By virtue of $a^h(1-\kappa_t) \le a^e$, we have that $r_t^{K,h} \le r_t^{K,e}$ so that experts always have a higher rate of return on capital relative to households. This implies that the non-negativity constraint on capital holdings for the household sector must bind. Since the capital market must clear in equilibrium, we have that experts hold all the capital, i.e. $\kappa_t = 1$. Hence the price of capital is given by

$$q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

which is a function of the net worth share of experts.

Since there is no leverage constraint, the experts instantly take out loans to finance the purchase of all available capital in the economy. Because of the capital market clearing condition

$$\theta_t^{K,e}\eta_t q_t K_t + \theta_t^{K,h}(1-\eta_t)q_t K_t = q_t K_t,$$

this implies that $\theta_t^{K,e} = 1/\eta_t$ instantaneously. Then, in the transition, experts repay their loans to households which in turn finances their consumption.

The non-negativity constraint on capital holdings for the household sector binding also implies that the risk-free rate must equal the return on capital if held by experts. This is since only experts will have an interior solution.

Step 3 Evolution of the net worth share

To find the evolution of the net worth share, recall that $\eta_t = N_t^e/N_t$ so that by Itô's lemma $\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$ since $\mu_t^N = \frac{1}{N_t}\frac{\mathrm{d}N_t}{\mathrm{d}t} = \eta_t\mu_t^{N^e} + (1-\eta_t)\mu_t^{N^h}$. The evolution of the net worth for experts is given by

$$\mu_t^{N^e} = -\rho^e + r_t + \theta^{K,e}(r_t^{K,e} - r_t) = -\rho^e + r_t,$$

since $r_t^{K,e} = r_t$. The evolution of the net worth for households is given by

$$\mu_t^{N^h} = -\rho^h + r_t + \theta^{K,h}(r_t^{K,h} - r_t) = -\rho^h + r_t,$$

since $\theta^{K,h} = 0$. Putting these together yields the evolution of the net worth share

$$\mu_t^{\eta} = -(1 - \eta_t)(\rho^e - \rho^h).$$

Step 4 Solving the ODE

Notice that the evolution of the net worth share can be written as the following ODE

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t} = -\eta_t(1 - \eta_t)(\rho^e - \rho^h)$$

Solving this with an initial condition for η_0 yields the following closed-form expression for η_t

$$\eta_t = \frac{e^{-(\rho^e - \rho^h)t}}{\frac{1-\eta_0}{\eta_0} + e^{-(\rho^e - \rho^h)t}}$$

Hence, for $\rho^e > \rho^h$ we have $\eta_t \to 0$ as $t \to \infty$ while $\eta_t \to 1$ for $\rho^e < \rho^h$. On the other hand, if $\rho^e = \rho^h$, then $\eta_t = \eta_0$ is constant over time.

3.2.2 Benchmark Model Conclusions

This frictionless model shows that

i) capital is always held by the most efficient sector which in this case are the experts,

ii) the consumption allocation is determined by the initial wealth distribution and wealth only moves due to differences in preferences for the timing of consumption, i.e. $\rho^e - \rho^h$. For $\rho^e = \rho^h$, every initial condition leads to a steady state, while for different time preferences, the model converges in the long run to a boundary

$\eta = 0$ for $\rho^e > \rho^h$ or $\eta = 1$ for $\rho^e < \rho^h$. However, these dynamics do not affect production,

iii) the price of capital is constant for when there are no differences in discount rates but otherwise, it rises over time because the agents with the lower marginal propensity to consume become richer as time goes by.

3.3 Basak-Cuoco Model

In this Section we study the heterogeneous agents model of Basak and Cuoco (1998), a simple yet classic model. This model is a special case of the general model in which there is no leverage constraint ℓ, and households cannot produce, i.e. $a^h \to -\infty$.

3.3.1 Solution Method

Step 1 Postulate aggregates, price processes and obtain return processes

In general, the aggregate capital stock is obtained by

$$K_t \equiv \int_0^1 k_t^{e,\tilde{i}} \mathrm{d}\tilde{i}.$$

With only aggregate risk, all experts (households) are identical (i.e., $k_t^{e,\tilde{i}} = k_t^e, \forall \tilde{i}$), so total capital stock, expert net worth and household net worth can be simply obtained by $K_t = k_t^e, N_t^e = n_t^e, N_t^h = n_t^h$.

Denote the price of capital by q_t. The total wealth of the economy is $q_t K_t$. The wealth share of experts is $\eta_t = N_t^e/(N_t^e + N_t^h) = N_t^e/q_t K_t$. We then *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Importantly, volatility of price loads on the same Brownian motion as capital stock. Given the price process and the consumption decision of experts, we can calculate the return rate to capital, $r_t^{K,e}$.[4] Using Itô's product rule,

$$\begin{aligned} dr_t^{K,e} &= \overbrace{\frac{a^e}{q_t}dt}^{\text{Dividend Yield}} + \overbrace{\frac{d(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\ &= \left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t. \end{aligned} \tag{3.1}$$

We then postulate that the stochastic discount factor (SDF, e.g., $\xi_t^i = e^{-\rho^i t}u'(c_t^i)$) is a diffusion process:

$$\frac{d\xi_t^i}{\xi_t^i} = \mu_t^{\xi^i}dt + \sigma_t^{\xi^i}dZ_t, \qquad i \in \{e, h\},$$

As we showed in Chapter 2, it turns out that its drift $\mu_t^{\xi^i}$ is the negative of the risk-free rate r_t, and its volatility loading $\sigma_t^{\xi^i}$ is the negative of the price of risk ς_t^i. As a result, the SDF follows

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \varsigma_t^i dZ_t, \qquad i \in \{e, h\}, \tag{3.2}$$

Step 2 For given SDF processes, derive individual equilibrium conditions

Since all experts are identical, aggregate capital stock K_t also follows

$$\frac{dK_t}{K_t} = g dt + \sigma dZ_t.$$

We use the Stochastic Maximum Principle to solve the optimization problems. Following the exposition in Chapter 2, the Hamiltonian for the experts is given by

$$\begin{aligned} \mathcal{H}_t^e &= e^{-\rho^e t}\log c_t^e + \xi_t^e n_t^e \mu_t^{n^e} - \varsigma_t^e \xi_t^e n_t^e \sigma_t^{n^e} \\ &= e^{-\rho^e t}\log c_t^e + \xi_t^e \left[-c_t^e + n_t^e r_t + n_t^e \theta_t^{K,e}\left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right)\right]dt - \varsigma_t^e \xi_t^e n_t^e \theta_t^{K,e}(\sigma + \sigma_t^q) \end{aligned}$$

[4] For superscripts, we use lowercase letters for different *types* and capital letters for different *assets*.

The first-order conditions for the expert's choices of consumption c_t^e and capital shares $\theta_t^{K,e}$ are given by

$$e^{-\rho^e t}\left(c_t^e\right)^{-1} = \xi_t^e$$
$$\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t = \varsigma_t^e(\sigma + \sigma_t^q)$$

The costate equation reads by virtue of the

$$\begin{aligned} d\xi_t^e &= -\frac{\partial H_t^e}{\partial n_t^e} dt - \varsigma_t^e \xi_t^e dZ_t \\ &= -r_t \xi_t^e - \varsigma_t^e \xi_t^e dZ_t \end{aligned}$$

where the equality follows from the first-order condition for $\theta_t^{K,e}$.

Using the first-order condition for consumption, we also find by Itô's lemma that

$$\frac{d\xi_t^e}{\xi_t^e} = \left[-\rho^e - \mu_t^{c^e} + (\sigma_t^{c^e})^2\right] dt - \sigma_t^{c^e} dZ_t$$

We already know that for the case of log-utility $c_t^e = \rho^e n_t^e$, so that $\mu_t^{c^e} = \mu_t^{n^e}$ and $\sigma_t^{c^e} = \sigma_t^{n^e}$. Then the price of risk is given by $\varsigma_t^e = \sigma_t^{c^e} = \sigma_t^{n^e} = \theta_t^{K,e}(\sigma + \sigma_t^q)$.

Similarly, the household sector will consume according to $c_t^h = \rho^h n_t^h$.

Finally, we combine two sectors' problems with market clearing conditions and solve the model. We will see the price of capital q_t is actually a constant in section Step 4.

Step 3 Evolution of state variable η_t

In this model, agents start with some initial endowments of capital. Over time, they allocate their wealth between the assets available to them by solving their respective utility maximization problems, subject to budget constraints and taking prices as given. Given prices, markets for capital and consumption goods have to clear.

Recall that

$$\eta_t = \frac{N_t^e}{q_t K_t} \in [0,1].$$

The total wealth of experts N_t follows

$$\begin{aligned}
\frac{dN_t^e}{N_t^e} = \frac{dn_t^e}{n_t^e} &= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left[dr_t^K - r_t dt\right] \\
&= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right]dt + (\sigma + \sigma_t^q)dZ_t\right\} \\
&= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\varsigma_t^e(\sigma + \sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{d(q_t K_t)}{q_t K_t} &= \left[\mu_t^q + g + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t \\
&= \left[r_t - \frac{a^e}{q_t} + \varsigma_t^e(\sigma + \sigma_t^q)\right]dt + (\sigma + \sigma_t^q)dZ_t.
\end{aligned}$$

Apply Itô's quotient rule to $\eta_t = N_t^e / q_t K_t$:

$$\frac{d\eta_t}{\eta_t} = \left[-\frac{c_t^e}{n_t^e} + \frac{a^e}{q_t} - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)\left(\varsigma_t^e - (\sigma + \sigma_t^q)\right)\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t. \tag{3.3}$$

Using what we found in the previous step for $c_t^e = \rho^e n_t^e$ and $\varsigma_t^e = \theta_t^{K,e}(\sigma + \sigma_t^q)$, we have

$$\frac{d\eta_t}{\eta_t} = \left[-\rho^e + \frac{a^e}{q_t} + (1 - \theta_t^{K,e})^2(\sigma + \sigma_t^q)^2\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t. \tag{3.4}$$

Step 4 Market clearing

Finally, we can close the model using market clearing conditions. Consumption good market clearing yields

$$C_t = \rho^e N_t^e + \rho^h N_t^h = \rho^e \eta_t q_t K_t + \rho^h (1-\eta_t) q_t K_t = a^e K_t \implies q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1-\eta_t)}$$

Capital market clearing yields

$$\theta_t^{K,e} = \frac{q_t K_t}{N_t^e} = \frac{1}{\eta_t}. \tag{3.5}$$

Then, the law of motion of η_t is

$$\frac{\mathrm{d}\eta_t}{\eta_t} = (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{1-\eta_t}{\eta_t^2}(\sigma + \sigma_t^q)^2\right]\mathrm{d}t + \frac{1-\eta_t}{\eta_t}(\sigma + \sigma_t^q)\mathrm{d}Z_t. \tag{3.6}$$

where by Itô's lemma

$$\sigma_t^q = -\frac{\rho^e - \rho^h}{\rho^e \eta_t + \rho^h (1-\eta_t)} \sigma_t^\eta \eta_t$$

Hence

$$\sigma_t^\eta = \frac{1-\eta_t}{\eta_t} \frac{\rho^e \eta_t + \rho^h (1-\eta_t)}{\rho^e} \sigma$$

and

$$\begin{aligned}\frac{\mathrm{d}\eta_t}{\eta_t} =& (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{1-\eta_t}{\eta_t^2} \frac{\left[\rho^e \eta_t + \rho^h (1-\eta_t)\right]^2}{(\rho^e)^2}\sigma^2\right]\mathrm{d}t \\ &+ \frac{1-\eta_t}{\eta_t}\frac{\rho^e \eta_t + \rho^h (1-\eta_t)}{\rho^e}\sigma \mathrm{d}Z_t\end{aligned}$$

a simple one-dimensional stochastic differential equation (SDE).

Numerical example. The figure below shows the price of capital, its volatility, the arithmetic drift and volatility of net worth as a function of net worth η. The parameter values are ρ^e = 0.06, ρ^h = 0.04, a^e = 0.11, σ = 0.10 and g=0.1.

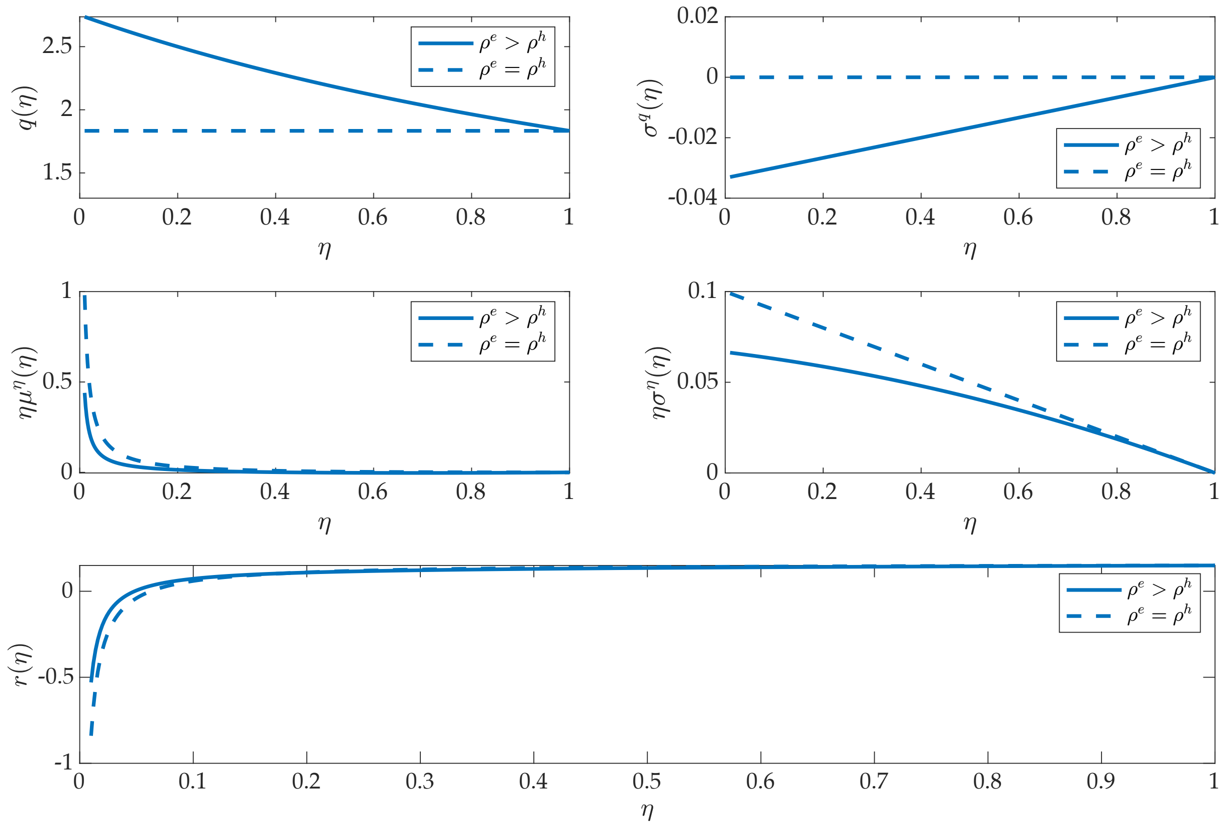

Figure 3.1: Basak-Cuoco numerical example

3.4 Kiyotaki-Moore Model

The third specific example of the general model is the Kiyotaki-Moore model. It is a deterministic model with a single zero probability shock rather than a model that features risk, i.e. $\sigma = 0$. Here we present a modified version of the model in Kiyotaki and Moore (1997). In particular, we convert the discrete-time model into a continuous-time setting.

3.4.1 Solution Method

Step 1 Postulate aggregates, price processes and obtain return processes

We *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t.$$

Importantly, there is no stochasticity as $\sigma = 0$. Given the price process and the consumption decision of experts, we can calculate their return rate to capital, $r_t^{K,e}$.[5] Using Itô's product rule,

$$\begin{aligned}
\mathrm{d}r_t^{K,e} &= \overbrace{\left(\frac{a^e}{q_t} + g\right)\mathrm{d}t}^{\text{Dividend Yield}} + \overbrace{\frac{\mathrm{d}(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\
&= \left[\frac{a^e}{q_t} + g + \mu_t^q\right]\mathrm{d}t. \qquad (3.7)
\end{aligned}$$

Similarly,

$$\mathrm{d}r_t^{K,h} = \left[\frac{a^h(1-\kappa)}{q_t} + g + \mu_t^q\right]\mathrm{d}t. \qquad (3.8)$$

We then postulate that the stochastic discount factor (SDF, e.g., $\xi_t^i = e^{-\rho^i t}u'(c_t^i)$) is the process:

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \mu_t^{\xi^i}\mathrm{d}t, \qquad i \in \{e, h\},$$

As we showed in Chapter 2, it turns out that its drift $\mu_t^{\xi^i}$ is the negative of the risk-free rate r_t. As a result, the SDF follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t, \qquad i \in \{e, h\}, \qquad (3.9)$$

[5] For superscripts, we use lowercase letters for different *types* and capital letters for different *assets*.

Step 2 For given SDF processes, derive individual equilibrium conditions

The Hamiltonians can be constructed as

$$\begin{aligned}\mathcal{H}_t^e =& e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left(1 - (1-\ell)\theta_t^{K,e}\right),\\ \mathcal{H}_t^h =& e^{-\rho^h t}u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h\left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right].\end{aligned}$$

where the ξ_t^i is the multiplier on the budget constraint and the $\xi_t^e n_t^e \lambda_t^\ell$ is the multiplier on the leverage constraint. Later we show that co-state variable ξ_t^i equals SDF, which for log-utility $= e^{-\rho^i t}\frac{1}{\rho^i n_t^i}$. Note that the Fisher Separation Theorem between consumption and portfolio choice applies. That is the first order conditions with respect to consumption, c_t^i, and portfolio choice, $\theta_t^{K,i}$ are independent. The first order conditions with respect to c_t^i and $\theta_t^{K,i}$ are given by

$$\begin{aligned}e^{-\rho^e t}u'(c_t^e) =& \xi_t^e\\ e^{-\rho^h t}u'(c_t^h) =& \xi_t^h\end{aligned} \Rightarrow c_t^i = \rho^i n_t^i, \text{under log utility}$$

and,

$$\begin{aligned}r_t^{K,e} - r_t &= (1-\ell)\lambda_t^\ell\\ r_t^{K,h} - r_t &= 0.\end{aligned}$$

The additional first order condition is given by the costate equation,

$$\mathrm{d}\xi_t^i = -\frac{\mathrm{d}H_t^i}{\mathrm{d}n_t^i},$$

which yields

$$\begin{aligned}\mathrm{d}\xi_t^e &= \xi_t^e\left(\left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right) + \lambda_t^\ell\left(1 - (1-\ell)\theta_t^{K,e}\right)\right),\\ \mathrm{d}\xi_t^h &= \xi_t^h\left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right).\end{aligned}$$

Plugging in the first order conditions for portfolio choices yields, as expected,

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t, \qquad i \in \{e, h\}. \tag{3.10}$$

Step 3 Evolution of state variable η_t

In this model, agents start with some initial endowments of capital. Over time, they allocate their wealth between the assets available to them by solving their respective utility maximization problems, subject to budget constraints and taking prices as given. Given prices, markets for capital and consumption goods have to clear. To think about dynamics first note that equilibrium objects are functions of the single state, the net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t K}$. The state dynamics are solved by noting

$$\mu_t^N dt = \frac{dN_t}{N_t} = \underbrace{\frac{N_t^e}{N_t}}_{\eta_t} \mu_t^{N^e} dt + \underbrace{\frac{N_t^h}{N_t}}_{(1-\eta_t)} \mu_t^{N^h} dt$$

such that,

$$\begin{aligned}
\mu_t^\eta &= \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h}) \\
&= (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}(r_t^{K,e} - r_t) - \theta_t^{K,h} \overbrace{(r_t^{K,h} - r_t)}^{=0 \text{ from above}}\right] \\
&= (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}\left(\underbrace{\frac{a^e}{q_t} - \frac{a^h(1-\kappa_t)}{q_t}}_{=r_t^{K,e} - r_t^{K,h}}\right)\right].
\end{aligned}$$

Step 4 Market clearing

Finally, we can close the model using market clearing conditions. Consumption good market clearing yields

$$q_t K_t[\rho^e \eta_t + \rho^h(1-\eta_t)] = [a^e \kappa_t^e + a^h(1-\kappa_t)(1-\kappa_t)]K_t.$$

Capital market clearing yields

$$\underbrace{\theta_t^{K,e}\eta_t}_{=\kappa_t} q_t K_t + \underbrace{\theta_t^{K,h}(1-\eta_t)}_{=1-\kappa_t} q_t K_t = q_t K_t \tag{3.11}$$

Equilibrium Conditions Summary. The equilibrium objects (κ, q, r) are functions of the state η_t. Following some algebra, they are pinned down by the following conditions,

$$\begin{aligned} q_t[(\rho^e - \rho^h)\eta_t + \rho^h] &= \kappa_t a^e + (1-\kappa_t)a^h(1-\kappa_t) \\ \kappa_t &\leq \frac{\eta_t}{1-\ell} \\ \mu_t^\eta &= (1-\eta_t)\left[-(\rho^e-\rho^h) + \frac{\kappa_t}{\eta_t}\frac{a^e - a^h(1-\kappa_t)}{q_t}\right]. \end{aligned}$$

3.4.2 Solution and Comments

Global Non-Linear Solution. Figure 3.2 plots the global solution to this model for the parameter values specified.

When experts have a low share of the total wealth their collateral constraint is binding. They put twice their wealth in capital ($\theta_t^{e,K} = 2$). They would like to lever up more but cannot and hence we observe an excess return over the households, which is increasing as the experts net worth share decreases. For larger wealth shares the portfolio choice of the experts becomes an interior solution. The excess return is then 0 as the constraint is slack.

Furthermore, the price of capital, q_t, is decreasing as wealth share decreases, in the binding collateral constraint region. The reason for this is that as the experts' wealth

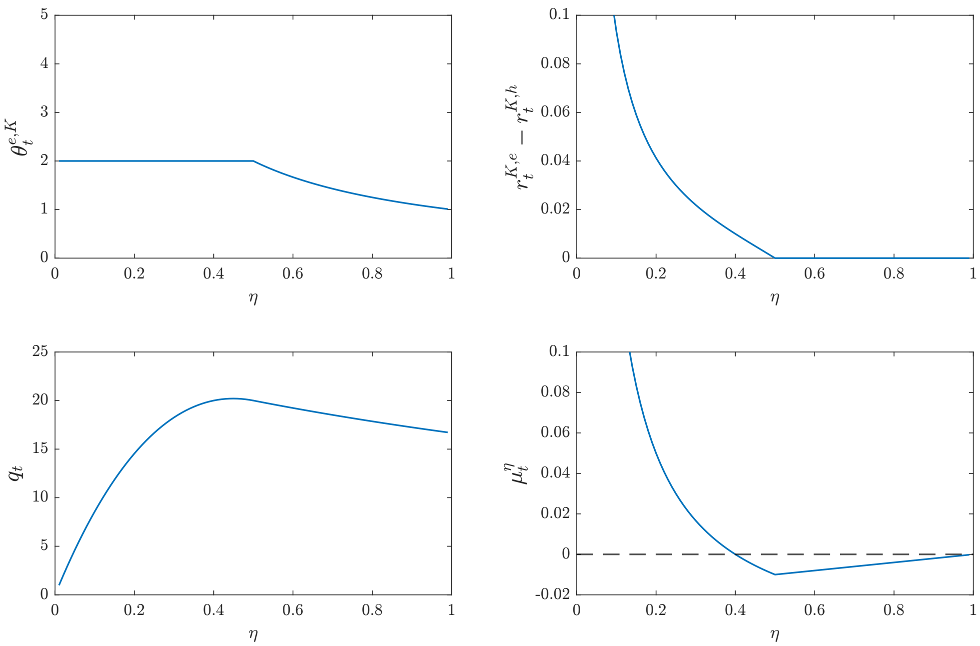

Figure 3.2: Global non-linear solution for parameters: $\rho^e = 0.06$, $\rho^h = 0.04$, $\ell = 0.05$, $a^e = 1.0$, $a^h(1-\kappa) = a^e\kappa$

share decreases they can hold less and less capital due to the collateral constraint. As capital is in fixed supply, households hold more and more, meaning each marginal value unit of capital becomes less productive, depressing the price. For high η values the decline in the price is from the fact that experts are less patient than households. The steady state is 0.4. For any value of η above this η drifts down, and for any value below η drifts up.

Impulse Responses. One can do analysis of an unanticipated shock to the steady state. Figure 3.3 plots impulse response functions for the same system above, for a 30% (of η) negative redistribution shock. We can observe the wealth share and price drifting back up to the steady state.

Aside: Understanding Asset Prices. With some proper initial conditions, the price dynamics are given by the solution of the following differential equation,

$$\frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} + \frac{a^h(1-\kappa_t)}{q_t} = r_t,$$

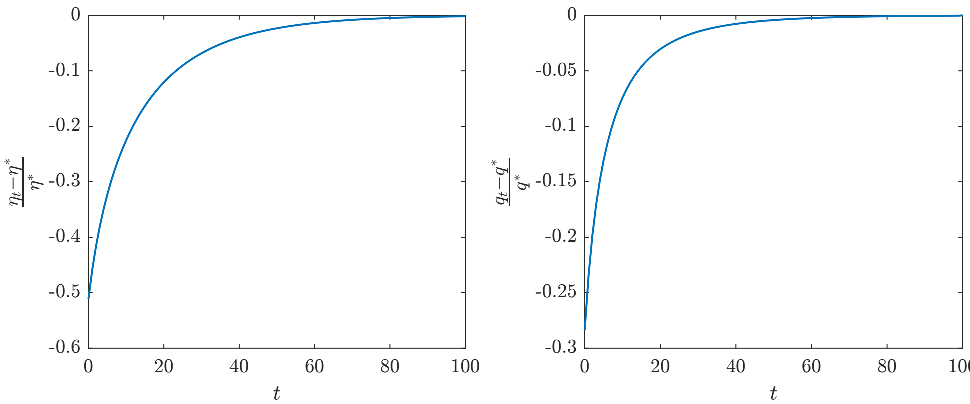

Figure 3.3: Impulse response function with 30% (of η) negative redistribution shock. Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

which is

$$q_t = \int_t^\infty e^{-\int_t^s r_u du} a^h(1 - appa_s^h)ds.$$

The discrete time analogy for this is the difference equation and it's solution,

$$\frac{q_{t+1} - q_t}{q_t} + \frac{a^h(1-\kappa_t)}{q_t} = r_t,$$

$$q_t = \sum_{s=0}^{\infty} \left[\prod_{u=0}^{s} \frac{1}{(1 + r_{t+u})} \right] a^h(1 - \kappa_{t+s}).$$

That is, the asset price is the sum of discounted dividend flows, and is solved backwards.

3.4.3 Log-linearized Dynamics around Steady State

The way the Kiyotaki-Moore model was solved originally, also in discrete time, was through a log-linearization around the steady state. We will now cover this and compare. First we derive the steady state with $\mu^\eta = 0$ and its' properties. Then we will log-linearize and characterize the dynamical system locally around the steady state.

The Steady State. First note that the collateral constraint always binds in the steady state. If the collateral constraint were not binding $\lambda_t^\ell = 0$ and hence $r^{K,e} = r^{K,h}$, i.e. $a^e = a^h(\cdot)$. The constraint does not need to bind *only if* $\kappa_t = 1$. Then $\mu_t^\eta = (1-\eta_t)(\rho^h - \rho^e)$

and since $\rho^e > \rho^h$, we have $\mu_t^\eta < 0$, i.e. η declines. Now, since the collateral constraint binds, the steady state capital share is given by

$$\kappa^{SS} = \frac{\eta^{SS}}{1-\ell}.$$

The expert sector's net worth share is $\eta_t := \frac{N_t^e}{q_t \bar{K}}$, is constant, i.e. $\mu_t^\eta := \frac{\mathrm{d}\eta_t}{\mathrm{d}t} = 0$. From the good's market clearing condition and the last equation of the equilibrium conditions,

$$\begin{aligned} q^{SS}[(\rho^e - \rho^h)\eta^{SS} + \rho^h] &= \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS}) \\ (\rho^e - \rho^h) &= \frac{\kappa^{SS}}{\eta^{SS}} \frac{a^e - a^h(1-\kappa^{SS})}{q^{SS}} \quad \text{for } \mu^\eta = 0. \end{aligned}$$

Combining the above,

$$\begin{aligned} \kappa^{SS} a^e - \kappa^{SS} a^h (1-\kappa^{SS}) + q^{SS}\rho^h &= \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS}) \\ \Rightarrow \quad q^{SS} &= a^h(1-\kappa^{SS})/\rho^h, \end{aligned}$$

where the steady state κ^{SS} is implicitly given by:

$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell} \frac{a^e - a^h(1-\kappa^{SS})}{a^h(1-\kappa^{SS})}.$$

Finally, for specific functional form $a^h(1-\kappa_t) = a^e \kappa_t$,

$$\kappa^{SS} = \frac{1}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1} \quad \Rightarrow \eta^{SS} = \frac{1-\ell}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1}.$$

One can do comparative statics with this derivation. For example for higher leverage, ℓ, (i.e. less tight collateral constraint)

- κ^{SS}, SS-capital share, is higher.
- η^{SS}, SS-net worth share, is lower.
- $q^{SS} = \frac{a^h}{\rho^h}$, price of capital, is higher. $q^{SS}\bar{K}$, total wealth in the economy, is higher too.

- $N^{e,SS}$ SS-experts' net worth, is higher (Check?)

Comparative statics analyzes a permanent (long-run) shift to new steady state, but not the dynamics between them. It compares two separate steady states. To analyze dynamics we can log-linearize.

Log-linearized Dynamics. Analytical solutions to η_t, q_t dynamics are hard to obtain. Expanding around the steady state,

$$\begin{aligned}\log(\eta_t/\eta^{SS}) &= \hat{\eta}_t \\ \log(q_t/q^{SS}) &= \hat{q}_t \\ \log(r_t/r^{SS}) &= \hat{r}_t \\ \log(a_t^h/a^{h,SS}) &= \hat{a}_t^h\end{aligned}$$

As an exercise one can derive an expression for $\hat{a}_t^h, \hat{q}_t^h$ as a function of $\hat{\eta}_t$ with a first order Taylor approximation. The state dynamics and price dynamics become,

$$\begin{aligned}\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} &= \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right) \\ \frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} &= r^{SS}(\hat{r}_t + \hat{q}_t - \hat{a}_t^h).\end{aligned}$$

This works well for small shocks around the steady state, where the drift of η is close to linear. However, further away from the steady state the log-linearization can lead to misleading conclusions about dynamics of the system.

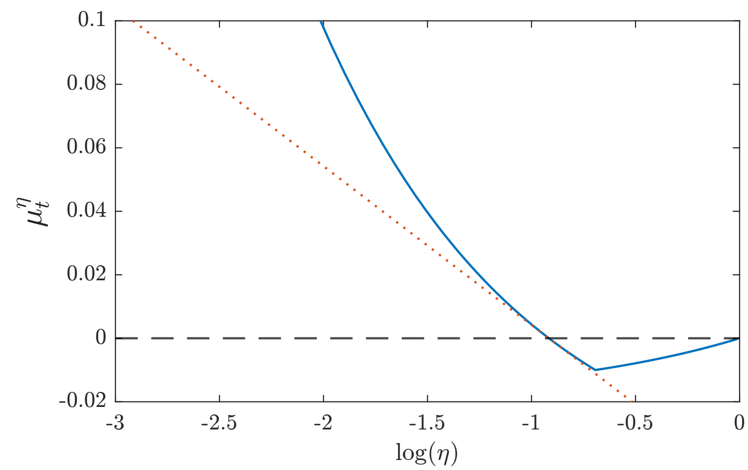

Figure 3.4: Global vs. Log-linearized Solution for η-drift. Note: x-axis is $\log(\eta)$, since we log-linearized.

3.4.4 Decomposing Amplification Effects

To decompose amplification effects, we first start at the steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$. We then shock the system by redistributing a fraction of experts' net worth share to households. In the original Kiyoataki-Moore model, the productivity shock lasts for one period (not for an instant), which causes initial redistribution. With determenistic recovery the immediate impacts, at $t = 0$, are as follows. First there is a direct redistributive effect/shock. Second, there is a price-net worth effect. The decline in q_t reduces experts' net worth share as they are levered, which feeds back into the price-net worth effect. Third, there is a price-collateral effect. The decline in q_t tightens collateral constraints, which also feeds back into price-net worth effect. Subsequent impact $t > 0$ feeds back into immediate impacts.

To carry out a decomposition, we switch off the price-collateral effect by assuming that the collateral constraint is determined by SS-price q^{SS} instead of the equilibrium price q_t. Formally, the collateral constraint, $\kappa_t \leq \frac{\eta_t}{1-\ell}$, becomes $\kappa_t \leq \frac{\eta_t}{1-\ell q^{SS}/q_t}$. Figure 3.5 plots the impulse response function of this decomposition, for a 30% (of η) negative redistribution shock. Because of this amplification, when all channels are on this translates to just over a 50% negative redistribution. If we switch off the the price-collateral effect the amplification translates to just under a 40% negative redistribution. The difference in amplification effects at $t = 0$ is also clear for the difference in price.

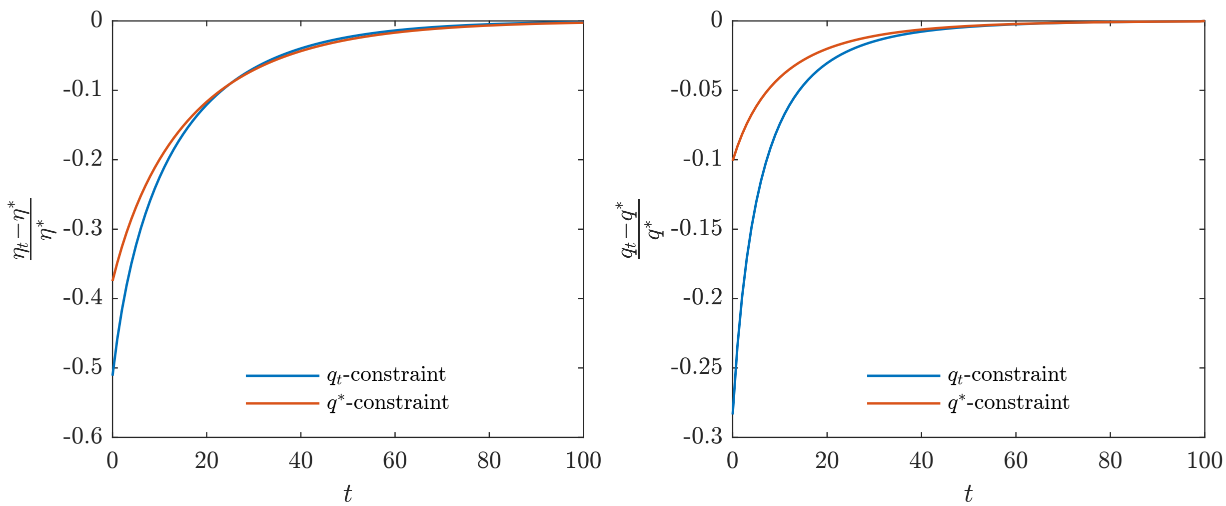

Figure 3.5: Impulse response function with 30% (of η) negative redistribution shock. Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = \kappa$

To show more clearly how this works, consider that economy is at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$. There is then a negative initial/direct redistributive shock $\eta' = (1 - \epsilon)\eta^{SS}$. The new price q', and capital holding κ' solve,

$$q' = \frac{\kappa' a^e + (1-\kappa')a^h(1-\kappa')}{(\rho^e - \rho^h)\eta' + \rho^h}, \qquad \text{(Goods market)}$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell}, \qquad (q_t\text{-constraint})$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell q^{SS}/q'}. \qquad (q^{SS}\text{-constraint})$$

However, the debt contract was signed by the old price q^{SS} so η drops further. The first round effect is done by considering the balance sheet,

$$\frac{\eta'}{1-\ell}q' = \frac{\ell}{1-\ell}\eta' q^{SS} + \eta'' q'.$$

To then get the full full convergence result, we need to do this procedure iteratively, both with the q_t-constraint and the q^{SS}-constraint.

The global solution for the $t > 0$ decomposition of amplification is plotted in Figure 3.6. The blue line (q_t-constraint) is the same as the global solution from before, in Figure 3.2. With the q^{SS}-constraint, the constraint is less binding as it takes into account the higher, steady-state, price of capital. This means that for lower η the excess return will be lower than the q_t-constraint since experts will hold more capital (as they are allowed to lever up more). It also means that the capital price is not so depressed as the experts hold a higher share of capital (and they are more productive). This portrays the additional amplification from keeping the price-collateral effect channel open. One interesting point is that with the q_t-constraint the system recovers faster. The drift of η is higher since the excess return is higher.

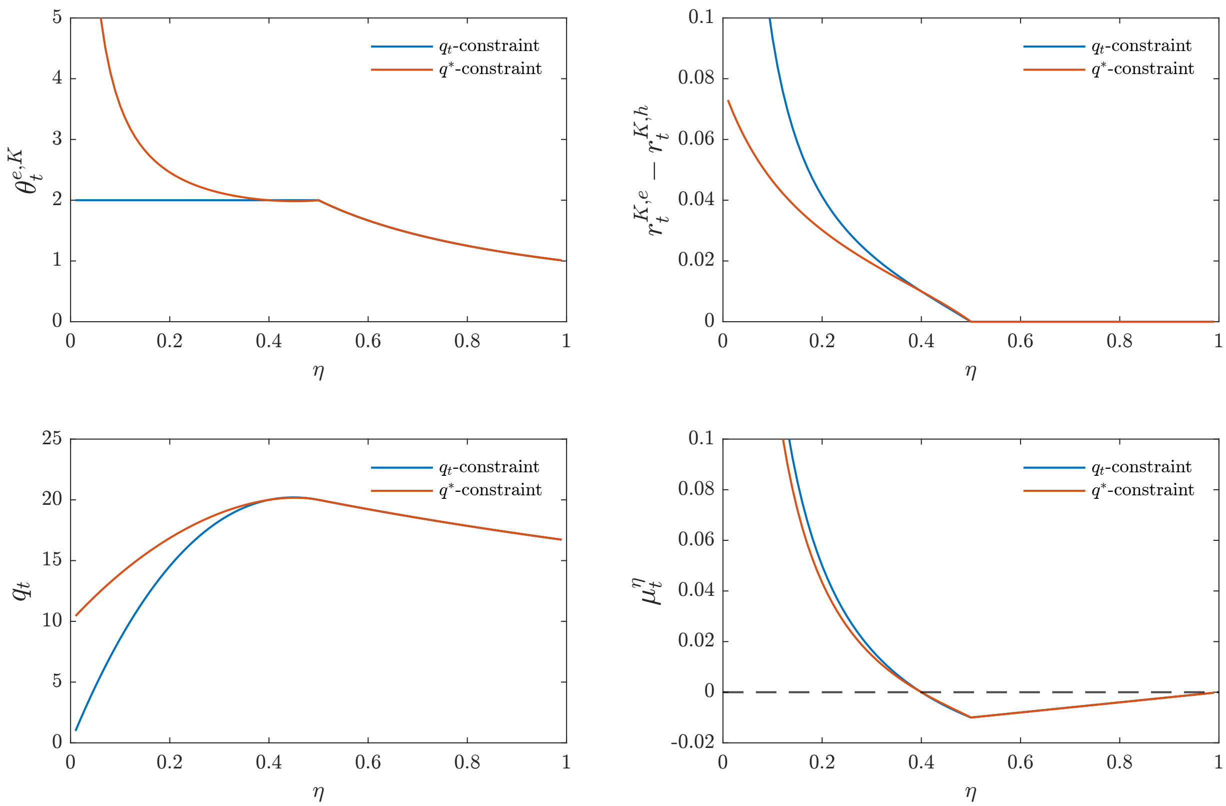

Figure 3.6: Decomposing Amplification for $t > 0$ (global solution). Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

For the log-linearized solution, we can also decompose the amplification for $t > 0$. The dynamics are given by

- Price dynamics:

$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = r^{SS}\hat{r}_t - r^{SS}\hat{a}_t^h + r^{SS}\hat{q}_t$$

- State dynamics with q_t-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

- State dynamics with q^{SS}-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{1}{1-\ell}\frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

Note that $\hat{q}_t, \hat{a}_t^h, \hat{r}_t$ are different with the different constraints. For large changes in η there will be a significant difference between these dynamics and the global solution.

3.5 Introducing Physical Investment

So far, we have considered a two sector model in which aggregate capital K_t evolves exogenously. We now add physical investment to it. Consumption goods can be converted in new capital. The conversion is not necessarily one-for-one, but concave in the investment rate ι, an agent's new physical investment divided by his capital.[6] This is captured by the concave capital conversion function $\Phi(\iota)$ with the assumption $\Phi'(\cdot) > 0, \Phi''(\cdot) < 0$. In addition, capital depreciates at a rate of δ. For example, consider an agent with capital k_t at an investment rate ι_t at time t. His real investment is $\iota_t k_t$, and the capital accumulation is $(\Phi(\iota_t) - \delta)k_t$. This is equivalent to a convex adjustment cost assumption. More generally, capital accumulation follows

$$\frac{\mathrm{d}k_t}{k_t} = \big(\Phi(\iota_t) - \delta\big)\,\mathrm{d}t + \sigma \mathrm{d}Z_t$$

[6]Assuming that $\Phi(\cdot)$ is concave in investment rate I_t/K_t rather than the investment I_t ensures tractable aggregate across agents.

where dZ_t is a Brownian motion capturing shocks to the capital accumulation process.

Note that investment is an intraperiod/static decision in models, in which capital does not take "time to build". Hence, maximizing agents' Hamiltonian with respect to ι_t only enters via individual agent's capital return r_t^k. The capital return consists of two terms: a dividend yield term as well as a capital gain term. We can write the return on capital as a function of the investment rate ι_t as

$$dr_t^k(\iota_t) = \underbrace{\frac{a-\iota_t}{q_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gain}} = \left[\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t.$$

The agent will now choose the investment rate ι_t to maximize their expected return on capital accumulation. That is,

$$\max_{\iota_t} \mathbb{E} dr_t^k(\iota_t)/dt$$

This yields the following Tobin Q optimality condition

$$\frac{1}{q_t} = \Phi'(\iota_t).$$

For the investment conversion function of the form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, this implies the following relationship between the investment rate and the price of capital

$$\phi\iota_t = q_t - 1.$$

This investment conversion function has a parameter ϕ that measures the degree of investment adjustment costs. As $\phi \to 0$, $\Phi(\iota) \to \iota$ so that there are no adjustment costs in capital investment. However, as $\phi \to \infty$ these costs become infinitely large and all new investment is lost, that is, $\Phi(\iota) \to 0$.

3.6 Exercises

3.6.1 Coding Continuous Time Kiyotaki-Moore Model

Please replicate the global solution figures (of Figure 3.2) for the case with ι-investments.

3.6.2 Capital (Quality) and Technology Shocks

Consider the simple model from chapter 3. There, we assumed that expert capital follows

$$\frac{dk_t^e}{k_t^e} = \left(\Phi(\iota_t^e) - \delta\right) dt + \sigma dZ_t \tag{3.12}$$

and produces an output flow[7]

$$y_t^e = \bar{a} k_t^e. \tag{3.13}$$

In this problem set you are asked to consider a different specification with consumption-specific technology shocks instead of capital shocks. Specifically, suppose instead of equations (3.12) and (3.13) that capital evolves according to

$$\frac{dk_t^e}{k_t^e} = \left(\Phi\left(\frac{\bar{a}}{a_t}\iota_t^e\right) - \delta\right) dt \tag{3.14}$$

and produces an output flow

$$y_t = a_t k_t \tag{3.15}$$

where a_t is now a stochastic process given by[8]

$$\frac{da_t}{a_t} = \psi\left(\log \bar{a} - \log a_t\right) dt + \sigma dZ_t. \tag{3.16}$$

[7] I use here $\bar{a}$ instead of a from the lecture to distinguish this more clearly from the process a defined below.

[8] One gets to this equation by imposing that $\log a_t$ follows an Ornstein-Uhlenbeck process, the continuous-time equivalent of a discrete-time AR(1) process, and correcting by a deterministic time drift, such that the long-run mean of A_t is not growing/shrinking over time. The equivalent in discrete time is often taken as a productivity process in standard macro models.

For $\psi = 0$ this specification implies a geometric Brownian motion for productivity, for $\psi > 0$, a_t mean-reverts to the level $\bar{a}$ in the long run. The additional term $\bar{a}/a_t$ in the Φ function implies that only consumption production is impacted by changes of a_t.

(a) Show that without productivity mean reversion ($\psi = 0$), the model with capital shocks (evolution (3.12) and output (3.13)) and the model with consumption-specific technology shocks (capital evolution (3.14), output (3.15) and productivity process (3.16)) are isomorphic in the sense that they imply the same dynamics for output, consumption, net worth, the expert wealth share η^e and the risk-free rate.

(b) How are the two models related, if $\psi > 0$?

(c) Explain economically, why the two shock types are not equivalent for neutral technology shocks (i.e. if the investment technology is $\Phi\left(\iota_t^e\right)$) and an inconstant ψ function (1-2 sentences are sufficient).

3.6.3 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the Basak-Cuoco Model of the slides. Now we introduce the investment conversion function $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$. Assume that households are more patient than experts, i.e. they have a discount rate $\rho^h < \rho^e$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

1. Derive closed-form expressions for ι, q, σ^q, μ^η and σ^η as functions of η and model parameters:

    (a) Start with goods market clearing condition and use $\hat{\rho}(\eta) = \rho^e\eta + \rho^h(1-\eta)$ to ease notation. Derive $q(\eta)$ and $\iota(\eta)$.

    (b) Use $q(\eta)$ and the law of motion for η to find $\sigma^q(\eta)$ and $\sigma^\eta(\eta)$.

    (c) Derive $\mu^\eta(\eta)$.

2. Replicate the figure 3.1, setting $\phi = 10$ and $\delta = 0.035$, then add to each plot the corresponding line for the model with $\rho^e = 5\%$ and $\rho^h = 2\%$ (and all other parameters as before).

3. Assume $\phi > 0$. Show that in this model asset price movements mitigate exogenous risk (i.e. $\sigma^q + \sigma < \sigma$). Explain economically why this happens and why the effect disappears if $\phi = 0$.

4. Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a formal proof).

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Chapter 4

A Macro-Model with Endogenous Risk Dynamics: Amplification, Fire-sales, and Speculation

In last chapter, we studied simple models to illustrate the basic structure of continuous-time macro-finance models.

In this chapter, we present a more complex model in which apart from risk-free debt, experts can also issue outside equity. In terms of economic insights, we enrich the model to obtain the following properties:

- The risk as well as the price of risk is endogenous and hence time-varying depending on the wealth distribution across the heterogeneous agents in the economy.
- Equilibrium dynamics contains two regimes – a normal regime around the steady state and a crisis regime. The economy should be relatively stable near the steady state, where experts are adequately capitalized and able to absorb most shocks. However, an unexpected large shock or a series of negative shocks can significantly damage the experts and bring the economy to the crisis regime. In a crisis, experts are undercapitalized and financially constrained. As a result, market liquidity can suddenly dry up and shocks do affect demand for and prices of assets.

This generates *endogenous risk and volatility* through feedback effects of fire-sales and financial constraints can become occasionally binding.

- Volatility is high in the crisis regime, which might push experts' net worth towards zero. In this case, the economy needs a long time to recover. Ex ante, the system will spend a large amount of time away from the steady state and the stationary distribution can be bimodal.

- Assets are more correlated during crises due to endogenous risk.

- Endogenous risk-taking gives rise to a *volatility paradox*, which means that the economy does not become more stable when the fundamental risk σ is lower. This is because when risk is lower, experts take on greater leverage, making the economy more prone to crises.

- Financial innovations (e.g., securitization) that improve risk-sharing among experts might destabilize the economy in equilibrium. The logic is similar. Being able to diversify (idiosyncratic) risk emboldens the experts, leading to higher leverage and amplifying systemic risk.

In terms of modeling, we highlight the essential techniques for solving large-scale macro-finance models in continuous time:

- We introduce an *occasionally binding constraint* in this setting. The "skin in the game constraint" is not binding in the normal regime, while it binds in the crisis regime in which fire sales occur and volatility spikes.

- We rely on the "Fisher separation theorem" in order to solve the model from the viewpoint of a "price-taking" social planner.

- We introduce a change from a consumption numeraire to a total wealth numeraire, which simplifies many algebraic steps.

This chapter builds on Brunnermeier and Sannikov (2016), which expands on Brunnermeier and Sannikov (2014).

4.1 Model Setup

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions. Upon death of an expert/household, a new agent takes their place, inherits their wealth, and becomes an expert with probability $\zeta^e \in (0, 1)$.

Experts. Experts have a CRS technology $y_t^e = a^e k_t^e$. Denote their consumption and investment rate by c_t^e, ι_t^e. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = (\Phi(\iota_t^e) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Still, we have only aggregate risk in the environment. Experts have a log utility function and they each maximize

$$\mathbb{E}_0 \left[\int_0^T e^{-\rho_0^e t} \log c_t^e \mathrm{d}t \right]$$

where T is exponentially distributed with parameter ρ_d^e. Define $\rho^e := \rho_0^e + \rho_d^e$. The objective is equivalent to infinite lifetime with higher discount rate ρ^e

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log c_t^e \mathrm{d}t \right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

We let households hold capital to capture fire-sales. Households are more patient than the experts, i.e., $\rho^h \leq \rho^e$. As we have discussed in section 3, assuming that households are more patient than the experts, i.e., $\rho^h \leq \rho^e$, is a modeling trick to ensure that the experts do not hold all the capital in the long run. However, here we achieve the same outcome by introducing death. The households maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^h t}\log c_t^h \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^h. Similarly, the objective is equivalent to infinite lifetime with higher discount rate $\rho^h := \rho_0^h + \rho_d^h$

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log c_t^h \mathrm{d}t\right].$$

Financial Friction. The financial friction in this chapter is due to incomplete markets (see, e.g., Dumas and Luciano, 2017). Although experts are allowed to issue equity, they must hold at least α fraction of their risk. The balance sheets of the two sectors are as following:

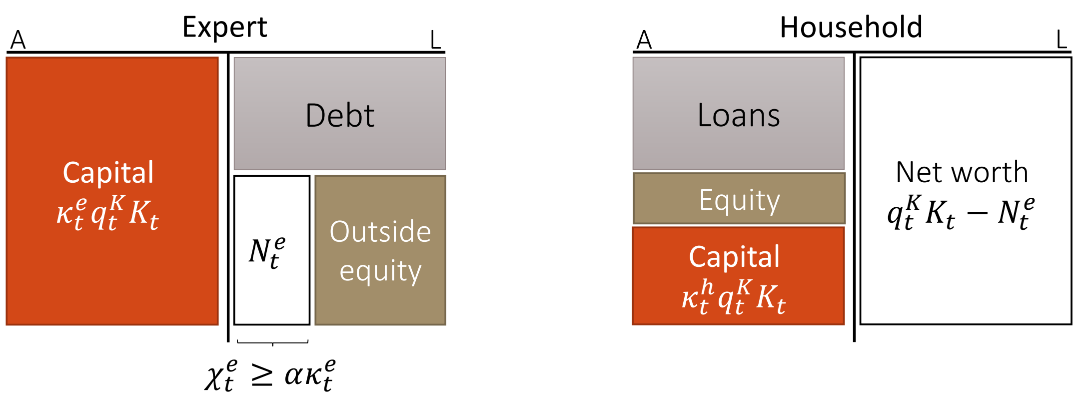

Figure 4.1: Balance sheets of experts and households

The skin-in-the-game constraint can be expressed as $\chi_t^e \geq \alpha\kappa_t^e$, where χ_t^e is the fraction of risk held by experts and κ_t^e is the fraction of capital held by experts. We will discuss carefully this relationship in later sections.

4.2 Solution Method

4.2.0 Postulate aggregates, price processes and obtain return processes

Again, with only aggregate risk, all experts (households) are identical, so total capital stock and net worth in each sector are obtained by $K_t^i = k_t^i$, $N_t^i = n_t^i, i = \{e, h\}$. Denote the price of capital by q_t. The total wealth of the economy is $q_t \sum_i K_t^i = q_t K_t = N_t = \sum_i N_t^i$. Define the capital shares as

$$\kappa_t^i = \frac{K_t^i}{\sum_{i'} K_t^{i'}},$$

and the net worth share as

$$\eta_t^i = \frac{N_t^i}{\sum_{i'} N_t^{i'}} = \frac{N_t^i}{q_t K_t}.$$

We then *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Given the price process and the consumption-investment decision of the expert, we can calculate the return rate to capital for both sectors, $r_t^{i,K}(\iota_t^i)$. Same as (3.7), we have

$$\mathrm{d}r_t^{i,K}(\iota_t^i) = \left[\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right]\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t. \tag{4.1}$$

We then postulate that SDF ($\xi_t^i = e^{-\rho^i t} u'(c_t^i)$) follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t, \tag{4.2}$$

where r_t is the risk-free rate.

4.2.1 For given SDF processes, derive individual equilibrium conditions

With an additional asset available, sector i's problem becomes

$$\begin{aligned} \max_{\{\iota_t^i, \theta_t^i, c_t^i\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho^i t} \log c_t^i \mathrm{d}t \right] \\ \text{s.t.} \quad & \frac{\mathrm{d}n_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}\mathrm{d}t + \theta_t^{i,K}\mathrm{d}r_t^{i,K}(\iota_t^i) + \theta_t^{i,OE}\mathrm{d}r_t^{OE} + \theta_t^{i,D} r_t \mathrm{d}t \\ & n_0^i \text{ given,} \end{aligned} \tag{4.3}$$

where r_t^{OE} is the return to outside equity. Note that the outside equity has the same risk (volatility) as capital but possibly different expected returns (drifts) due to the skin-in-the-game constraint. The experts' allocation satisfies

$$\theta_t^{e,K} \geq 0, \quad -(1-\alpha)\theta_t^{e,K} \leq \theta_t^{e,OE} \leq 0, \quad \theta_t^{e,D} \leq 0, \quad \text{and} \quad \theta_t^{e,K} + \theta_t^{e,OE} + \theta_t^{e,D} = 1. \tag{4.4}$$

The households' allocation satisfies

$$\theta_t^{h,K} \geq 0, \quad \theta_t^{h,OE} \geq 0, \quad \theta_t^{h,D} \geq 0, \quad \text{and} \quad \theta_t^{h,K} + \theta_t^{h,OE} + \theta_t^{h,D} = 1. \tag{4.5}$$

Optimal investment ι. The choice of investment rate is still a static and time-separable problem. An agent chooses ι_t^i to maximize her return $r_t^K(\iota_t^i)$. The first-order condition yields the Tobin's q equation

$$\frac{1}{q_t} = \Phi'(\iota_t^i). \tag{4.6}$$

With the special functional form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, $\phi\iota_t^i = q_t - 1$.

Asset and risk allocation. We solve the portfolio choice problem via the "price-taking planner's problem", which is widely applicable to environments with multiple assets. Intuitively, the **price-taking planner's Theorem** means that a social planner that takes prices as given chooses a real asset (capital) allocation κ_t and risk allocation χ_t that coincides with the choices implied by all individuals' portfolio choices. The planner's problem is often of the form

$$\max_{\{\kappa,\chi\}} \ \mathbb{E}\left[\text{Capital Return}\right] - (\text{weighted ave. price of risk}) \times (\text{incremental capital risk}),$$
$$\text{s.t.} \qquad \text{Financial Friction(s).}$$

Let's see how this seemingly magical result works in our environment. The price-taking planner's problem is

$$\max_{\{\kappa_t,\chi_t\}} \left\{ \underbrace{\mathbb{E}_t\left[\mathrm{d}r_t^K(\kappa_t)\right]/\mathrm{d}t}_{\text{Expected Return}} - \overbrace{\underbrace{\left(\sum_{i=\{e,h\}} \varsigma_t^i \chi_t^i\right)}_{\text{(Weighted) Price of Risk}} \times \underbrace{\sigma^{r^K}}_{\text{Risk}}}^{\text{(Weighted) Risk Premium}} \right\} \qquad \text{s.t.} \qquad \underbrace{\chi_t^e \geq \alpha \kappa_t^e}_{\text{Financial Friction}} .$$

Here $r_t^K(\kappa_t)$ is the overall return to capital: $(\iota_t = \iota_t^e = \iota_t^h = \frac{1}{\phi}(q_t - 1))$

$$\mathrm{d}r_t^K(\kappa_t) = \sum_i \kappa_t^i \mathrm{d}r_t^{i,K}(\iota_t^i) = \left[\frac{\sum_i \kappa_t^i a^i - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right]\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t.$$

Hence, the planner's problem can be written as[1]

$$\max_{\{\kappa_t,\chi_t\}} \left\{ \frac{\sum_i \kappa_t^i a^i - \iota_t}{q_t} - \left(\sum_i \varsigma_t^i \chi_t^i\right)(\sigma + \sigma_t^q) \right\} \quad \text{s.t.} \quad \chi_t^e \geq \alpha \kappa_t^e. \tag{4.7}$$

Theorem 4.1. *The equilibrium allocation of physical capital, κ_t^e, as well as the allocation of risk, χ_t^e, that arises from agents' portfolio decisions can be more directly obtained by solving the "price-taking social planner problem" (4.7).*

Proof. The proof takes three steps

1. By **Fisher's Separation Theorem**[2], each individual's portfolio maximization is equivalent to the following maximization problem of a "firm". In our model,

[1] Note that $\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q$ does not depend on κ_t, χ_t.

[2] We postpone the proof of this result till chapter 7. See also Kelsey and Milne (2006) for more details.

individuals in sector i solve[3]

$$\max_{\{\theta_t^{i,K},\,\theta_t^{i,OE},\,\theta_t^{i,D}\}} \quad \theta_t^{i,K}\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t+\theta_t^{i,OE}\mathbb{E}_t\left[\mathrm{d}r_t^{OE}\right]/\mathrm{d}t+\theta_t^{i,D}r_t-\varsigma_t^i(\theta_t^{i,K}+\theta_t^{i,OE})\sigma^{r^{i,K}}$$
$$\text{s.t.} \quad \begin{cases}(4.4) & \text{if } i=e\\ (4.5) & \text{if } i=h\end{cases}.$$

2. Aggregate $\{\boldsymbol{\eta}_t\}$-weighted sum of the two sectors' problems:[4]

$$\max_{\{\boldsymbol{\theta}_t^i\}_{i=\{e,h\}}} \quad \sum_i \eta_t^i\theta_t^{i,K}\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t+\sum_i \eta_t^i\theta_t^{i,OE}\mathbb{E}_t\left[\mathrm{d}r_t^{OE}\right]/\mathrm{d}t+\sum_i \eta_t^i\theta_t^{i,D}r_t$$
$$-\sum_i \varsigma_t^i\eta^i(\theta_t^{i,K}+\theta_t^{i,OE})\sigma_t^{r^K}, \qquad \text{s.t.} \quad (4.4) \text{ and } (4.5).$$

3. Market clearing conditions are

$$\begin{aligned}\text{Capital:} &\quad \eta_t^i\theta_t^{i,K}=\kappa_t^i, \quad \eta_t^i(\theta_t^{i,K}+\theta_t^{i,OE})=\chi_t^i,\\ \text{Outside Equity:} &\quad \sum_i \eta_t^i\theta_t^{i,OE}=0,\\ \text{Debt:} &\quad \sum_i \eta_t^i\theta_t^{i,D}=0.\end{aligned}$$

Note that (4.4) togerther with the capital market clearing condition implies

$$\chi_t^e=\eta_t^e[\theta_t^{e,K}+\theta_t^{e,OE}]\geq\eta_t^e[\theta_t^{e,K}-(1-\alpha)\theta_t^{e,K}]=\alpha\kappa_t^e.$$

Therefore, the aggregated problem can be simplified to

$$\max_{\{\kappa_t,\chi_t\}} \quad \sum_i \kappa_t^i\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t-\left(\sum_i \varsigma_t^i\chi_t^i\right)\sigma_t^{r^K}, \qquad \text{s.t.} \quad \chi_t^e\geq\alpha\kappa_t^e,$$

which is equivalent to the planner's problem (4.7).

□

[3] Recall that outside equity and capital have the same risk (volatility).
[4] Note that $\sigma^{r^{i,K}}=\sigma^{r^K}$ as the two sectors face the same aggregate risk.

Although we proved the theorem for this specific model, the three-step argument is generally valid for more complicated models. Now we can solve the planner's problem (4.7) to obtain the risk/capital allocations. The KKT conditions are

$$\chi_t : \quad \min\left\{\varsigma_t^e - \varsigma_t^h, \chi_t^e - \alpha\kappa_t^e\right\} = 0, \tag{4.8}$$

$$\kappa_t : \quad \min\left\{\frac{a^e - a^h}{q_t} - \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q), 1 - \kappa_t^e\right\} = 0. \tag{4.9}$$

The derivation of the KKT conditions and how these conditions overlap may not be obvious, first notice that apart from the financial friction constraint stated in 4.7, we have another constraint on the capital holding by experts, which is $\kappa_t^e \leq 1$. Hence the Lagrangian reads

$$\mathcal{L} = \frac{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t}{q_t} - \left(\chi_t^e \varsigma_t^e + (1-\chi_t^e)\varsigma_t^h\right)(\sigma + \sigma_t^q) + l_1(\chi_t^e - \alpha\kappa_t^e) + l_2(1-\kappa_t^e)$$

where l_1, l_2 are the Lagrangian multipliers.

Then we focus on χ, taking FOC and the complementary slackness gives

$$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq 0, \chi_t^e - \alpha\kappa_t^e \geq 0, (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)(\chi_t^e - \alpha\kappa_t^e) = 0$$

Hence we have two cases here,

$$\text{Case 1}: \varsigma_t^e(\sigma + \sigma_t^q) > \varsigma_t^h(\sigma + \sigma_t^q), \chi_t^e = \alpha\kappa_t^e,$$
$$\text{Case 2}: \varsigma_t^e(\sigma + \sigma_t^q) = \varsigma_t^h(\sigma + \sigma_t^q), \chi_t^e > \alpha\kappa_t^e.$$

Now under Case 1, we plug $\chi_t^e = \alpha\kappa_t^e$ into the Lagrangian and take FOC and complementary slackness regarding to κ_t^e, we get cases 1a and 1b,

$$\text{Case 1a: } \frac{a^e - a^h}{q_t} > \alpha(\varsigma_t^e - \varsigma_t^h), \kappa_t^e = 1,$$
$$\text{Case 1b: } \frac{a^e - a^h}{q_t} = \alpha(\varsigma_t^e - \varsigma_t^h), \kappa_t^e < 1.$$

Under Case 2, there is no such relationship between χ_t^e and κ_t^e, so the conditions are

$$\text{Case 2a: } \frac{a^e - a^h}{q_t} > 0, \kappa_t^e = 1,$$
$$\text{Case 2b: } \frac{a^e - a^h}{q_t} = 0, \kappa_t^e < 1.$$

where 2b is impossible because $a^e > a^h$.

Intuitively, $\frac{a^e - a^h}{q_t}$ is the benefit of shifting a unit of capital from households to experts and $\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ is the associated cost, which is the difference in the required risk premium. When the "skin in the game" constraint binds, there might be an interior solution where the cost and benefit equalizes, and experts do not hold all the capital, which is case 1a, there might also be a corner solution where experts already hold all the capital but the benefit still exceeds the cost, which is case 1b. If the "skin in the game" constraint does not bind, then the cost of shifting capital is zero as they require the same risk premium, hence there is only a corner solution given by case 2a.

The FOCs can be visualized as following:

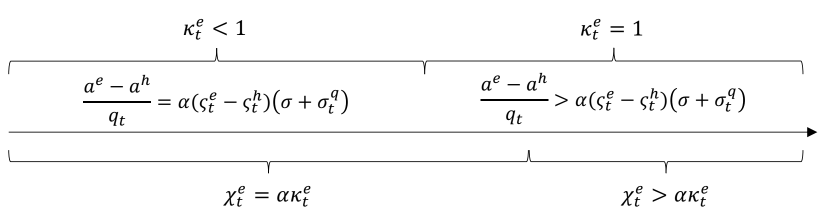

Figure 4.2: First-order conditions of the planner's problem

4.2.2 Evolution of state variable η_t

The definition of equilibrium is similar to the previous chapter – a map from histories of shocks to equilibrium prices.

Definition 4.1. *Given any initial allocation of capital among the agents, an* ***equilibrium*** *is a map from histories* $\{Z_s, s \in [0,t]\}$ *to price* q_t*, investment rate* ι_t^i*, price of risk* ς_t^i *and capital/risk allocation* κ_t^i, χ_t^i*, such that*

(1) all agents choose portfolios and consumption rates to maximize utility,

(2) all markets, for capital, equity and consumption goods, clear.

Drift of η_t^i. One might solve for the process of the state variable $\eta_t^i = N^i/(q_t K_t^i)$ by brute-force, i.e., combining process (6.1) and the market clearing conditions by Itô's quotient rule. However, this method involves a formidable amount of algebra and is thus error-prone. As an alternative, we introduce a new method – "change of numeraire" – to compute the *drift* of wealth shares, $\mu_t^{\eta^i}$.

Note that as a ratio of two equilibrium quantities, the wealth share η_t^i (and its volatility) remain unchanged under a different numeraire, so the change-of-numeraire method does not provide any new information on $\sigma_t^{\eta^i}$.

Change of Numeraire.

Consider two different numeraires – call them dollars (\$) and euros (€). Let x_t^A be the value of a self-financing strategy in \$. Denote the exchange rate by Y_t:

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t.$$

Then, x_t^A/Y_t is the value of the self-financing strategy in €. Applying the martingale approach, $\xi_t^{\$} x_t^A$ and $\xi_t^{€}(x_t^A/Y)$ are both martingales. Recall that for any two assets A, B, the martingale approach implies

$$\mu_t^A - \mu_t^B = \varsigma_t^{\$}(\sigma_t^A - \sigma_t^B),$$
$$\mu_t^{A/Y} - \mu_t^{B/Y} = \varsigma_t^{€}(\sigma_t^{A/Y} - \sigma_t^{B/Y}).$$

By Itô's quotient rule,

$$\mu_t^{A/Y} - \mu_t^{B/Y} = (\mu_t^A - \mu_t^B) - \sigma_t^Y(\sigma_t^A - \sigma_t^B),$$
$$\sigma_t^{A/Y} - \sigma_t^{B/Y} = (\sigma_t^A - \sigma_t^Y) - (\sigma_t^B - \sigma_t^Y) = \sigma_t^A - \sigma_t^B.$$

Hence,

$$(\mu_t^A - \mu_t^B) - \sigma_t^Y(\sigma_t^A - \sigma_t^B) = \varsigma_t^{€}(\sigma_t^A - \sigma_t^B)$$

$$\iff (\varsigma_t^{\$} - \sigma_t^Y)(\sigma_t^A - \sigma_t^B) = \varsigma_t^{€}(\sigma_t^A - \sigma_t^B)$$
$$\iff \boxed{\varsigma_t^{€} = \varsigma_t^{\$} - \sigma_t^Y}. \tag{4.10}$$

We change the numeraire from consumption goods to the total wealth in the economy $N_t = \sum_i N_t^i$. Consider two assets:

- Asset A: sector i's portfolio return in terms of total wealth, that is $N_t^i/N_t = \eta_t^i$. Extra drift terms are included due to reshuffling (death). The return to this asset is

$$\frac{d\eta_t^i + (C_t^i/N_t)dt}{\eta_t^i} = \left(\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} + \rho_d^i \zeta^{\neg i} - \rho_d^{\neg i} \zeta^i \frac{N_t^{\neg i}}{N_t^i}\right) dt + \sigma_t^{\eta^i} dZ_t.$$

- Asset B: a benchmark asset that everyone can hold (e.g., risk-free asset or money in terms of total wealth). In this chapter, asset B is the risk-free loan from the households to the experts, which has return $r_t dt$. However, in new numeraire, the previously riskfree asset is not riskfree and does not earn the same expected return. Under N-numeraire, its risk is $-\sigma_t^N$ and we denote its expected return as r_t^m.

The martingale asset pricing formula implies

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} + \rho_d^i \zeta^{\neg i} - \rho_d^{\neg i} \zeta^i \frac{N_t^{\neg i}}{N_t^i} - r_t^m = \underbrace{(\varsigma_t^i - \sigma_t^N)}_{\substack{\text{price of risk} \\ \text{under } N_t \text{ numeraire}}} (\sigma_t^{\eta^i} + \sigma_t^N). \tag{4.11}$$

Aggregate $\{\boldsymbol{\eta}_t\}$-weighted sum of the two sectors

$$\underbrace{\sum_{i'} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma_t^N)(\sigma_t^{\eta^{i'}} + \sigma_t^N), \tag{4.12}$$

where the first item equals zero because it is the drift of $\sum_{i'} \eta^{i'} = 1$. Subtracting (4.12) from (4.11), the drift of η_t^i is

$$\mu_t^{\eta^i} = (\varsigma_t^i - \sigma_t^N)(\sigma_t^{\eta^i} + \sigma_t^N) - \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma_t^N)(\sigma_t^{\eta^{i'}} + \sigma_t^N) - \left(\frac{C_t^i}{N_t^i} - \frac{C_t}{N_t}\right) - \rho_d^i \zeta^{\neg i} + \rho_d^{\neg i} \zeta^i \frac{N_t^{\neg i}}{N_t^i}$$

$$= (\varsigma_t^i - \sigma - \sigma_t^q)(\sigma_t^{\eta^i} + \sigma + \sigma_t^q) - \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma - \sigma_t^q)(\sigma_t^{\eta^{i'}} + \sigma + \sigma_t^q) - \left(\frac{C_t^i}{N_t^i} - \frac{C_t}{N_t} \right) - \rho_d^i \zeta^{\neg i} + \rho_d^{\neg i} \zeta^i \frac{N_t^{\neg i}}{N_t^i},$$

where the second equality holds because $N_t = q_t K_t$, and hence $\sigma_t^N = \sigma + \sigma_t^q$.

Volatility of η_t^i. Recall that the wealth share η_t^i is numeraire invariant, so we still use Itô's quotient rule to solve for its volatility. Since $\eta_t^i = N_t^i / N_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = \left[\frac{\chi_t^i}{\eta_t^i} - \sum_{i'} \eta_t^{i'} \frac{\chi_t^{i'}}{\eta_t^{i'}} \right] (\sigma + \sigma_t^q) = \frac{\chi_t^i - \eta_t^i}{\eta_t^i} (\sigma + \sigma_t^q),$$

where the third equality follows from (6.1) and market clearing conditions, as

$$\sigma_t^{N^i} = \sigma_t^{n^i} = (\theta_t^{i,K} + \theta_t^{i,OE})(\sigma + \sigma_t^q) = \frac{\chi_t^i}{\eta_t^i} (\sigma + \sigma_t^q).$$

Amplification. Applying Itô's lemma to $q(\eta_t^e)$,

$$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} (\eta_t^e \sigma_t^{\eta^e}) = \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q).$$

The total volatility is

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \dfrac{q'(\eta_t^e)}{q/\eta_t^e} \dfrac{\chi_t^e - \eta_t^e}{\eta_t^e}} > \sigma. \tag{4.13}$$

The amplification effect arises due to fire-sales of capital from the experts to the households. A negative shock increases market illiquidity, leading to expert losses on their capital stock. Since the experts are levered, when hit by a negative shock, they are forced to sell their capital stock to the households, causing a further price drop, and so on – a *loss spiral*.

4.2.3 Goods market clearing

Since both experts and households may hold capital, the goods market clearing condition takes the form:

$$\sum_i \kappa_t^i a^i - \iota_t = \sum_i \frac{C_t^i}{K_t} = q_t \sum_i \eta_t^i \frac{C_t^i}{N_t^i} \tag{4.14}$$

4.2.4 The special case of log-utility and the Inner Loop

So far we have characterized the optimal investment decision (4.6), the capital and risk allocation (4.8 and 4.9), the volatility of capital price (4.13) and the goods market clearing condition (4.14). If we know the prices of risk ς_t^i and consumption-to-wealth ratios C_t^i/N_t^i, we can already solve the model. We see that under log-utility $\varsigma_t^i = \sigma_t^{n^i} = \frac{\chi_t^i}{\eta_t^i}(\sigma + \sigma_t^q)$ and $C_t^i/N_t^i = \rho^i$. It is straightforward to show that (4.8) and (4.9) reduce to the following two conditions:

$$\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2, \text{ with equality if } \kappa_t^e < 1 \tag{4.15}$$

$$\chi_t^e = \max\left\{\alpha\kappa_t^e, \eta_t^e\right\}, \tag{4.16}$$

It follows that for the case of log-utility we only need to solve a system of non-linear equations, one of which is an ODE (4.13). We refer to the algorithm as the "inner loop", as it will also be part of solution procedure for more general types of preferences.

Inner loop. Our goal is to solve for equilibrium variables $(\chi_t^i, \kappa_t^i, \sigma_t^q, \iota_t^i, q_t)$ as functions of η_t^e using conditions (4.6), (4.13), (4.14), (4.15), (4.16), and given expressions for ς_t^i and C_t^i/N_t^i. The algorithm goes as follows:

Start at $q(0)$ (the autarky economy) and solve to the right. Use different procedures for two η^e regions:

i. If $\kappa_t^e < 1$, solve ODE for $q(\eta_t^e)$ using conditions (4.13), (4.14) and (4.15). Specifically, we have to first plug (4.6) into (4.14) and (4.16) into (4.15). Then solve the three equilibrium conditions to the right using Newton's method.

ii. If $\kappa_t^e = 1$, (4.15) is no longer informative about σ_t^q. Instead, we solve (4.6) and (4.14) for $q(\eta_t^e)$, again, using Newton's method.

Newton's Method.

The classic one-dimensional Newton's Method finds the root of a real-valued function by successively computing the intercept of the tangent line approximation of this function. Mathematically, it iteratively computes

$$z_{n+1} = z_n - [f'(z_n)]^{-1} f(z_n).$$

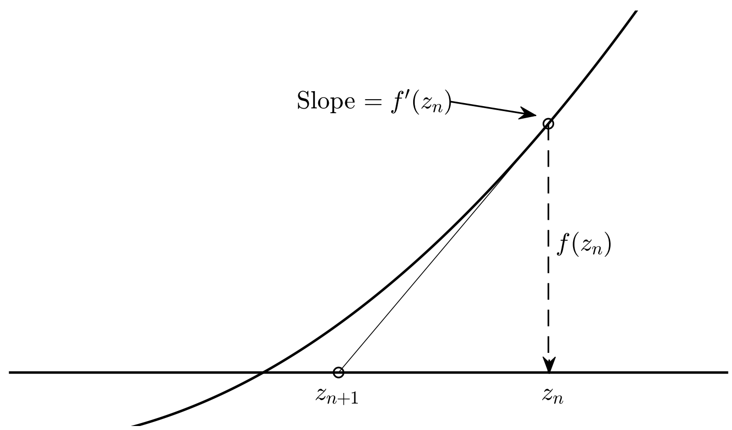

Figure 4.3: One-dimensional Newton's method

For a multi-dimensional system $F(\mathbf{z}) = \mathbf{0}$, the Newton's method proceeds as

$$\mathbf{z}_{n+1} = \mathbf{z}_n - J_n^{-1} F(\mathbf{z}_n), \tag{4.17}$$

where J_n is the Jacoban matrix, i.e., $J_{i,j} = \partial f_i(\mathbf{z})/\partial z_j$.

In the algorithm above, we use Newton's method to approximate the solution to our ODE. Specifically, we let $F(\cdot)$ be the the three equilibrium conditions and $\mathbf{z} = \{q, \kappa^e, (\sigma + \sigma^q)\}$. Starting from the autarky solution $\mathbf{z}_0$, we then iteratively compute to the right $\mathbf{z}_n$ by (4.17) on the η^e grid. Note that for each grid point, we essentially only conduct the first step in the Newton's method. It can be shown that the error is of order

$O((\mathbf{z}-\mathbf{z}^*)^2)$, where $\mathbf{z}^*$ is the true solution. Since our η^e grid will be very dense and all variables are continuous in η^e, the error should be negligible in practice.

If using only the first step is insufficient, one can easily switch to a multi-step Newton's method. For example, in the inner loop function presented in Section 4.2.5, one only needs to add an additional loop for codes between line 21 and 34. However, it is important to keep in mind that Newton's method does not guarantee global convergence. In practice, a multi-step Newton's method might diverge, and one should check the gain from each additional step to deter possible divergence.

4.2.5 Implementation in MATLAB: log-utility

Following is main code executing the algorithm under the assumption of log-utility.

```
%% Parameters and grid
a_e = 0.11; a_h = 0.03;       % production rates
rho_0 = 0.04;                 % time preference
rho_e_d = 0.01;rho_h_d = 0.01;% death rates
rho_e = rho_0 + rho_e_d;      % expert's discount rate
rho_h = rho_0 + rho_h_d;      % household's discount rate
zeta = 0.05;                  % probability of becoming an expert
delta = 0.05; sigma = 0.1;    % decay rate/volatility
phi = 10; alpha = 0.5;        % adjustment cost/equity constraint

N = 501;                           % grid size
eta = linspace(0.0001,0.999,N)'; % grid for \eta

%% Solution
% Solve for q(0)
q0 = (1 + a_h*phi)/(1 + rho_h*phi);

% Inner loop
[Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h, sigma, phi, alpha);

S = (Chi - eta).*SSQ; % \sigma_{\eta^e} -- arithmetic volatility of \eta^e
Sg_e = S./eta;        % \sigma^{\eta^e} -- geometric volatility of \eta^e
Sg_h = -S./(1-eta);   % \sigma^{\eta^h} -- geometric volatility of \eta^h

VarS_e = Chi./eta.*SSQ;          % \varsigma^e -- experts' price of risk
VarS_h = (1-Chi)./(1-eta).*SSQ;  % \varsigma^h -- households' price of risk

CN_e = rho_e; % experts' consumption-to-networth ratio
CN_h = rho_h; % households' consumption-to-networth ratio

```

```
MU = eta .* (1-eta) .* ((VarS_e - SSQ).*(Sg_e + SSQ) -  (VarS_h - SSQ).*(Sg_h + SSQ) - (CN_e - CN_h) + (rho_h_d.*zeta.*(1-eta)-rho_e_d.*(1-zeta).*eta)./(eta.*(1-eta))); % \mu_{\eta^e} -- arithmetic drift of \eta^e
```

The inner loop procedure is implemented in the function `inner_loop_log.m`:

```
function [Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h, sigma, phi, alpha)

N = length(eta);
deta = [eta(1); diff(eta)]; % imposes the correct grid step for numerical derivative at \eta^e = 0

% variables
Q = ones(N,1);     % price of capital q
SSQ = zeros(N,1);   % \sigma + \sigma^q
Kappa = zeros(N,1); % capital fraction of experts \kappa

Rho = eta*rho_e + (1-eta)*rho_h; % auxiliary variable: average consumption-to-networth ratio

% Initiate the loop
kappa = 0; q_old = q0; q = q0; ssq = sigma;

% Iterte over eta
% At each step apply Newton's method to F(z) = 0 where z = [q, kappa, ssq]'
% Use chi = alpha*kappa
for i = 1:N
    % Compute F(z_{n-1})
    F = [kappa*(a_e - a_h) + a_h - (q-1)/phi - q*Rho(i);
         ssq*(q - (q - q_old)/deta(i) * (alpha*kappa - eta(i))) - sigma*q;
         a_e - a_h -  q*alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq^2];

    % Construct Jacobian J^{n-1}
    J = zeros(3,3);
    J(1,:) = [-1/phi - Rho(i), a_e - a_h, 0];
    J(2,:) = [ssq*(1 - (alpha*kappa - eta(i))/deta(i)) - sigma, ...
          -ssq*(q-q_old)/deta(i)*alpha, q - (q-q_old)/deta(i)*(alpha*kappa - eta(i))];
    J(3,:) = [- alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq^2, ...
          -q*alpha^2/(eta(i)*(1-eta(i)))*ssq^2, -2*q*alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq];

    % Iterate, obtain z_{n}
    z = [q, kappa, ssq]' - J\F;

    % If the new kappa is larger than 1, break
    if z(2) >= 1
        break;
    end

    % Update variables
    q = z(1); kappa = z(2); ssq = z(3);

```

```
    % save results
    Q(i) = q; Kappa(i) = kappa; SSQ(i) = ssq;
    q_old = q;
end

% Set kappa = 1, use chi = max(alpha, eta) and compute the rest
n1 = i;
for i = n1:N
    q = (1 + a_e*phi)/(1 + Rho(i)*phi);
    qp = (q - q_old)/deta(i);

    Q(i) =  q; Kappa(i) = 1;
    SSQ(i) = sigma/(1 - (max(alpha, eta(i)) - eta(i))*qp/q);
    q_old = q;
end

% Compute chi, iota
Chi  = max(alpha*Kappa, eta);
Iota = (Q - 1)/phi;
```

4.3 Stationary Distribution and Fan Charts

4.3.1 Stationary distribution

Recall that in the simple model in Chapter 2, experts hold all the capital in the long run. In this chapter, we introduce death for experts and households to avoid a degenerate stationary distribution. Figure 4.4 shows the drift and volatility of η^e. There exists an η^* where the drift of η^e becomes zero. η^* can be viewed as the "steady state" of this model. In the absence of shocks, the system will converge to and stay at the steady state. In response to small shocks, drifts of η^e can still push the economy back to the steady state. Moving away from the crisis regime, risk premia decline, which boosts experts' consumption and lowers the drift of η^e. At η^*, risk premia decline sufficiently so experts' income is exactly offset by their consumption propensity, and hence their wealth share stays constant.

The region where $\eta^e \geq \alpha = 0.5$ reflects perfect risk sharing between experts and households, where the volatility of η^e is zero. Since the drift is negative, the system will never stay in this region. If we start there, the system deterministically moves to $\eta^e = \alpha$.

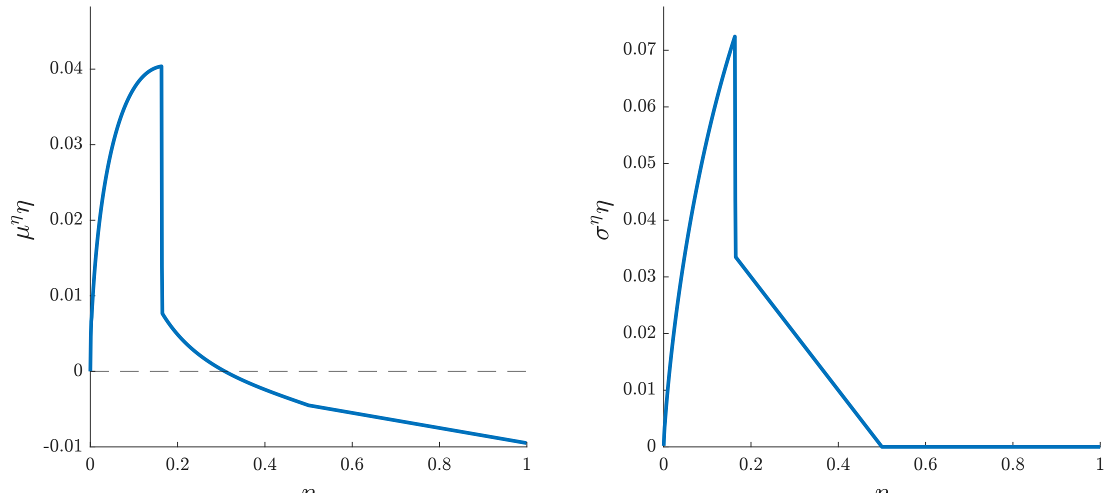

Figure 4.4: Drift and Volatility of η^e

The stationary distribution of the system can be derived using Kolmogorov Forward Equation. Consider a n-dimensional Itô diffusion X with law of motion

$$\mathrm{d}X = \mu(X)\mathrm{d}t + \sigma(X)\mathrm{d}Z, \tag{4.18}$$

where $\mu : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and Z is a m-dimensional Brownian motion. The stationary KFE for f_X is

$$0 = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu_i(x) f_X(x)\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(\left(\sigma(x)\sigma(x)^T\right)_{ij} f_X(x)\right).$$

This is a linear equation for the function f_X, namely $Tf_X = 0$ with the differential operator T defined by

$$Tf := -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu_i f\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(\left(\sigma\sigma^T\right)_{ij} f\right). \tag{4.19}$$

One can show by integration by parts that

$$\int \left(Tf\right)(x) g(x)\mathrm{d}x = \int f(x) \underbrace{\left(\sum_{i=1}^{n} \mu_i(x)\frac{\partial}{\partial x_i} g(x) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sigma\sigma^T\right)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} g(x)\right)}_{=:(Sg)(x)} \mathrm{d}x,$$

so T is the adjoint of the operator S. We then discretize the differential operator S by a matrix A. In finite dimensions, forming adjoints means taking transposes, so that A^T is a finite-dimensional approximation of T. We interpret A as the transition matrix of a continuous-time Markov chain. Then its basis $\boldsymbol{y}$ (i.e., $A^T\boldsymbol{y} = 0$) is a multiple of the invariant distribution of this Markov chain, so one can divide $\boldsymbol{y}$ by its (unweighted) sum to obtain invariant probabilities. Then we can form the cumulative sum to get the CDF and approximate the density by taking finite differences.

In next section we provide MATLAB program `KFE.m` to solve stationary and time-dependent Kolmogorov forward equation. Figure 4.5 plots the stationary distribution. Note that any monotone transformation of η^e is also a valid state variable, including the CDF of η^e.

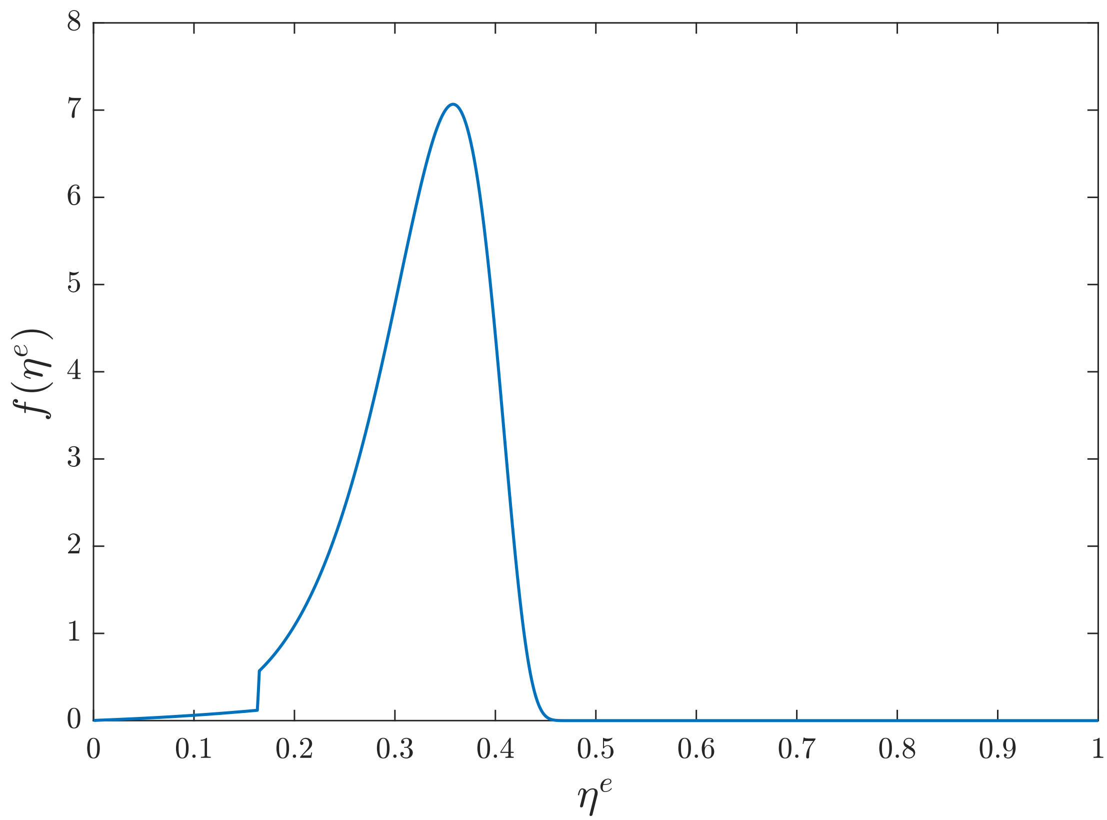

Figure 4.5: Stationary distribution

4.3.2 Fan Charts

In standard macroeconomic models with a fixed steady state, it is commonplace to start at the deterministic steady state and shock the state variable with a one standard deviation (negative) shock (bad unanticipated realization). Subsequently one can observe how the system/economy converges back to the steady state, i.e., by plotting the the impulse response function.

The mathematical tool for studying transition paths in a continuous-time environment is given by the time-dependent Kolmogorov forward equation. For a process X with the evolution (4.18), the KFE is

$$\frac{\partial}{\partial t} f_X(x,t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu_i(x) f_X(x,t)\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(\left(\sigma(x)\sigma(x)^T\right)_{ij} f_X(x,t)\right)$$

or, shorter, $\partial f_X(x,t)/\partial t = \left[T f_X(\cdot,t)\right](x)$ with T as defined in (4.19). To solve such a linear parabolic PDE, we can borrow the A^T approximation matrix from the stationary case and iterate the right hand side of KFE forward.[5]

The following MATLAB function outlines this solving method for both stationary and time-dependent Kolmogorov forward equation:

```
function [pdf_stat,varargout] = KFE(X,MU,S,T,F0)
% KFE Solve one dimensional stationary and time-dependent KFE by finite
% difference method The process being studied is dX = MU(X)dt + S(X)dZ_t.

% REQUIRED INPUT:
% X: [X(1), X(2) ... X(N)]' is the state space (can be uneven grid)
% MU: a drift vector of length N, with MU(1) >= 0 and MU(N) <= 0
% S: a volatility vector of length N, with S(1) = S(N) = 0
% OPTIONAL INPUT FOR TIME-DEPENDENT KFE:
% T: time grid, M*1
% F0: initial distribution vector, N*1

% IMPLEMENT
% 1. For stationary distribution, [pdf_stat] = KFE(X,MU,S);
% 2. For distribution diffusion, [pdf_stat,cdf] = KFE(X,MU,S,T,F0);

% NOTE: 1. Fokker Planck operator (KFE) is the adjoint operator of Feynman Kac
% operator (KBE). We first build Feynman Kac operator and then transpose it.
% 2. We use upwind scheme and implicit scheme for monotonicity and stability.

```

[5] In the one-dimensional case can also be solved with MATLAB built-in solver pdepe.

```
N = length(X);
dX = X(2:N) - X(1:N-1);
%% 1. Build Fokker-Planck operator
% approximate drift terms with an upwind scheme
% upper diagonal
AU = max(MU(1:N-1),0)./dX;
% lower diagonal
AD = - min(MU(2:N),0)./dX;
% main diagonal
A0 = zeros(N,1); A0(1:N-1) = A0(1:N-1) - AU; A0(2:N) = A0(2:N) - AD;
% matrix A
A = sparse(1:N,1:N,A0,N,N) + sparse(1:N-1,2:N,AU,N,N) + sparse(2:N,1:N-1,AD,N,N);

% approximate volatility terms
% sigma^2/(x_{n+1} - x_{n-1})
S0 = zeros(N,1); S0(2:N-1) = S(2:N-1).^2./(dX(1:N-2) + dX(2:N-1));
% upper diagonal
BU = S0(1:N-1)./dX;
% lower diagonal
BD = S0(2:N)./dX;
% main diagonal
B0 = zeros(N,1); B0(1:N-1) = B0(1:N-1) - BU; B0(2:N) = B0(2:N) - BD;
% matrix B
B = sparse(1:N,1:N,B0,N,N) + sparse(1:N-1,2:N,BU,N,N) + sparse(2:N,1:N-1,BD,N,N);

% Fokker Planck operator
FP = (A+B)';

%% 2. Find stationary distribution.
% MATLAB doesn't have build-in kernel solver for sparse matrix, for higher
% efficiency one can use online package like spnull, etc.
F_stat = null(full(FP));
cdf_stat = cumsum(F_stat(:,1)./sum(F_stat(:,1)));
pdf_stat = [0;(cdf_stat(2:end)-cdf_stat(1:end-1))./dX];

%% 3. Solve the time-dependent KFE.
if nargin == 5
    F = F0;
    DT = [0 T(2:end)-T(1:end-1)];
    pdf_diffusion = zeros(length(T),length(F));

    for i = 1:length(T)
        F = (speye(N,N) - DT(i)*FP)\F;
        pdf_diffusion(i,:) = F;
    end
    varargout{1} = pdf_diffusion;
end

end
```

To visualize the transition, we can use fan charts originally introduced by the Bank of England. The first type of fan chart plots the evolution of (the distribution of) the state variable after a shock – the “distributional impulse response”. The idea is similar to the impulse response function in the DSGE literature, but instead of imposing a one-time shock, it studies the dynamics of the whole system. Figure 4.6 plots the convergence back to the stationary distribution after a shock to an economy originally at the median of the stationary distribution. To simulate a negative shock, we set the original Brownian shock at its 1% quantile ($\mathrm{d}Z_t = -2.32\mathrm{d}t$) for a period of $\Delta t = 1$. The dashed lines indicate different quantiles of the distribution and the solid line is the median response. The color is monotone in density.

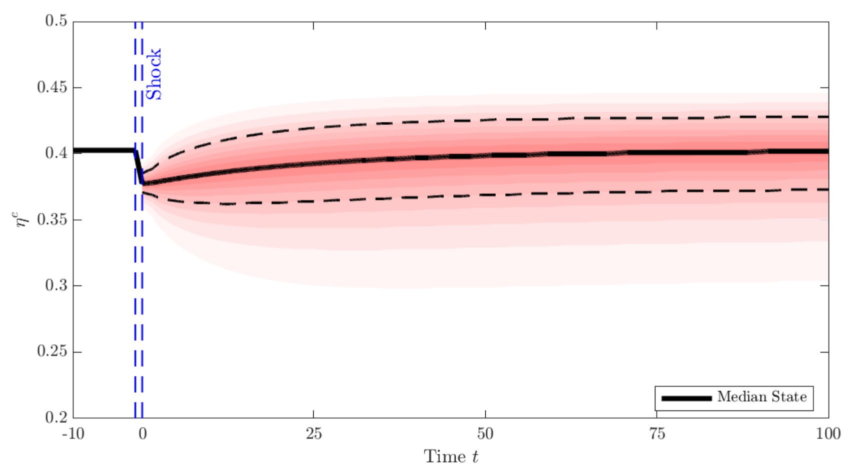

Figure 4.6: Distributional impulse response at stochastic steady state

More interestingly, the second type of fan chart plots the *difference* between distributions with and without the shock. As we find in Figure 4.7, the difference converges to zero in the long run.

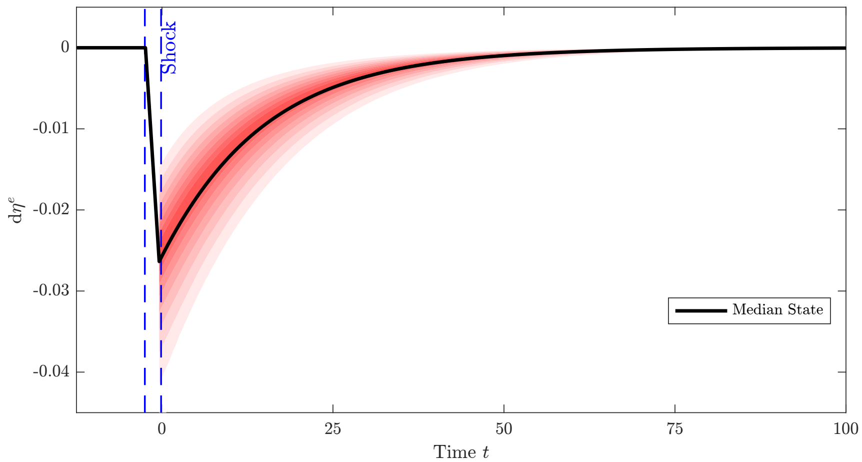

Figure 4.7: Distributional impulse response (difference to unshocked path), $\sigma = 0.1$

4.4 Discussions

In this section we study how specific parameters (a^h, σ, α) affect the equilibrium.

Figure 4.8 shows the effect of σ on the equilibrium. The steady state η^* drops as σ declines, while risk premia fall in the normal region, until η^* reaches the boundary of the crisis region (the kink in q).[6] A volatility paradox emerges: endogenous risk σ_t^q does not necessarily fall as σ declines. Note that as $\sigma \to 0$, the boundary of the crisis region does not converge zero – there is always some positive endogenous risk.

[6]This happens for $\sigma = .01$ in Figure 4.8.

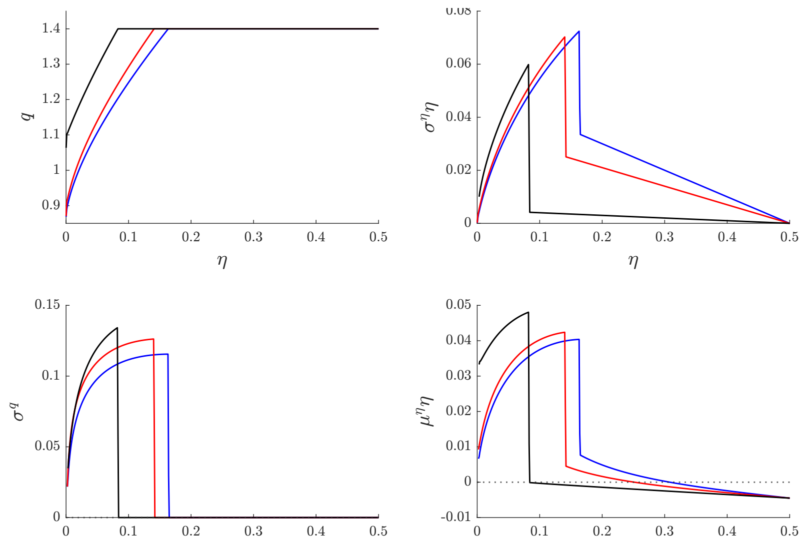

Figure 4.8: Equilibrium for σ= .1 (blue), .07 (red) and .01 (black)

The effect of relaxed financial friction is similar. As shown in Figure 4.9, endogenous risk σ_t^q increases as α falls.

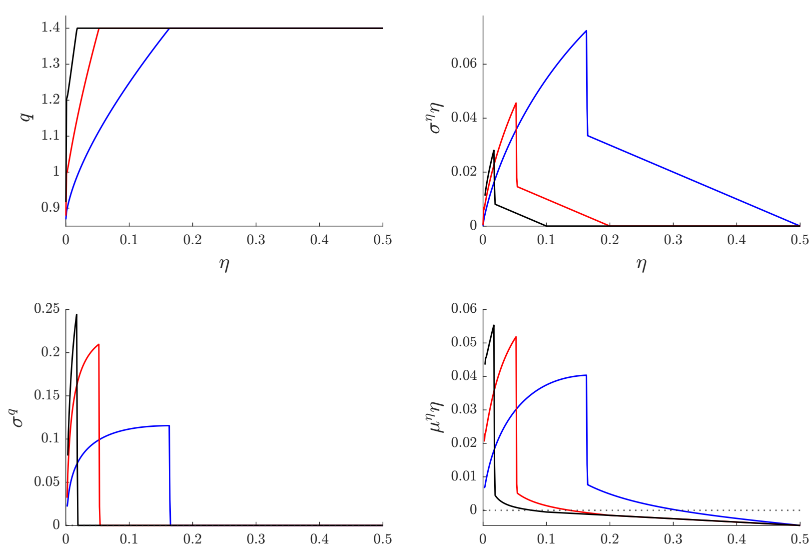

Figure 4.9: Equilibrium for α= .5 (blue), .2 (red) and .1 (black)

The household productivity a^h has a major impact on stability of the system. a^h reflects how households value capital, when they are forced to hold it. Figure 4.10 shows the equilibrium dynamics for different values of a^h. Endogenous risk significantly increases as a^h declines while the behavior in the normal regime and η^* are insensitive to a^h.[7] It is surprising that although expert leverage responds to fundamental risk σ, it does not respond to endogenous tail risk.

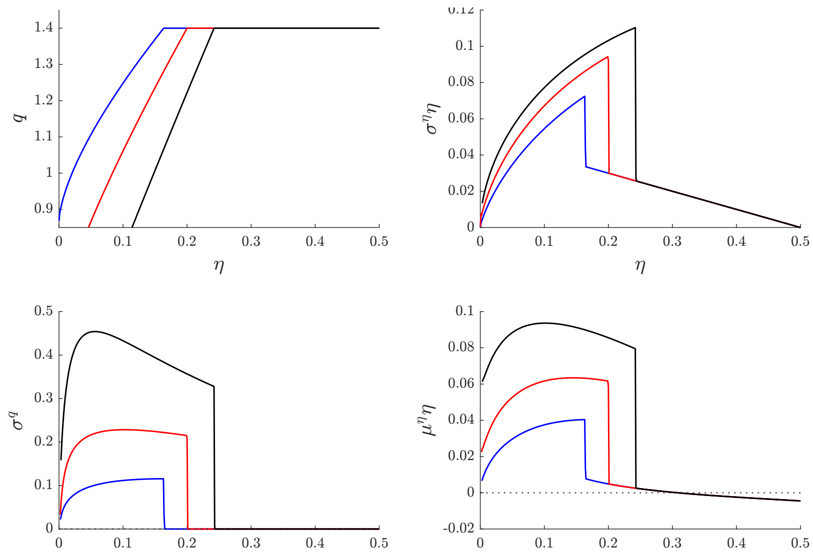

Figure 4.10: Equilibrium for a^h= .03 (blue), -.02 (red) and -.07 (black)

4.5 Exercises

4.5.1 Fire Sales

In this exercise you will solve the model from Lecture 04 numerically, under the assumption of log utility.

1. Our goal is to construct functions $q(\eta), \iota(\eta), \kappa(\eta)$ and $\sigma^q(\eta)$ on the $[0, 1]$ grid. Slides 47-48 provide the parameter values, and slide 46 provides the set of equations and the algorithm.

   (a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$.

[7] As noted in Brunnermeier and Sannikov (2016), for log utility, it can be analytically proved that the dynamics in the normal regime is independent of a^h.

(b) Create a uniform grid for $\eta \in [0.0001, 0.9999] = \{\eta_1 = 0.0001, \eta_2, \dots, \eta_N = 0.9999\}$.

(c) Using the implicit method with the one-step Newton's algorithm, solve the system of equations on slide 46 for $\eta_1, \eta_2, \dots$ and so on.

(d) Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$, solve for q and σ^q.

(e) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$ and comparing it with the graph on slide 50. Do your functions converge to the boundary solution for $\eta = 1$ that you obtained in (a) as $\eta \to 1$?

(f) Plot the remaining variables: $\iota(\eta), \kappa(\eta)$.

(g) We can also look at the experts' balance sheet: derive expression for the scaled version of issued debt: $\frac{D_t^e}{q_t K_t}$ and plot it against η.

2. Recall from the lecture that drift and volatility of η in the general case are given by:

$$\mu_t^\eta = (1-\eta_t)\Bigg[(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) -$$

$$(\varsigma_t^h - \sigma - \sigma_t^q)\left(-\frac{\eta_t}{1-\eta_t}\sigma_t^\eta + \sigma + \sigma_t^q\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right) + \frac{\rho_d^h \zeta(1-\eta_t) - \rho_d^e(1-\zeta)\eta_t}{\eta_t(1-\eta_t)}\Bigg]$$

$$\sigma_t^\eta = \frac{\kappa_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q)$$

(a) Which terms in the above equations can we simplify/substitute because of log utility and why? Perform these substitutions and derive the drift and volatility of η under log utility.

(b) Verify your solution by plotting $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$ and comparing them with the graph on slide 50.

4.5.2 Brunnermeier and Sannikov (2014)

In Brunnermeier and Sannikov (2014), experts and less productive households are risk neutral. However, while consumption of households can go negative, for experts it has to stay non-negative. Hence, experts become extremely risk averse when consumption approaches zero. As a result, the stationary distribution is bimodal.

Try to replicate the paper with tools studied in this chapter.

Bibliography

Brunnermeier, Markus K. and Yuliy Sannikov, "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 2014, *104* (2), 379–421.

_ **and** _ , "Macro, money, and finance: A continuous-time approach," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 1497–1545.

Dumas, Bernard and Elisa Luciano, *The Economics of Continuous-Time Finance*, MIT Press, 2017.

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Chapter 5

Contrasting Financial Frictions

In this chapter, we extend the previous model by including a leverage contrainst on the amount of debt the expert sector can issue.

5.1 Model Setup

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions. Upon death of an expert/household, a new agent takes their place, inherits their wealth, and becomes an expert with probability $\zeta^e \in (0, 1)$.

Experts. Experts have a CRS technology $y_t^e = a^e k_t^e$. Denote their consumption and investment rate by c_t^e, ι_t^e. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = (\Phi(\iota_t^e) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,e}$$

Still, we have only aggregate risk in the environment. Experts have a log utility function and they each maximize Experts have a log utility function and they each maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^e t}\log c_t^e \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^e. Define $\rho^e := \rho_0^e + \rho_d^e$. The objective is equivalent to infinite lifetime with higher discount rate ρ^e

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^e t}\log c_t^e \mathrm{d}t\right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,h}.$$

We let households hold capital to capture fire-sales. As we have discussed in section 3, assuming that households are more patient than the experts, i.e., $\rho^h \leq \rho^e$, is a modeling trick to ensure that the experts do not hold all the capital in the long run. However, here we achieve the same outcome by introducing death. The households maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^h t}\log c_t^h \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^h. Similarly, the objective is equivalent to infinite lifetime with higher discount rate $\rho^h := \rho_0^h + \rho_d^h$

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log c_t^h \mathrm{d}t\right].$$

Financial Friction. The financial friction in this chapter is due to incomplete markets (see, e.g., Dumas and Luciano, 2017). Although experts are allowed to issue equity, they must hold at least α fraction of their risk. At the same time, experts are subject to a leverage constraint by which they can issue risk-free debt up to a fraction ℓ of the value of their capital holdings. As for households, they are subject to a no-short-selling constraint on capital. The balance sheets of the two sectors are as following:

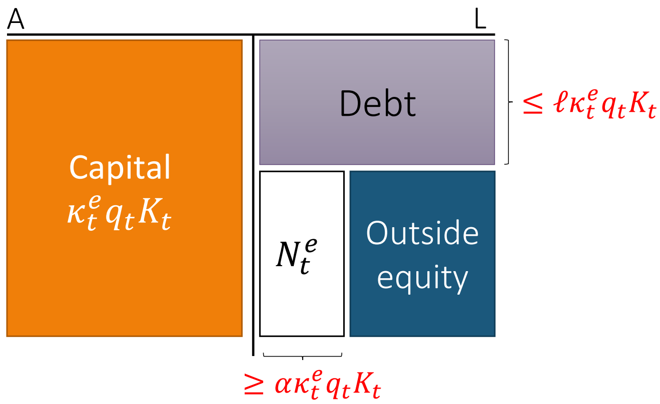


Figure 5.1: Balance sheets of experts and households

Letting $\theta^{e,K} = \kappa^e q K / N^e, \theta^{e,D} = -D^e / N^e$ and $\theta^{e,OE} = OE^e / N^e$, the skin-in-the-game constraint can be expressed as $(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geq 0$ and the leverage constraint as $(1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leq 1$. In addition, the no-short-selling constraint for the household sector is $\theta_t^{h,K} \geq 0$.

Leverage Constraints in Practice. The international regulation for leverage is imposed by the Basel Accords. In 1988 Basel I was published, setting minimal capital requirements for credit risk. In 2004 this was updated to Basel II which introduced risk dependent weights (which are not time varying) on capital requirements. Basel III, introduced in 2010/11, tightened capital requirements and set a leverage ratio requirement.

5.2 Solution Method

We solve this problem using the Stochastic Maximum Problem.

5.2.0 Agents' portfolio choice

Experts' portfolio choice. Let $r_t^{e,j} := \mathbb{E}[dr_t^{e,j}]/dt$. Then the experts solve the following optimization problem

$$\begin{aligned}
&\max_{c_t^e,\iota_t^e,\theta_t^{e,K},\theta_t^{e,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^e t} u(c_t^e)\mathrm{d}t\right] \quad \text{subject to} \\
&\mathrm{d}n_t^e = \left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]\mathrm{d}t \\
&\quad + n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t \\
&\quad (1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geq 0 \text{ (skin-in-the-game constraint)} \\
&\quad (1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leq 1 \text{ (leverage constraint)}
\end{aligned}$$

Denote by λ_t^ℓ the multiplier on the leverage constraint and by λ_t^χ the multiplier on the skin-in-the-game constraint. The Hamiltonian can be constructed as

$$\begin{aligned}
\mathcal{H}_t^e = {}& e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} \\
& - \varsigma_t^e \xi_t^e \overbrace{n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left[1 - (1-\ell)\theta_t^{e,K} - \theta_t^{e,OE}\right] \\
& + \xi_t^e n_t^e \lambda_t^\chi \left[(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE}\right]
\end{aligned}$$

Notice that the objective function is linear in θ, hence the solution is of bang-bang type: the agents are either indifferent or at a constraint. In addition to this, the Fisher Separation Theorem applies between c_t^e, $\grave{}_t^e$, ι_t^e since their first-order conditions are decoupled.

Households' portfolio choice. The households solve the following optimization problem

$$\max_{c_t^h,\iota_t^h,\theta_t^{h,K},\theta_t^{h,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t} u(c_t^h)\mathrm{d}t\right], \; s.t.$$

$$
\begin{aligned}
\mathrm{d}n_t^h &= \left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K} - r_t) + \theta_t^{h,OE}(r_t^{h,OE}(\iota_t^h) - r_t)\right)\right]\mathrm{d}t \\
&\quad + n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t \\
&\quad \theta_t^{h,K} \geq 0 \text{ (household short-sale constraint)}
\end{aligned}
$$

Denote the multiplier on the no-short-selling constraint on capital as λ_t^h. The Hamiltonian can be constructed as

$$
\begin{aligned}
\mathcal{H}_t^h = e^{-\rho^h t}u(c_t^h) + \xi_t^h &\overbrace{\left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t) + \theta_t^{h,OE}(r_t^{h,OE} - r_t)\right)\right]}^{\mu_t^{n^h} n_t^h} \\
- \varsigma_t^h \xi_t^h &\overbrace{n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^h} n_t^h} \; + \; \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}
\end{aligned}
$$

which is also linear in θ_t^h and Fisher Separation Theorem also applies.

θ-choice. The first-order conditions with respect to θs are

- Experts' sector

$$
\begin{cases}
r_t^{e,K} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi \\
r_t^{OE} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi
\end{cases}
$$

- Households' sector

$$
\begin{cases}
r_t^{h,K} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) - \lambda_t^h \\
r_t^{OE} - r_t = \varsigma_t^h(\sigma + \sigma_t^q)
\end{cases}
$$

Taking the difference between them, we obtain

$$
\begin{aligned}
\frac{a^e - a^h}{q_t} &= (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^h + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi, \\
0 &= (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi,
\end{aligned}
$$

Hence, if we focus on the return gaps $r_t^{OE} - r_t^{h,K}$ and $r_t^{e,K} - r_t^{OE}$, we can write the

FOCs as

$$\begin{cases} r_t^{e,K} - r_t^{OE} = \alpha\lambda_t^{\chi} - \ell\lambda_t^{\ell} \\ r_t^{OE} - r_t^{h,K} = \lambda_t^h \end{cases}$$

Let us now consider different cases in which the constraints may or may not bind.

- Household short-selling constraint not binding ($\lambda_t^h = 0 \implies r_t^{OE} = r_t^{h,K}$)
  - $\lambda_t^{\chi} = 0, \lambda_t^{\ell} > 0$ is impossible because $r_t^{e,K} > r_t^{h,K} = r_t^{OE}$
  - $\lambda_t^{\chi} > 0, \lambda_t^{\ell} > 0$ or $\lambda_t^{\chi} > 0, \lambda_t^{\ell} = 0$ are both possible. The skin-in-the-game constraint always binds while the leverage constraint may or may not bind.
- Household short selling constraint binding ($\lambda_t^h > 0$)
  - If we define $\eta^{e,*}$ to be the smallest η_t^e such that $\lambda_t^h > 0$, then $\lambda_t^{\ell} > 0$ is impossible because $1/\eta_t^e < 1/\eta^{e,*}$. Hence, only the skin-in-the-game constraint may bind. The intuition behind this is that outside equity cannot generate a higher return than physical capital.

$(\kappa, \chi)-$ **Asset/Risk Allocation.** We now translate the constraints from the θ space to the $\kappa - \chi$ space. These become

$$\text{Skin-in-the-game constraint } \chi_t^e = \eta_t^{e,K}\theta_t^e + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\geq -(1-\alpha)\kappa_t^e} \geq \alpha\kappa_t^e$$

$$\text{Leverage constraint } \chi_t^e = \eta_t^e\theta_t^{e,K} + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\leq (1-(1-\ell)\theta_t^{e,K})} \leq \ell\kappa_t^e + \eta_t^e$$

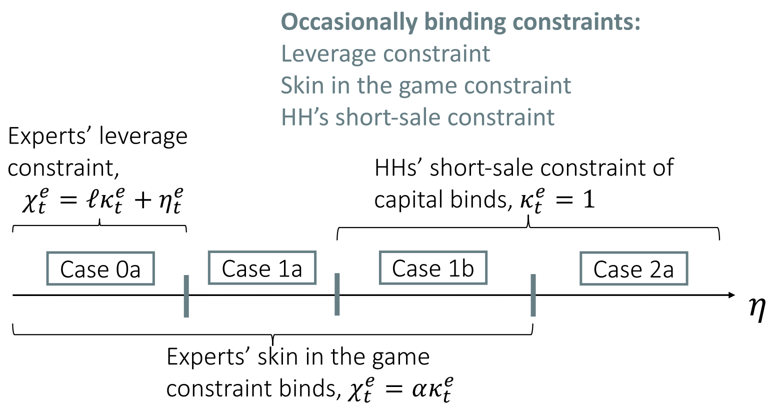
And the FOCs can be rewritten as

$$\frac{a^e - a^h}{q_t} \geq \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta-\text{risk premia}}, \qquad \text{with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

$$\varsigma_t^e \geq \varsigma^h, \qquad \text{with equality if } \chi_t^e > \alpha\kappa_t^e$$

With this in mind, we can now consider four different cases across η which are summarized in the table and figure below

Cases	0a	1a	1b	2a
Leverage	$\chi_t^e = \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$
Skin-in-the-game	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e > \alpha\kappa_t^e$
Short-sale	$\kappa_t^e < 1$	$\kappa_t^e < 1$	$\kappa_t^e = 1$	$\kappa_t^e = 1$
Δ-risk premia	$>$	$=$	$>$	$>$
Risk-sharing	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$



5.2.1 Market Clearing

Determination of κ, χ. The determination of κ is based on the difference in risk premia $\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$. Hence, κ is determined by the FOC

$$\boxed{\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.}$$

The risk share of experts χ_t^e is given by

$$\boxed{\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}}$$

while the determination of κ_t^e in the leverage constrained region follows from

$$\boxed{\kappa_t^e = \frac{\eta_t^e}{\alpha - \ell}}$$

Investment and capital prices. Optimal investment dictates

$$\boxed{\phi\iota = q - 1}$$

and from the previous lecture on amplification, we have that

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}\frac{\chi_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

Output good market clearing. This requires

$$(\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota_t)K_t = C_t$$

$$\Rightarrow \quad \boxed{\kappa_t^e a^e + (1 - \kappa_t^e)a^h - \iota_t = q_t[\eta_t \rho^e + (1 - \eta_t)\rho^h]}$$

5.2.2 Algorithm – Static Step

We have five static conditions,

1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e: $\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Planner condition for χ_t^e: $\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$
4. $\kappa_t^e a_t^e + (1 - \kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t \rho^e + (1 - \eta_t)\rho^h] = 0$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$
   $\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

The algorithm strategy is to start at $q(0)$, solve to the right, and use a different procedure for the two η regions depending on κ:

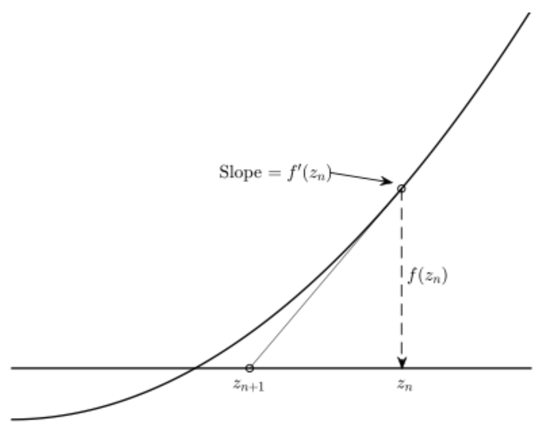
1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (4)

- Plug in the Planner's condition of χ_t
- Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method
- if $\chi^e_t \geq \ell\kappa^e_t + \eta^e_t$, replace κ^e_t by $\frac{\eta^e_t}{\alpha-\ell}$, solve (3) (4) (5) for $\chi(\eta^e)$,$q(\eta^e)$,$\sigma^q(\eta^e)$

2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (4) for $q(\eta^e)$
   (HINT: When constraint binds, we directly substitute in κ^e)

Aside: Newton's Method The system of equations to use for Newton's method in this case is as follows,

$$\mathbf{z}_n = \begin{bmatrix} q_t \\ \kappa^e_t \\ \sigma + \sigma^q_t \end{bmatrix}, F(\mathbf{z}_n) = \begin{bmatrix} \kappa^e_t a^e_t + (1-\kappa^e_t)a^h - \iota(q_t) - q_t[\eta_t\rho^e + (1-\eta_t)\rho^h] \\ q'(\eta^e_t)(\chi^e_t - \eta^e_t)(\sigma+\sigma^q_t) - \sigma^q q(\eta^e_t) \\ (a^e - a^h) - \alpha q_t \frac{\chi^e_t - \eta^e_t}{(1-\eta^e_t)\eta^e_t}\left(\sigma+\sigma^q_t\right)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{Planner.} \end{bmatrix}$$



5.3 Numerical Solution and Model Properties

In this Section we cover the numerical solution, which was computed with the algorithm covered above.

5.3.1 Capital Price and Volatility

We first analyze the price of capital and amplification. Figure 5.2 plots these, for the displayed parameters. Note that $\ell = 0.55$, which corresponds to one being able to borrow up to 55% of their collateral. We observe four regions split by the dashed lines. In the left most region both the leverage constraint ad outside equity constraint are binding. In the second left most region the leverage constraint no longer binds. These two regions make up the crisis regimes, where fire-sales are occurring and capital price is increasing as a function of η. The third region is where 100% of the capital is held by the experts, and households are short-sale constrained. In the final, right most, region there is perfect risk sharing, where $\chi = \kappa$.

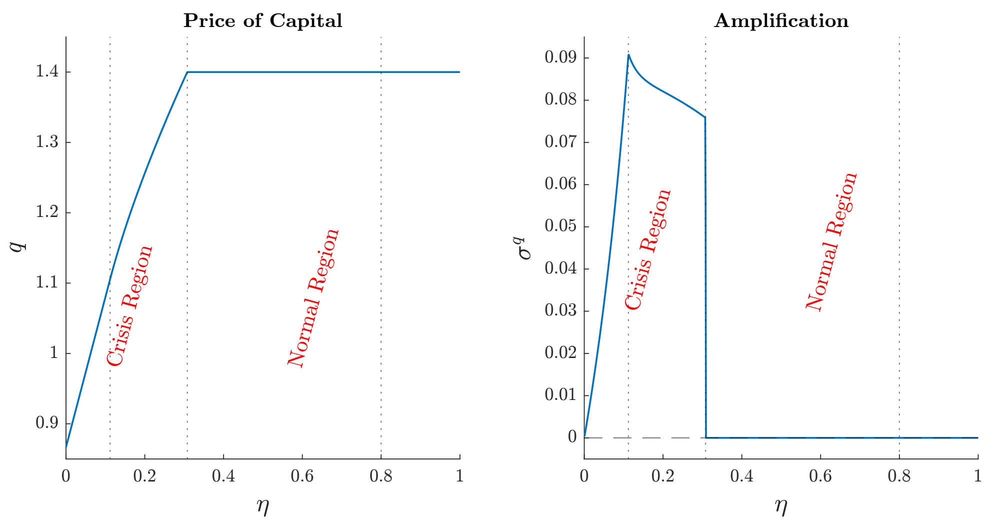


Figure 5.2: $\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

It is interesting to compare the volatility of the price of capital in the model over different cases. Figure 5.2 plots this for the full model, with outside equity and leverage. Figure 5.3 plots σ^q for a benchmark model (dashed black line) with no outside equity and no leverage, as well as a model with outside equity only (red line). As outside equity only is added to the model, the volatility goes up in the fire sale region. As the leverage constraint is added on top of this, the fire-selling region in which it does not bind (the second left most region) experiences more price volatility.

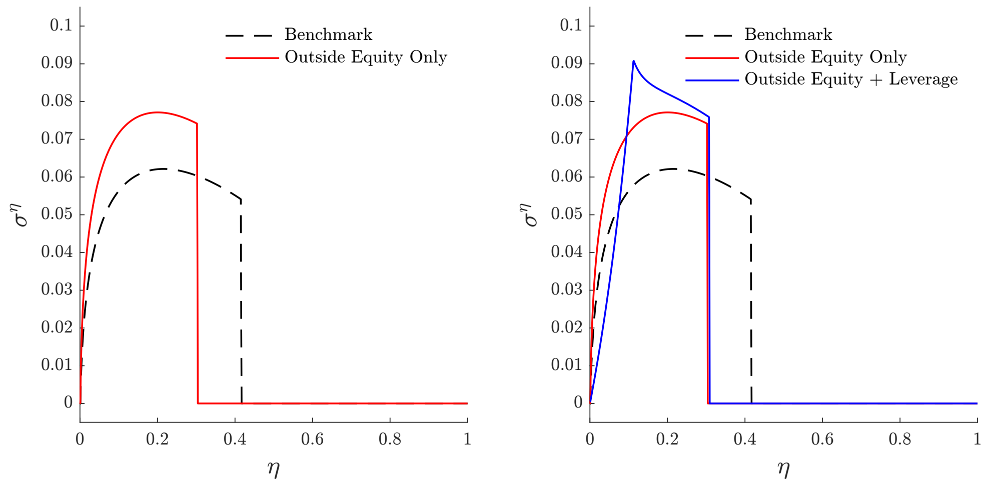


Figure 5.3: Same parameters as Figure 5.2, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

5.3.2 Net Worth Evolution: Drift & Volatility

Figure 5.4 plots the drift and volatility of η in arithmetic terms. When the drift of the state variable is 0 we are at the steady state. Below the steady state the system drifts up and above it it drifts down. The volatility is highest in the region where the leverage constraint is not binding but fire-sales are still happening.

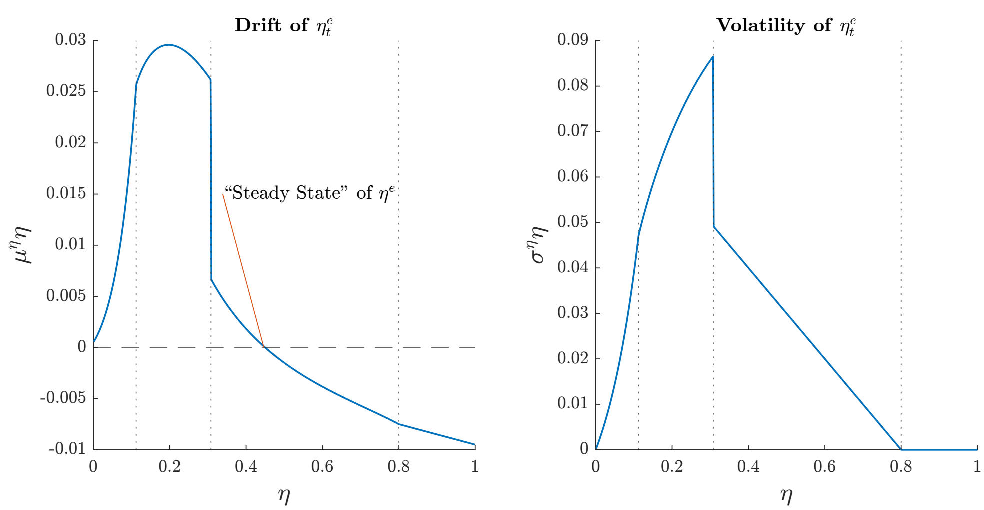


Figure 5.4: $\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, δ =0.05, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

5.3.3 Risk Allocation & Leverage

Figure 5.5 plots the risk allocation and leverage for the full model, with both outside equity and the leverage constraint. The risk allocation plot shows what proportion of risk is held by the experts. It is increasing in η, and we observe perfect risk sharing in the right most region. When the experts hold, for example, 90% of the wealth, they also hold 90% of the risk. Therefore from $\eta = 0.8$ onwards the risk holding plot is the identity line. In the second right most region, the experts already hold all of the capital but only 80% of the risk. This is because the outside equity constrain of $\alpha = 0.8$ is binding. In the capital net worth ratio plot we observe the leverage constraint kicking in in the left most region. Experts would like to lever up more but cannot. Note that the lever up less as they capture more of the wealth share.

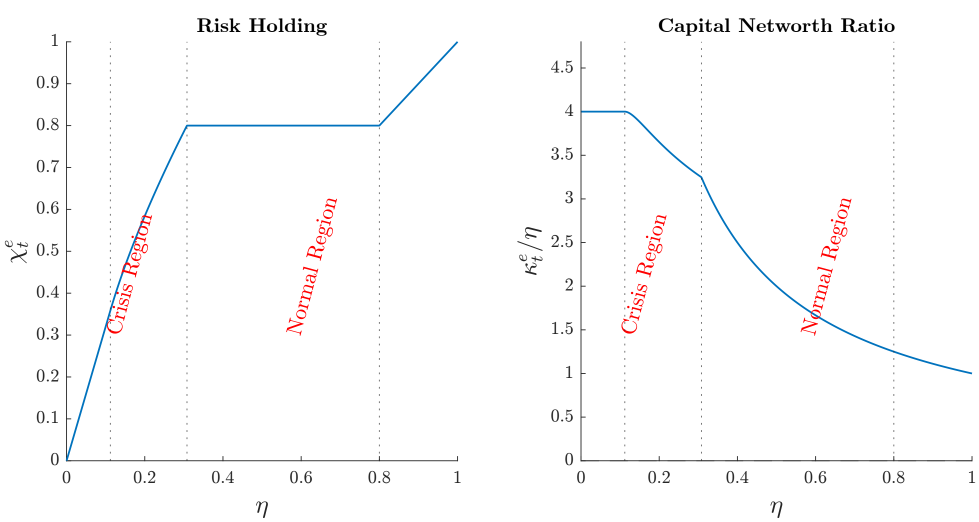


Figure 5.5: $\rho^{e,h}$ = 0.05, $\rho_0^{e,h}$ = 0.04, $\rho_d^{e,h}$ = 0.01, ζ^e = 0.05, δ =0.05, a^e = 0.11, a^h = 0.03, σ = 0.10, ϕ = 10, $\alpha = 0.8$, $\ell = 0.55$.

Risk Allocation Comparison. Figure 5.6 plots risk allocation for the benchmark model (dashed black line) with no outside equity and no leverage, as well as the model with outside equity only (red line). The full model is again plotted in blue. The addition of outside equity allows experts to offload a proportion of their risk, which is why the red line is lower.

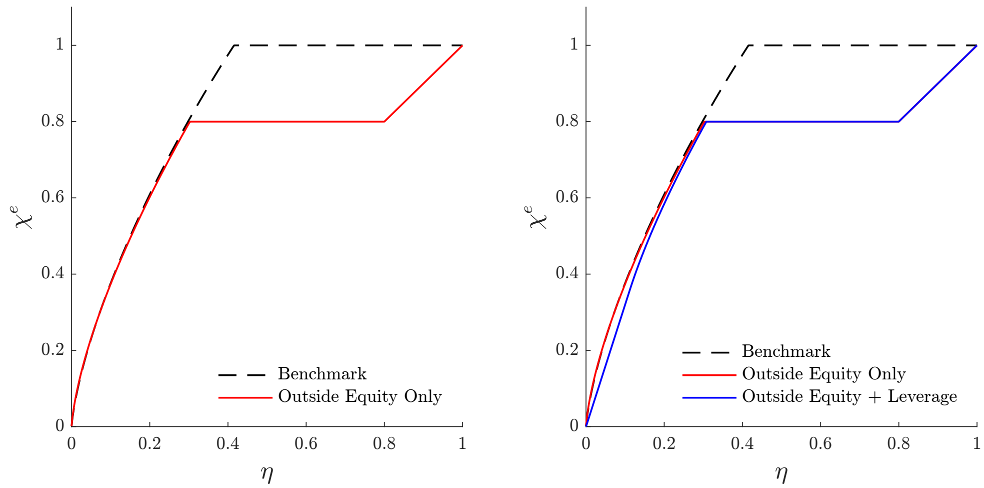


Figure 5.6: Same parameters as Figure 5.5, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

Leverage Comparison. Figure 5.6 plots leverage for the benchmark model (dashed black line) with no outside equity and no leverage, as well as the model with outside equity only (red line). The full model is again plotted in blue. Allowing for outside equity lets experts hold even more capital in relation to net worth, but if there is a leverage constraint on top this mechanically tightened again.

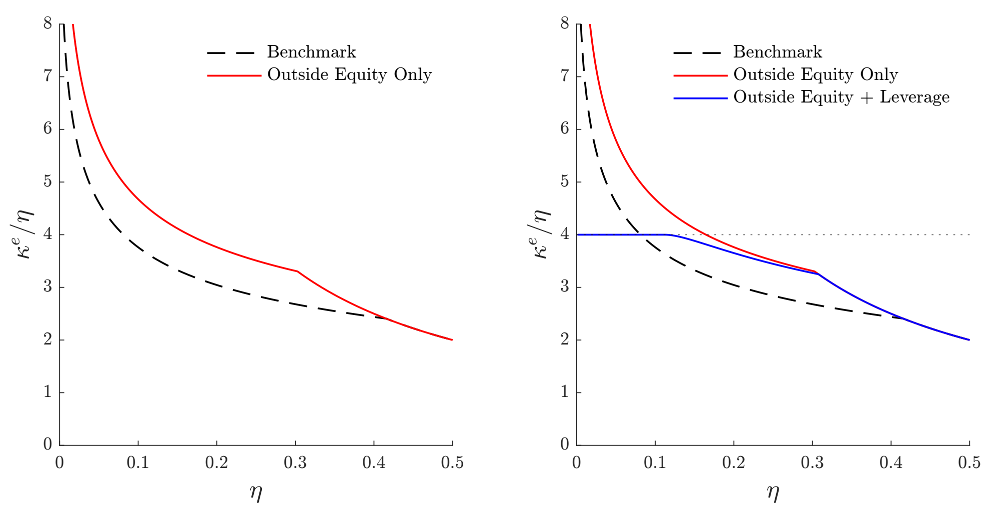


Figure 5.7: Same parameters as Figure 5.5, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

5.3.4 Volatility Paradox

We now consider the model with outside equity $\alpha = 0.8$, but no leverage constraint. Then the *volatility paradox* is a phenomenon which shows that σ^η (as well as $\sigma + \sigma^q$) stays roughly constant as σ varies (even when $\sigma \to 0$). Figure 5.8 plots various objects for different σ. We observe that the regions shift slightly, since the fire-selling regions start for lower η if there is lower σ. However, we observe that the total volatility, at the point where fire sales start in each parameterization, is roughly the same. This is because as you lower the fundamental volatility, σ, the price volatility, σ^q, increases.

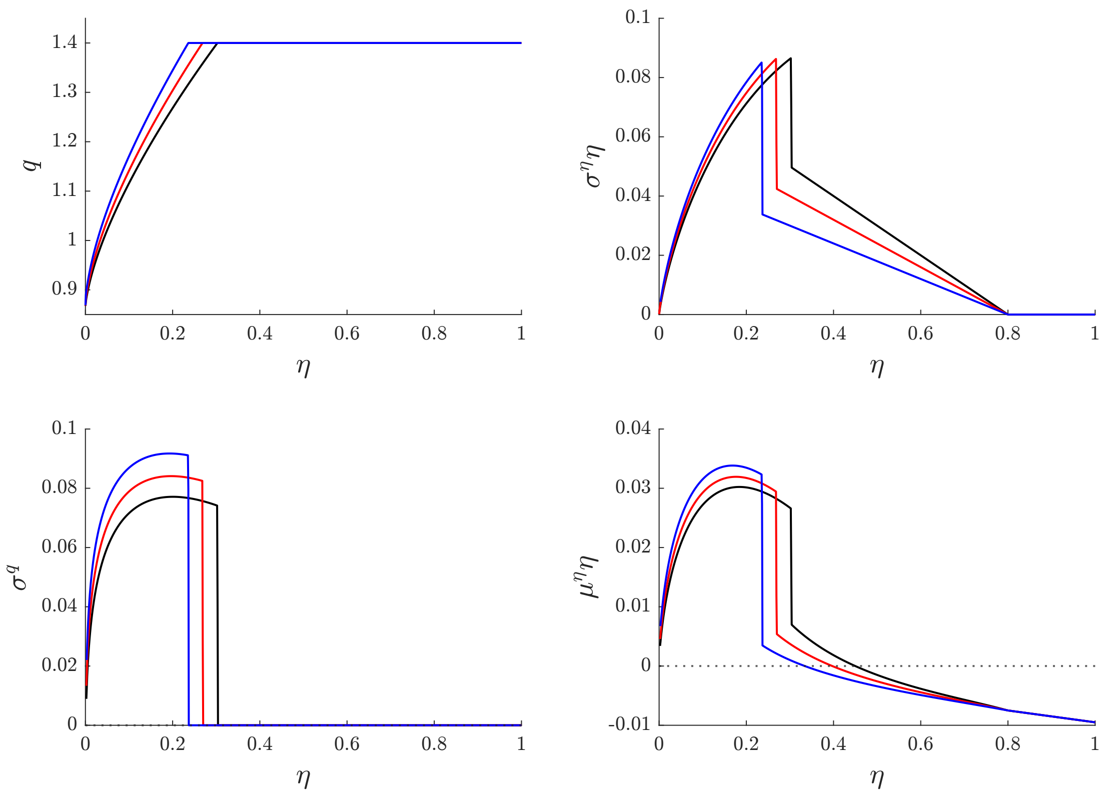


Figure 5.8: Volatility Paradox $\alpha = 0.8$, $\sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$

If we add a leverage constraint on top of the model $\ell = 0.55$ we get even more interesting results. Figure 5.9 plots the same objects as before for different σ. Now σ^q is really increasing in σ for the second left most region, where fire-sales are happening but the leverage constraint does not bind. In the left most region of the binding leverage constraint and fire-sales, the $\sigma^\eta \eta$ and σ^q are decreasing in σ. The leverage constraint binds earlier on for lower σ and makes the volatility go down. This was not the case in the model without the leverage constraint. Interestingly, the drift also collapses compared to the case of no leverage constraint. this is because experts cannot lever up as they wish to take advantage of investment opportunities. The upwards drift in the left most region is lower for lower σ. The lower the σ the longer it takes to get out of the leverage constrained region.

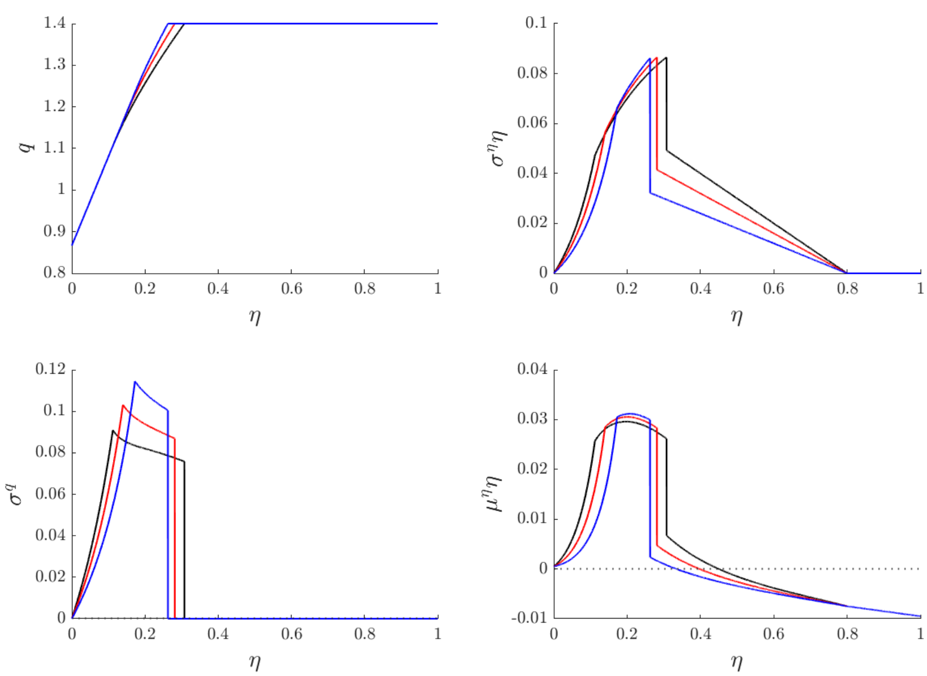


Figure 5.9: Volatility Paradox $\alpha = 0.8, \ell = 0.55, \sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$

5.4 Stationary Distribution & Net Worth Trap

Recall from Chapter 4 that one can find the stationary distribution from the "Kolmogorov Forward Equation". Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows

$$\frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}.$$

A corollary is that if stationary distribution $f(\eta)$ exists, it satisfies ODE

$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2},$$

which has the closed form solution,

$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x\right).$$

Aside: KFE Analytical Example.

- Reflected Geometric Brownian Motion (Reflecting barrier at $x = d$):

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}Z_t - \mathrm{d}U_t, X_t \in (0, d]$$

- KFE:

$$\frac{\partial f}{\partial t} = -\frac{\partial(\mu x f)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2 x^2 f)}{\partial x^2}$$

- Stationary distribution

$$f(x) = \frac{\text{Const}}{\sigma^2 x^2} \exp\left(\int_0^x \frac{2\mu y}{\sigma^2 y^2}\mathrm{d}y\right) = \frac{\frac{2\mu}{\sigma^2} - 1}{d^{\frac{2\mu}{\sigma^2}-1}} x^{\frac{2\mu}{\sigma^2}-2}$$

Net Worth Trap. Figure 5.10 plots the stationary distribution for different σ. The net worth trap is that the stationary distribution, for certain values of σ (in this case $\sigma = 0.05$), is double humped shape. The system lives a lot of time in the center, but also lives a lot of time in the low η regime, or the crisis regime. Without the leverage constraint this phenomenon does not occur.

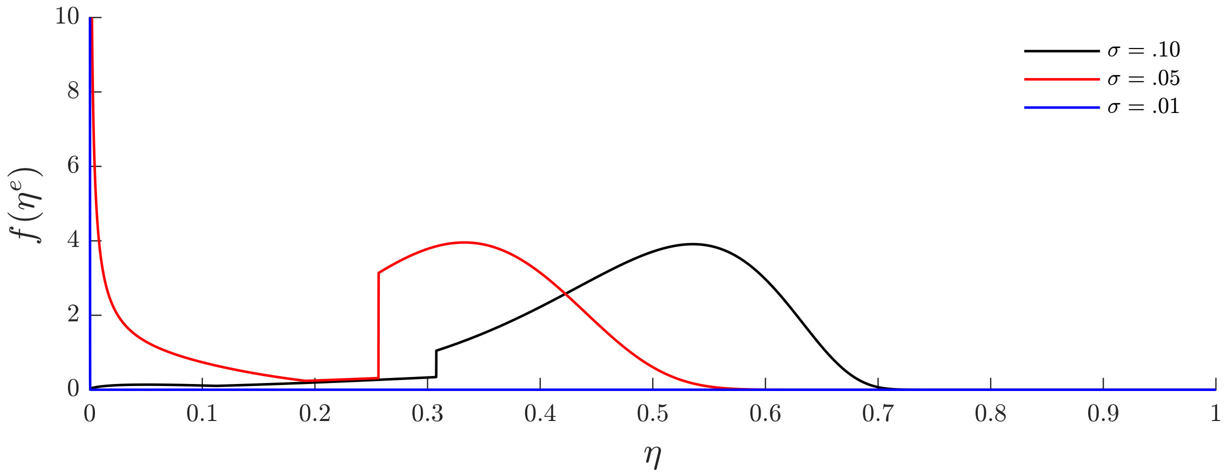


Figure 5.10: Stationary Distribution for Different σ. Fundamental volatility: $\sigma = .1$, $\sigma = .05$, $\sigma = .01$

For low values of σ, like 0.01 in this case, the stationary distribution is degenerate. Once experts become under capitalized they cannot escape the crisis regime and are pushed to hold no wealth. So when does the invariant distribution exist? The asymptotic solution (as $\eta \to 0$) follows

$$f(\eta) \sim \left(\frac{2\mu(0)}{\sigma^2(0)} - 1 \right) \eta^{\frac{2\mu(0)}{\sigma^2(0)} - 2}.$$

We then have the following cases,

- $\frac{2\mu(0)}{\sigma^2(0)} \geq 2$: $f(\eta)$ is finite at $\eta = 0$.
- $2 \geq \frac{2\mu(0)}{\sigma^2(0)} > 1$: $f(\eta)$ is infinite at $\eta = 0$, but still normalizeable ($\int f d\eta < \infty$).
- $1 \geq \frac{2\mu(0)}{\sigma^2(0)}$: $f(\eta)$ is infinite at $\eta = 0$, and stationary distribution does not exist.

5.4.1 Implementation in MATLAB: log-utility

Following is main code executing the algorithm under the assumption of log-utility.

```
%% Parameter values
rho_0 = 0.04; rho_e_d = 0.01;rho_h_d = 0.01;
rho_e = rho_0 + rho_e_d; rho_h = rho_0 + rho_h_d;
zeta = 0.05; delta = 0.05; a_e = 0.11; a_h = 0.03;
```

```
sigma = 0.1;alpha = .8; phi = 10;ell = .55;

N = 10001;

%% Grids
eta = linspace(0.0,1,N); deta = eta(2)-eta(1);

rho = rho_e.*eta+ rho_h.*(1-eta);

q = (1+phi* a_e)./(ones(N,1)+phi*rho'); q=q';
iota = (ones(N,1)*a_e - rho')./(ones(N,1)+phi.*rho');iota = iota';
chi = zeros(1,N);sig_q = zeros(1,N);kappa = zeros(1,N);

q(1) = (1+a_h*phi)/(1+rho_h*phi);
iota(1)=(a_h-rho_h)/(1+rho_h*phi);

tor = 1e-5;
max_it = 100;
ind_lv = 1;
ind_fl = 1;
flag = 0;
%% Model solution: Newton's method
for i = 2:N
    ind_fl = ind_fl +1;
    iter = 0;
    error = 1.0;
    etai = eta(i);
    F = @(x)[(a_e-a_h)/x(1)-alpha*((alpha*x(2)-etai)/etai/(1-etai))*...
        (sigma+x(3))^2;
        x(2)*a_e + (1-x(2))*a_h-(x(1)-1)/phi-x(1)*rho(i);...
        (x(1)-q(i-1))/deta* (alpha *x(2)-etai)*(sigma+x(3))-x(1)*x(3)];
    J = @(x) [-(a_e-a_h)/x(1)^2, -alpha^2*(sigma +x(3))^2/etai/(1-etai), ...
        -2*alpha*(alpha*x(2)-etai)/etai/(1-etai)*(sigma+x(3));
        -1/phi - rho(i), a_e - a_h, 0;...
        1/deta*(alpha*x(2)-etai)*(sigma+x(3))-x(3), alpha/deta*...
        (x(1)-q(i-1))*(sigma+x(3)), -1/deta*(x(1)-q(i-1))...
        *(alpha*x(2)-etai)-x(1)];
    z0 = [q(i-1), kappa(i-1),sig_q(i-1)];
    while error > tor
        iter = iter + 1;
        z1 = z0 - reshape(J(z0)\F(z0),[1,3]);
        error = norm(z1-z0)/norm(z0);
        z0 = z1;
        %disp(error);
        if iter>max_it
            disp("HAVE trouble!")
            break
        end
    end
    if z0(2)>etai/(alpha-ell)
        % if leverage constraint is violated, solve kappa by leverage
        % constraint
```

```
        kappai= etai/(alpha-ell); % alpha * kappa = ell * kappa + eta
        kappa(i) = kappai;
        q(i) = (a_e * kappai + a_h * (1-kappai) +1/phi)/(1/phi + rho(i));
        chi(i) = alpha * kappa(i);
        sig_q(i) = sigma/(1-(q(i)-q(i-1))/deta/q(i) * (chi(i)-etai))-sigma;
        iota(i)= (q(i)-1)/phi;
        ind_lv  = ind_lv+1;
    else
        q(i) = z0(1);
        kappa(i) = z0(2);
        chi(i)=alpha*kappa(i);
        sig_q(i) = z0(3);
        iota(i)= (q(i)-1)/phi;
    end

    if kappa(i)>1
        flag = i;
        break
    end
end

for i = flag:N
    etai = eta(i);
    q(i)=(1+phi*a_e)/(1+phi*rho(i));
    iota(i)=(a_e - rho(i))/(1+phi*rho(i));
    kappa(i)=1;
    if etai <alpha
        ind_fl = ind_fl +1;
        chi(i) =alpha;
        sig_q(i) = sigma *(q(i)-q(i-1))/deta/q(i)*(alpha-etai)/...
            (1-(q(i)-q(i-1))/deta/q(i)*(alpha-etai));
    else
        chi(i)=etai;
        sig_q(i)=0;
    end
end
 sig_q(1)=sig_q(2);

%% Compute drift and volatility of eta
mu_eta = (ones(1,N)-eta).*(((chi.^2-chi.*eta)./eta.^2-(eta-chi).*...
    (ones(1,N)-chi)./(ones(1,N)-eta).^2) ...
    .*(sigma.*ones(1,N)+sig_q).^2-(rho_e-rho_h).*ones(1,N)+(rho_h_d.*zeta.*ones(1,N)
    .*(1-eta)-rho_e_d.*(1-zeta).*ones(1,N).*eta)./(eta.*(ones(1,N)-eta)));
sig_eta = (sigma*ones(1,N)+sig_q).*(chi-eta)./eta;

%% Compute stationary distribution
M = buildM(eta ,mu_eta .*eta,sig_eta.*eta);

g = ones(N,1);
Mt = M';
g(2:N) = - (Mt(2:N,2:N)\(Mt(2:N,1))*g(1));
g = g/sum(g)/deta;
```

5.5 Net Worth Trap & Volatility Paradox Interaction

The net worth trap is based on the volatility paradox interaction with the leverage constraint. The leverage constraint depresses μ^η and σ^η when η is close to 0 since experts are constrained and cannot take on any risk. Furthermore, there is higher volatility of q and η in the fire-sale region outside the binding leverage constraint.

Regulation, like the Basel Accords, is important for keeping volatility low and maintaining stability in the economy. However, a leverage constraint which is not varying in η can induce a net worth trap, making it hard for experts to recover from low wealth shares.

Bibliography

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Part III

Immersion Chapters

Chapter 6

A More General Macro-model with Endogenous Risk Dynamics

In Chapter 4, we studied a macro model with endogenous risk dynamics under log-utility. In this chapter, we present a generalization to other utility functions, namely constant relative risk aversion (CRRA) utility and Epstein-Zin (EZ) utility. We begin with the CRRA case.

6.1 CRRA Utility and Value Functions

Now with CRRA utility, sector i agents now have the following optimization problem

$$\begin{aligned} \max_{\{\iota_t^i, \boldsymbol{\theta}_t^i, c_t^i\}_{t=0}^{\infty}} \quad & \mathbb{E}_0\left[\int_0^\infty e^{-\rho^i t}\frac{c_t^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \\ \text{s.t.} \quad & \frac{\mathrm{d}n_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}\mathrm{d}t + \theta_t^{i,K}\mathrm{d}r_t^{i,K}(\iota_t^i) + \theta_t^{i,OE}\mathrm{d}r_t^{OE} + \theta_t^{i,D} r_t \mathrm{d}t \\ & n_0^i \text{ given,} \end{aligned} \tag{6.1}$$

Once we deviate from the assumption of log-utility, we no longer have readily available expressions for prices of risk and consumption-to-wealth ratios. We can however express these variables in terms of agents' value functions, which we discuss in this Section.

Martingale approach works well in an endowment economy, where the consumption stream is exactly the endowment stream, hence the marginal utility is given exogenously by stochastic discount factor. While for our production economy case, we mix the martingale approach with value function method.

By the same arguments presented in Section 3.3.1, CRRA utility implies that c_t^i/n_t^i-ratio is invariant in n_t^i, that is[1]

$$V^i(n_t^i;\boldsymbol{\eta_t},K_t) = \frac{u\left(\omega^i(\boldsymbol{\eta_t},K_t)n_t^i\right)}{\rho^i} = \frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma}, \qquad \frac{c_t^i}{n_t^i} = (\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}. \tag{6.2}$$

Let's first take a look at a special case. For constant investment opportunities $\omega_t^i = \omega$, c_t^i/n_t^i is constant, and hence $\mu_t^{c^i} = \mu_t^{n^i}$, $\sigma_t^{c^i} = \sigma_t^{n^i}$. Furthermore, by Itô's lemma,

$$\frac{\mathrm{d}(c_t^i)^{-\gamma}}{(c_t^i)^{-\gamma}} = \left[-\gamma\mu_t^{c^i} + \frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2\right]\mathrm{d}t - \gamma\sigma_t^{c^i}\mathrm{d}Z_t.$$

Because $\xi_t^i = e^{-\rho^i t}u'(c_t^i) = e^{-\rho^i t}(c_t^i)^{-\gamma}$, Itô's product rule implies that

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -\rho^i\mathrm{d}t + \frac{\mathrm{d}(c_t^i)^{-\gamma}}{(c_t^i)^{-\gamma}} = \left[-\rho^i - \gamma\mu_t^{c^i} + \frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2\right]\mathrm{d}t - \gamma\sigma_t^{c^i}\mathrm{d}Z_t.$$

Recall the SDF process (4.2), and we now see that $\varsigma_t^i = \gamma\sigma_t^{c^i} = \gamma\sigma_t^{n^i}$ and

$$r_t = \rho^i + \gamma\mu_t^{c^i} - \frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2. \tag{6.3}$$

Consider a self-financing strategy that reinvests consisting of an agent's net worth with

[1] The value function for individuals i donates $V^i(n_t^i;\boldsymbol{\eta_t},K_t)$, where $(\boldsymbol{\eta_t},K_t)$ are state variables. For n-sector problem, $\boldsymbol{\eta_t}$ is a $n-1$ vector, while in this chapter $\boldsymbol{\eta_t}$ is a scalar η_t^e. For simplicity, later we use the notations $V_t^i = V^i(n_t^i;\boldsymbol{\eta_t},K_t), \omega_t^i = \omega^i(\boldsymbol{\eta_t},K_t), v_t^i = v^i(\boldsymbol{\eta_t})$.

consumption reinvested. By construction, the value of this strategy $p_t^{n^i}$ follows

$$\frac{\mathrm{d}p_t^{n^i}}{p_t^{n^i}} = \frac{\mathrm{d}n_t^i}{n_t^i} + \frac{c_t^i}{n_t^i}\mathrm{d}t.$$

The martingale approach tells us that $\xi_t^i p_t^{n^i}$ follows a martingale and the following asset pricing equation holds

$$\mu_t^{p^{n^i}} - r_t = \varsigma_t^i \sigma_t^{p^{n^i}} \iff \mu_t^{n^i} + \frac{c_t^i}{n_t^i} - r_t = \varsigma_t^i \sigma_t^{n^i} = \frac{(\varsigma_t^i)^2}{\gamma}.$$

The net worth then follows

$$\frac{\mathrm{d}n_t^i}{n_t^i} = \mu_t^{n^i}\mathrm{d}t + \sigma_t^{n^i}\mathrm{d}Z_t = \left[r_t + \frac{(\varsigma_t^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right]\mathrm{d}t + \frac{\varsigma_t^i}{\gamma}\mathrm{d}Z_t.$$

Since $\mu_t^{c^i} = \mu_t^{n^i}$, $\sigma_t^{c^i} = \sigma_t^{n^i}$, (6.3) implies

$$r_t = \rho^i + \gamma\left[r_t + \frac{(\varsigma_t^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right] - \frac{\gamma(1+\gamma)}{2}\frac{(\varsigma_t^i)^2}{\gamma^2} \implies \frac{c_t^i}{n_t^i} = \rho^i + \frac{\gamma-1}{\gamma}\left[r_t - \rho^i + \frac{(\varsigma_t^i)^2}{2\gamma}\right]. \tag{6.4}$$

This will turn out to be a useful relationship for the next chapter, but now we focus on more general investment opportunity processes.

For arbitrary opportunity processes ω_t^i, we still have $\varsigma_t^i = \gamma\sigma_t^{c^i}$ and that $\xi_t^i p_t^{n^i}$ follows a martingale. By Itô's product rule,

$$\frac{\mathrm{d}(\xi_t^i p_t^{n^i})}{\xi_t^i p_t^{n^i}} = \frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i} + \frac{c_t^i}{n_t^i}\mathrm{d}t.$$

Rewrite (6.2) as

$$(c_t^i)^{-\gamma} = \frac{1}{\rho^i}(\omega_t^i)^{1-\gamma}(n_t^i)^{-\gamma} \iff e^{\rho^i t}\underbrace{e^{-\rho^i t}(c_t^i)^{-\gamma}}_{\xi_t^i} n_t^i = \underbrace{\frac{1}{\rho^i}(\omega_t^i)^{1-\gamma}(n_t^i)^{1-\gamma}}_{(1-\gamma)V_t^i}. \tag{6.5}$$

Hence,

$$\frac{\mathrm{d}V_t^i}{V_t^i} = \frac{\mathrm{d}(e^{\rho^i t}\xi_t^i n_t^i)}{e^{\rho^i t}\xi_t^i n_t^i} = \left(\rho^i - \frac{c_t^i}{n_t^i}\right)\mathrm{d}t + \underbrace{\frac{\mathrm{d}(\xi_t^i p_t^{n^i})}{\xi_t^i p_t^{n^i}}}_{\text{Martingale}}. \tag{6.6}$$

Unfortunately, we can not use Itô's formula on V_t^i to get the drift of $\mathrm{d}V_t^i/V_t^i$, as $n_t^i(\eta_t)$ and $\omega_t^i(\eta_t)$ are not differentiable when $q_t(\eta_t)$ has a kink[2]. Instead, we can de-scale the value function with regard to K_t and define the "de-scaled value function" v_t^i:

$$V_t^i = \frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{\left(w_t^i n_t^i/K_t\right)^{1-\gamma}}{\rho^i}}_{v_t^i :=}\frac{K_t^{1-\gamma}}{1-\gamma}, \tag{6.7}$$

By such a de-scaling, we separate two state variables η_t^i and K_t[3], hence can work on them independently.

By Itô's product rule,

$$\frac{\mathrm{d}V_t^i}{V_t^i} = \frac{\mathrm{d}\left[v_t^i K_t^{1-\gamma}\right]}{v_t^i K_t^{1-\gamma}} = \left[\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i}\right]\mathrm{d}t + [\cdots]\mathrm{d}Z_t.$$

Recall (6.6), the drift of V_t^i equals

$$\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = \rho^i - \frac{c_t^i}{n_t^i}.$$

This gives us the following backward stochastic differential equation (BSDE)

$$\frac{\mathrm{d}v_t^i}{v_t^i} = \left[\rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)(\Phi(\iota_t)-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2 - (1-\gamma)\sigma\sigma_t^{v^i}\right]\mathrm{d}t + \sigma_t^{v^i}\mathrm{d}Z_t. \tag{6.8}$$

This is a BSDE that we can solve using standard numerical methods (Tourin, 2011). But before that, we need to study the evolution of the state η_t^i in order to pin down terms

[2]As we will see in section 6.4, this is indeed the case.

[3]We will see later v_i only depends on state variable η_t^i, and is twice differentiable in η_t^i. Besides, state variable K_t is easy to handle due to scale invariance.

like c_t^i/n_t^i and ι_t^i.

Price of risk ς_t^i. The value function (6.2) implies

$$\begin{aligned} V^i(n_t^i; \boldsymbol{\eta_t}, K_t) &= \frac{u\left(\omega^i(\boldsymbol{\eta_t}, K_t) n_t^i\right)}{\rho^i} \\ \Rightarrow \frac{\partial V^i(n_t^i; \boldsymbol{\eta_t}, K_t)}{\partial n_t^i} &= \frac{\left(\omega^i(\boldsymbol{\eta_t}, K_t)\right)^{1-\gamma}}{\rho^i}(n^i)^{-\gamma} = \underbrace{\frac{(\omega_t^i n_t^i/K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} \left(\frac{K_t}{n_t^i}\right)^{1-\gamma} (n_t^i)^{-\gamma} \end{aligned}$$

Applying the envelop condition $\frac{\partial V_t}{\partial n_t} = u'(c_t)$,

$$\frac{\partial V_t^i}{\partial n_t^i} = v_t^i \left(\frac{K_t}{n_t^i}\right)^{1-\gamma} (n_t^i)^{-\gamma} = (c_t^i)^{-\gamma} = \frac{\partial u(c_t^i)}{\partial c_t^i}.$$

In equilibrium $N_t^i = n_t^i$ and $C_t^i = c_t^i$, plugging in $N_t^i = \eta_t^i q_t K_t$, the condition ends up becoming

$$\frac{C_t^i}{K_t} = \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.9}$$

Applying Itô's quotient rule and comparing the volatility terms, we have[4]

$$\sigma_t^{c^i} - \sigma = \frac{1}{\gamma}\left(-\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q\right).$$

The prices of risk are then

$$\varsigma_t^i = \gamma \sigma_t^{c^i} = -\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q + \gamma\sigma. \tag{6.10}$$

[4]Note that $\sigma_t^K = \sigma$ because $K_t = \sum_i \kappa^i k_t^i$ and $\sigma_t^{k^i} = \sigma,\ \forall i$.

Consumption propensity C^i_t/N^i_t. Note that we can express C^i_t/N^i_t in terms of η_t, q_t and v^i_t. Plug in $K_t = N^i_t/\eta^i_t q_t$ and rewrite (6.9) as

$$\frac{C^i_t}{N^i_t} = \frac{c^i_t}{n^i_t} = \frac{(\eta^i_t q_t)^{1/\gamma-1}}{(v^i_t)^{1/\gamma}}. \tag{6.11}$$

On the aggregate level,

$$\frac{C_t}{N_t} = \sum_i \eta^i_t \frac{C^i_t}{N^i_t} = \frac{1}{q_t}\sum_i \left(\frac{\eta^i_t q_t}{v^i_t}\right)^{1/\gamma}. \tag{6.12}$$

6.1.1 Value function iteration

To apply the finite difference method, we *postulate* that $v^i_t = v^i(\eta^e_t, t)$ (Note in two-sector model we only use η^e as state variable). By Itô's formula, it follows

$$\frac{\mathrm{d}v^i_t}{v^i_t} = \frac{\partial_t v^i_t + (\eta^e_t \mu^{\eta^e}_t)\partial_\eta v^i_t + \frac{1}{2}(\eta^e_t\sigma^{\eta^e}_t)^2\partial_{\eta\eta}v^i_t}{v^i_t}\mathrm{d}t + \frac{(\eta^e_t\sigma^{\eta^e}_t)\partial_\eta v^i_t}{v^i_t}\mathrm{d}Z_t.$$

Comparing with the BSDE (6.8), we get the growth equation

$$\begin{aligned}
&\partial_t v^i_t + \left[\eta^e_t \mu^{\eta^e}_t\right]\partial_\eta v^i_t + \left[\frac{1}{2}\left(\eta^e_t\sigma^{\eta^e}_t\right)^2\right]\partial_{\eta\eta}v^i_t \\
&\quad = \left\{\rho^i - \frac{c^i_t}{n^i_t} - (1-\gamma)\left[(\Phi(\iota^i_t)-\delta) - \frac{1}{2}\gamma\sigma^2 + \sigma\left(\eta^e_t\sigma^{\eta^e}_t\right)\frac{\partial_\eta v^i_t}{v^i_t}\right]\right\} v^i_t,
\end{aligned} \tag{6.13}$$

where

$$\mu^{\eta^i}_t = (\varsigma^i_t - \sigma - \sigma^q_t)(\sigma^{\eta^i}_t + \sigma + \sigma^q_t) - \sum_{i'}\eta^{i'}_t(\varsigma^{i'}_t - \sigma - \sigma^q_t)(\sigma^{\eta^{i'}}_t + \sigma + \sigma^q_t) \tag{6.14}$$

$$- \left(\frac{C^i_t}{N^i_t} - \frac{C_t}{N_t}\right) - \rho^i_d \zeta^{\neg i} + \rho^{\neg i}_d \zeta^i \frac{N^{\neg i}_t}{N^i_t}, \tag{6.15}$$

$$\sigma^{\eta^i}_t = \frac{\chi^i_t - \eta^i_t}{\eta^i_t}(\sigma + \sigma^q_t). \tag{6.16}$$

In order to solve this PDE, we need to know all the terms in red. Luckily, we already have all the building blocks from the previous sections:

- Tobin's q gives us the investment rate ι_t^i:

$$\iota_t^i = \frac{1}{\phi}(q_t - 1). \tag{6.17}$$

- The price of risk ς_t^i is given by (6.10):

$$\varsigma_t^i = -\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q + \gamma\sigma \qquad \text{where} \qquad \sigma_t^{v^i} = \frac{(\eta_t^e \sigma_t^{\eta^e})\partial_\eta v_t^i}{v_t^i}. \tag{6.18}$$

- The amplification equation yields σ_t^q:

$$\sigma_t^q = \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q). \tag{6.19}$$

- The first-order conditions to the planner's problems gives χ_t^i, κ_t^i. It can be shown that the FOCs (4.8)-(4.9) are equivalent to the following[5]

$$\min\left\{\frac{a^e - a^h}{q_t} - \alpha\left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2, 1 - \kappa_t^e\right\} = 0, \tag{6.20}$$

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}. \tag{6.21}$$

- In (6.11) and (6.12), we have obtained the consumption ratios $c_t^i/n_t^i, C_t/N_t$ from the optimal consumption condition:

$$\frac{C_t^i}{N_t^i} = \frac{c_t^i}{n_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma - 1}}{(v_t^i)^{1/\gamma}}, \qquad \frac{C_t}{N_t} = \sum_i \eta_t^i \frac{C_t^i}{N_t^i} = \frac{1}{q_t}\sum_i \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.22}$$

[5]Proof will be added soon. For now, please see relevant parts of the lecture slides.

- Finally, goods market clearing jointly constraints q_t and κ_t^i:

$$\sum_i \kappa_t^i a^i - \iota_t = \sum_i \frac{C_t^i}{K_t} = \sum_i \left(\frac{\eta_t^i q_t}{v_t^i} \right)^{1/\gamma}. \tag{6.23}$$

In the following algorithm, we first guess two functions $v^e(\eta_t^e, T), v^h(\eta_t^e, T)$ and use them as *terminal conditions*. We then solve the PDE (6.13) *backwards* on a discretized time grid. In each step (time t), we solve for time-t equilibrium quantities as functions of η_t using the Inner Loop procedure, introduced above.

More specifically, the algorithm goes as following

1. Start by guessing two functions $v^e(\eta^e, T), v^h(\eta^e, T)$ over a grid of η^e
2. Loop over $t = \{T, T - \Delta t, \cdots, 0\}$ until changes in v^e-functions are small. In each step, do the following:
   (a) Compute $\partial_\eta v_t^i$ by first-order differences
   (b) Perform the Inner Loop procedure, using appropriate conditions for capital and risk allocation (6.20 and 6.21 instead of 4.15 and 4.16)
   (c) compute $\mu_t^{\eta^e}(\eta_t^e), \sigma_t^{\eta^e}(\eta_t^e), \mu_t^{v^i}(\eta_t^e), \sigma_t^{v^i}(\eta_t^e)$ using equations (6.15) and (6.16)
   (d) make time-step – back in time – and update the $v_t^i(t, \cdot)$ functions to $v_t^i(t - \Delta t, \cdot)$

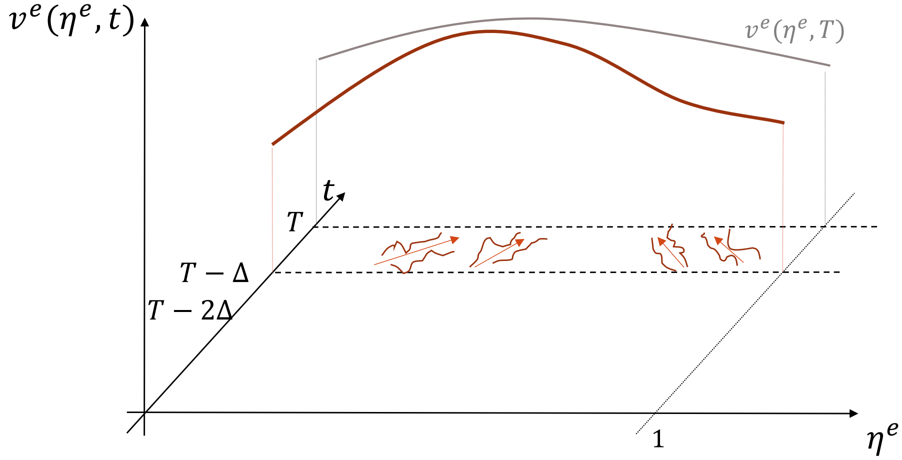

Figure 6.1: Visualization of the time step

6.2 Implementation in MATLAB

Following is main code executing the algorithm under the assumption of CRRA utility.

```
%% Parameters and grid
a_e = 0.11; a_h = 0.03;        % production rates
rho_0 = 0.04;                  % time preferences
rho_e_d = 0.01;rho_h_d = 0.01;% death rates
rho_e = rho_0 + rho_e_d;       % expert's discount rates
rho_h = rho_0 + rho_h_d;       % household's discount rates
zeta = 0.05;                   % probability of becoming an expert
delta = 0.05; sigma = 0.1;     % decay rate/volatility
phi = 10; alpha = 0.5;         % adjustment cost/equity constraint
dt = 100; tol = 1e-6;          % time step/convergence criterion
gamma = 2;                     % CRRA parameter

fsolveOptions = optimoptions('fsolve','Display','off','FunctionTolerance',1e-8);

N = 501;                          % grid size
eta = linspace(0.0001,0.999,N)'; % grid for \eta

%% Solution
deta = eta(2)-eta(1); % grid step

% Initial guess for the value functions
v_e = a_e^(-gamma)*eta.^(1-gamma);
v_h = a_e^(-gamma)*(1-eta).^(1-gamma);

```

```
vp_e = [v_e(2) - v_e(1); v_e(2:N) - v_e(1:N-1)]./deta;   % \partial{v_e}
vp_h = [v_h(2) - v_h(1); v_h(2:N) - v_h(1:N-1)]./deta;   % \partial{v_h}

ev_e = (eta./v_e).^(1/gamma);       % auxiliary variable for goods-market clearing
ev_h = ((1 - eta)./v_h).^(1/gamma); % auxiliary variable for goods-market clearing

%% Iteration

for i = 1:10000
    ev = ev_e + ev_h; % auxiliary variable for goods-market clearing
    vpv = -vp_e./v_e + vp_h./v_h + 1./(eta.*(1 - eta)); % auxiliary variable for kappa FOC

    % Solve for q(0), approximating value functions at the left boundary by
    % the values at the first grid point. Initial guess: q(0) under
    % log-utility
    q0 = fsolve(@(x) a_h - (x-1)/phi - x.^(1/gamma)*ev(1), (1 + a_h*phi)/(1 + rho_h*phi), fsolveOptions);

    % Inner loop
    [Q, SSQ, Kappa, Chi, Iota] = inner_loop_crra(eta, q0, ev, vpv, a_e, a_h, sigma, phi, alpha, gamma);

    S = (Chi - eta).*SSQ; % \sigma_{\eta^e} -- arithmetic volatility of \eta^e
    Phi = log(Q)/phi;     % Investment (net of costs)

    Sg_e = S./eta;      % \sigma^{\eta^e} -- geometric volatility of \eta^e
    Sg_h = -S./(1-eta); % \sigma^{\eta^h} -- geometric volatility of \eta^h

    vp_e = [v_e(2) - v_e(1); v_e(2:N) - v_e(1:N-1)]./deta;   % \partial{v_e}
    vp_h = [v_h(2) - v_h(1); v_h(2:N) - v_h(1:N-1)]./deta;   % \partial{v_h}

    ev_e = (eta./v_e).^(1/gamma);       % auxiliary variable for goods-market clearing
    ev_h = ((1 - eta)./v_h).^(1/gamma); % auxiliary variable for goods-market clearing

    Sv_e = S.*vp_e./v_e; % \sigma^{v^e}
    Sv_h = S.*vp_h./v_h; % \sigma^{v^h}

    VarS_e = -Sv_e + Sg_e + SSQ - (1-gamma)*sigma; % \varsigma^e -- experts' price of risk
    VarS_h = -Sv_h + Sg_h + SSQ - (1-gamma)*sigma; % \varsigma^h -- households' price of risk

    CN_e = ev_e.*Q.^(1/gamma - 1)./eta;       % experts' consumption-to-networth ratio
    CN_h = ev_h.*Q.^(1/gamma - 1)./(1 - eta); % households' consumption-to-networth ratio

    MU = eta .* (1-eta) .* ((VarS_e - SSQ).*(Sg_e + SSQ) -  (VarS_h - SSQ).*(Sg_h + SSQ) - (CN_e - CN_h) + (rho_h_d.*zeta.*(1-eta) - rho_e_d.*(1-zeta).*eta) ./ (eta.*(1-eta))); % \mu_{\eta^e} -- arithmetic drift of \eta^e

    S([1, N]) = 0;          % ensures volatiliuty is zero at the boundaries
```

```
    MU(1) = max(MU(1), 0);  % ensures drift at the left boundary is non-negative
    MU(N) = min(MU(N), 0);  % ensures drift at the right boundary is non-positive

    u_e = (CN_e + (1 - gamma)*(Phi - delta - gamma*sigma^2/2 + Sv_e*sigma)).*v_e; % flow term in experts' HJB
    u_h = (CN_h + (1 - gamma)*(Phi - delta - gamma*sigma^2/2 + Sv_h*sigma)).*v_h; % flow term in households' HJB

    v_e1 = update_v(v_e, eta, rho_e, u_e, MU, S, dt); % update experts' value function
    v_h1 = update_v(v_h, eta, rho_h, u_h, MU, S, dt); % update households' value function

    d = max(abs(v_e1 - v_e) + abs(v_h1 - v_h))/dt; % convergence
    if d <= tol
        break
    end

    v_e = v_e1;
    v_h = v_h1;
end
```

The inner loop is implemented in the function inner_loop_crra.m, which is identical to the one under log-utility, except for a couple of lines in the construction of the $F(\cdot)$ function for the Newton's method and its Jacobian J.

```
function [Q, SSQ, Kappa, Chi, Iota] = inner_loop_crra(eta, q0, ev, vpv, a_e, a_h, sigma, phi, alpha, gamma)

N = length(eta);
deta = [eta(1); diff(eta)]; % imposes the correct grid step for numerical derivative at \eta^e = 0

% variables
Q = ones(N,1);    % price of capital q
SSQ = zeros(N,1);   % \sigma + \sigma^q
Kappa = zeros(N,1); % capital fraction of experts \kappa

% Initiate the loop
kappa = 0; q_old = q0; q = q0; ssq = sigma;

% Iterate over eta
% At each step apply Newton's method to F(z) = 0 where z = [q, kappa, ssq]'
% Use chi = alpha*kappa
for i = 1:N
    % Compute F(z_{n-1})
    F = [kappa*(a_e - a_h) + a_h - (q-1)/phi - q^(1/gamma)*ev(i); % Compare with log-utility: C/N is no longer \rho
         ssq*(q - (q - q_old)/deta(i) * (alpha*kappa - eta(i))) - sigma*q;
         a_e - a_h -  q*alpha*(alpha*kappa - eta(i))*ssq^2*vpv(i)];  % Compare with log-utility: price of risk is no longer \sigma^n

```

```
    % Construct Jacobian J^{n-1}
    J = zeros(3,3);
    J(1,:) = [-1/phi - q^(1/gamma - 1)/gamma * ev(i), a_e - a_h, 0];
    J(2,:) = [ssq*(1 - (alpha*kappa - eta(i))/deta(i)) - sigma, ...
          -ssq*(q-q_old)/deta(i)*alpha, q - (q-q_old)/deta(i)*(alpha*kappa - eta(i))];
    J(3,:) = [-alpha*(alpha*kappa - eta(i))*ssq^2*vpv(i), ...
          -q*alpha^2*ssq^2*vpv(i), -2*q*alpha*(alpha*kappa - eta(i))*ssq*vpv(i)];

    % Iterate, obtain z_{n}
    z = [q, kappa, ssq]' - J\F;

    % If the new kappa is larger than 1, break
    if z(2) >= 1
        break;
    end

    % Update variables
    q = z(1); kappa = z(2); ssq = z(3);

    % save results
    Q(i) = q; Kappa(i) = kappa; SSQ(i) = ssq;
    q_old = q;
end

% Set kappa = 1, use chi = max(alpha, eta) and compute the rest
n1 = i;
for i = n1:N
    F = a_e - (q-1)/phi - q^(1/gamma)*ev(i); % Compare with log-utility: C/N is no longer \rho
    J = -1/phi - q^(1/gamma - 1)/gamma * ev(i);
    q = q - F/J;
    qp = (q - q_old)/deta(i);

    Q(i) =  q; Kappa(i) = 1;
    SSQ(i) = sigma/(1 - (max(alpha, eta(i)) - eta(i))*qp/q);
    q_old = q;
end

% Compute chi, iota
Chi  = max(alpha*Kappa, eta);
Iota = (Q - 1)/phi;
```

The time step is implemented in the function `update_v.m`:

```
function [v] = update_v(v, x, rho, u, mu, sig, dt)

N = length(x); % Grid size

% Constrict the M matrix
M = buildM(x, mu, sig);

B = (1 + dt*rho)*speye(N) - dt*M;
v = B\(u*dt + v); % update v
```

with matrix M constructed in function `biuldM.m`:

```
function [M] = buildM(x, mu, sig)
% Construct the M matrix using three vectors (dM, dD, dU), corresponding to
% the main diagonal, the diagonal below the main one, the diagonal above
% the main one:
% M = [dM(1) dU(1)     0      0  ....      0       0      0
%       dD(2) dM(2) dU(2)     0  ....      0       0      0
%          0  dD(3) dM(3) dU(3)  ....      0       0      0
%       ....  ....  ....  ....  ....    ....    ....   ....
%          0     0     0     0  .... dD(N-1) dM(N-1) dU(N-1)
%          0     0     0     0  ....      0    dD(N)   dM(N)]
% =========================================================================
% Input:
% x - grid (N-by-1), equally spaced
% mu - drift term (N-by-1)
% sig - volatility term (N-by-1)

N = length(x); % Grid size
dx = x(2)-x(1); % Grid step
dx2 = dx^2; % Grid step squared

% Constrict the diagonals
dD = -min(mu, 0)/dx + sig.^2/(2*dx2);
dM = -max(mu, 0)/dx + min(mu, 0)/dx - sig.^2/dx2;
dU = max(mu, 0)/dx + sig.^2/(2*dx2);

% Construct the M matrix
M = spdiags([dD dM dU],[1 0 -1],N,N)';
```

To ensure numerical stability, it is important to:

- Use implicit method in step 2.(b)
- Use an upwind scheme when taking derivatives in step (a):

$$\partial_\eta v^i_t(t,\eta^i(n)) = \begin{cases} \dfrac{v^i(t,\eta^i(n+1)) - v^i(t,\eta^i(n))}{\eta^i(n+1)-\eta^i(n)} & \text{if } \mu^{\eta^i}_t \eta^i_t > 0 \\[2ex] \dfrac{v^i(t,\eta^i(n)) - v^i(t,\eta^i(n-1))}{\eta^i(n)-\eta^i(n-1)} & \text{if } \mu^{\eta^i}_t \eta^i_t < 0 \end{cases} \tag{6.24}$$

A good initial guess is usually crucial to the success of a numerical procedure. Here are some common ways of choosing initial guesses for v^i:

- Take an arbitrary constant, e.g. a vector of ones. It is the easiest way, but doesn't always work / may take a long time to converge.

- Take a specific constant, namely the value at the boundary steady state ($\eta^e = 0$ or $\eta^e = 1$) where only one type exists (if that is a valid equilibrium) – this is typically also very easy.
- Assume there are no financial contracts and compute for each η^e the autarky value of the agent types, when the initial wealth distribution is described by η^e.
- Along the same lines, assume complete markets and compute first-best utility as a function of η^e (this certainly bounds utility from above).
- If the log utility model is simple to solve, solve it first. Use the consumption path of agents in that model, but compute the implied CRRA utility.
- If you have solved the model for different parameters that are "close", use that solution as an initial guess.

6.3 Epstein-Zin Preferences

The Epstein-Zin utility function allows to specify the elasticity of intertemporal substitution (EIS) and the degree of relative risk aversion (RRA) separately. It is defined in a recursive way:

$$U_t = \mathbb{E}_t \left[\int_t^\infty f(c_s, U_s) ds \right]$$

$$f(c, U) = \frac{1-\gamma}{1-\psi^{-1}} \rho U \left(\left(\frac{c}{((1-\gamma)\rho U)^{1/(1-\gamma)}} \right)^{1-\psi^{-1}} - 1 \right)$$

with EIS ψ and RRA γ. Setting $\gamma = \psi^{-1}$ recovers the CRRA utility function $U_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho t} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right]$. The model solution follows the same steps as under CRRA utility, with some differences that we highlight below.

1. Naturally, consumption-to-wealth ratio (6.2) is now given by:

$$\frac{c_t^i}{n_t^i} = (\rho^i)^\psi (\omega_t^i)^{1-\psi}$$

2. Stochastic discount factor now satisfies (compare with (6.5)):

$$e^{-\int_0^t \frac{\partial f^i(c_s,V_s)}{\partial U} ds} \xi_t^i n_t^i = (1-\gamma) V_t^i$$

and hence, analogously to (6.6):

$$\frac{dV_t^i}{V_t^i} = \left(-\frac{\partial f^i(c_s, V_s)}{\partial U} - \frac{c_t^i}{n_t^i} \right) dt + \text{martingale}$$

with $\frac{\partial f^i(c,U)}{\partial U} = \frac{\rho^i}{1-\psi^{-1}} \left[(\psi^{-1} - \gamma) \left(\frac{c}{((1-\gamma)\rho^i U)^{1/(1-\gamma)}} \right)^{1-\psi^{-1}} - (1-\gamma) \right]$. This affects the BSDE for v_t^i (6.8) by changing its drift $\mu_t^{v^i}$.

Finally, a particularly useful special case is when IES = 1 (as under log-utility), which implies:

$$\frac{c_t^i}{n_t^i} = \rho^i$$
$$f(c,U) = \rho U \left[(1-\gamma) \log c - \log((1-\gamma)\rho U) \right]$$
$$\frac{\partial f(c,U)}{\partial U} = \rho \left[(1-\gamma) \log c - \log((1-\gamma)\rho U) - 1 \right]$$

6.4 Numerical Results

In this section, we demonstrate the solutions generated by the code in Section 6.2 and discuss their implications. We use CRRA utility and set the baseline parameters are as follows.

$\rho_0^{e,h}$	$\rho_d^{e,h}$	ζ^e	a^e	a^h	δ	σ	α	γ	ϕ
0.04	0.01	0.05	0.11	0.03	0.05	0.10	0.50	2	10

Figure 6.2 illustrates the equilibrium with baseline parameter values. Note that $q(\eta_t^e)$ indeed has a kink, which marks the boundry between the cirsis region near $\eta^e = 0$ and the normal region near $\eta^e = 1$. In the crisis region, $\kappa^e < 1$, and households hold

some capital, while in the normal region, experts hold all capital in the economy. Unlike the model under log utility in Chapter 4, the price of capital under the CRRA utility is not flat in the normal region. This difference is not due to risk aversion, but rather the elasticity of intertemporal substitution (EIS) different from 1. The consumption to wealth ratio is dependent on investment opportunities. One can see the role of the EIS if one generalizes the CRRA utility functions to EZ with EIS as we outline in Section 6.3.

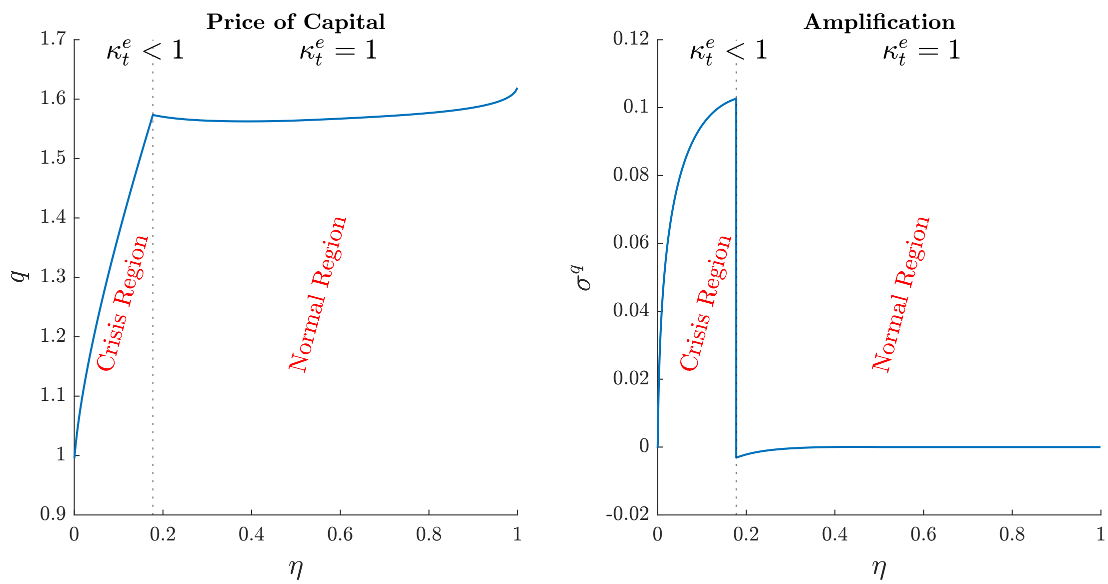

Figure 6.2: Equilibrium for the baseline set of parameters (η should read η^e)

6.5 Exercises

6.5.1 He and Krishnamurthy (2013)

The goal of this problem is to characterize equilibria in the model of He-Krishnamurthy ("Intermediary Asset Pricing") and to use the iterative method to compute equilibria. There are two agent types: experts and households. Households have log utility, while experts have CRRA utility with relative risk aversion γ, and both with discount rate ρ. New households are born continuously, and the newborn receive labor income at rate lK_t.

Aggregate capital follows the law of motion

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Capital produces dividend of aK_t. The price of capital per unit is denoted by q_t and follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Only experts can hold capital, and they can finance capital by borrowing through risk-free debt and by issuing equity to households, but they must retain fraction of at least $\underline{\chi} = 1/(1+m)$ of risk.

(a) Write down the expression for $\mathrm{d}r_t^K$ for the return on capital.

Experts make optimal consumption and portfolio decisions: they choose how much to borrow and how much outside equity to issue (up to fraction $1-\underline{\chi}$) to buy capital. Denote the required risk premium of experts by ς_t^e and recall that $\varsigma_t^e = -\gamma \sigma_t^{Ce}$, where σ_t^{Ce} is the volatility of aggregate consumption of experts. Denote the value function of a representative expert by

$$v_t^e \frac{K_t^{1-\gamma}}{1-\gamma}.$$

(b) Write down the law of motion of aggregate net worth of experts N_t^e as a function of the risk-free rate r_t, the experts' equity share $\underline{\chi}_t^e$, the experts' net worth share η_t^e, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' risk premium ς_t^e and process v_t^e. To write the law of motion of N_t^e, you need to express the experts' consumption rate C_t^e/N_t^e as a function of η_t^e, q_t and v_t^e.

(c) He and Krishnamurthy assume that inside and outside equity of experts earn the same returns. Thus, the experts' equity held by households earns the risk premium of ς_t^e, even though households' required risk premium is higher. Under this assumption, write down the law of motion of world wealth $q_t K_t$, as a function of the risk-free rate r_t, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' risk premium ς_t^e and output parameter a.

(d) From your answers to parts (b) and (c), derive the law of motion of the experts' wealth share $\eta_t = N_t^e/(q_t K_t)$.

(e) Write down the market-clearing condition for output. Hint: Recall that total world output is $(a+l)K_t$, including dividend and labor income of newborn households.

Next, you should determine the size of the "constrained region" where $\chi_t^e = \underline{\chi}$ and the size of the unconstrained region where $\chi_t^e > \underline{\chi}$. To do that, you should use the following assumptions of He and Krishnamurthy. Assume that fraction λ of households (i.e. the net worth share of these households is $(1-\eta_t^e)\lambda$) are "debt" households who can only hold the risk-free asset. Fraction $1-\lambda$ are "equity" households who can hold outside equity of experts and the risk-free asset. He and Krishnamurthy furthermore assume that equity households cannot use leverage, i.e. the risk of their net worth can be at most equal to the risk of experts' net worth (who hold their own inside equity). Assume (you can verify this later), that it is this constraint that determines the amount of equity that experts can issue.

(f) Derive the value of χ_t^e as a function of η_t^e implied by the constraint that equity households cannot use leverage.

The goal of the next questions is to formulate a procedure to compute equilibria using Matlab, using the value function iteration in section 6.2. You should use Matlab function `payoff_policy_growth.m` to perform the "time step" of the iterative procedure.

(g) Formulate a procedure for the static step. That is, suppose you are given function $v^e(t, \eta^e)$ for all η^e at time t. Find the price of capital $q(t, \eta^e)$ for all η^e at time t. Then, given this function, derive the law of motion of η^e. Provide an expression for $\mu_t^{v^e}$.

(h) Formulate a procedure for the time step. That is, for the function

```
F = payoff_policy_growth(X, R, MU, S, G, V, lambda0),
```

what values of `X, R, MU, S, G, V, lambda0` should you use?

(i) Program the iterative procedure using the terminal condition $v^e(T,\eta) = a^{-\gamma}(\eta^e)^{1-\gamma}$. Use $N = 1000$. Compute an example for the parameters of He and Krishnamurthy, $\rho = 0.04$, $g = 0.02$, $m = 4$, $a = 1$, $l = 1.84$, $\sigma = 0.09$, $\gamma = 2$ and $\lambda = 0.6$. (See Table 2 of HK - for these parameters you should be able to get convergence by setting lambda0 for payoff_policy_growth aggressively to 0.9).

Plot, as a function of η^e, the price of capital q, the risk-free rate r_t, the drift and volatility of $\eta_t\eta$ (i.e. $\sigma_t^{\eta^e}\eta^e$ and $\mu_t^{\eta^e}\eta^e$), the fraction of equity χ_t^e held by experts and the experts' consumption rate C_t^e/N_t^e.

(j) Replicate Figure 2 from He and Krishnamurthy, where the vertical axis displays the risk premium for capital, i.e. $\varsigma_t^e(\sigma + \sigma_t^q)$.

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Chapter 7

A Model with Jumps

In Chapter 4, we studied a benchmark model with financial frictions and endogenous risk dynamics. Despite having highly non-linear dynamics, the model is driven by Brownian shocks, and the paths of all variables are *continuous*. Idiosyncratic death or type switching jumps (with log-utility) do not lead to jumps in endogenous prices and state variables.

Nevertheless, there is a long tradition of modeling both unanticipated and anticipated jumps in macro-finance models. Some notable examples include models of bank runs (Diamond and Dybvig, 1983), liquidity spirals (Brunnermeier and Pedersen, 2009), sudden stops (Calvo, 1998; Mendoza, 2010), currency attacks (Obstfeld, 1996; Morris and Shin, 1998), twin crises (Kaminsky and Reinhart, 1999), and the loss of safe asset status.

In this chapter, we illustrate how to incorporate self-fulfilling jumps into the model outlined in Chapter 4, based on a simplified version of Mendo (2020).

7.1 Jump Processes

Previously, we focused on Itô processes in the form

$$dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t,$$

where the Brownian "shocks" $\mathrm{d}Z_t$ are i.i.d. and small, such that time paths are continuous. For non-normal shocks within dt one needs discontinuities. In this chapter, we allow for discontinuities by considering a more general class of processes with i.i.d. increments: Levy processes.

The Levy-Itô decomposition states that any Levy process L_t can be additively decomposed into three independent components: a linear time drift, a scaled Brownian motion and a Levy jump process, that is

$$\mathrm{d}L_t = a\mathrm{d}t + b\mathrm{d}Z_t + \mathrm{d}J_t,$$

where a, b are constants, $\mathrm{d}Z_t$ is a Brownian motion and $\mathrm{d}J_t$ is a Levy jump process. Processes driven by Levy-noise therefore look formally like Itô processes with an additional jump component

$$\mathrm{d}X_t = \mu_t^X X_t \mathrm{d}t + \sigma_t^X X_t \mathrm{d}Z_t + j_t^X X_{t-} \mathrm{d}J_t.$$

There is a fairly rich class of Levy jump processes $\mathrm{d}J_t$. Here we restrict our attention to Poisson processes.[1] Consider a Poisson process with intensity (or arrival rate) $\lambda > 0$: J_t takes only values in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and increments $J_{t+\Delta t} - J_t$ are Poisson distributed with parameter $\lambda \Delta t$. The following are some important properties of this Poisson process:

- J_t is weakly increasing. That is, J_t is locally constant or has a jump of size 1.
- Conditional on $J_t = n$, the random time to the next jump, $\tau := \inf\{s \geq 0 \mid J_{t+s} > n\}$, is exponentially distributed with parameter λ (i.e. the expected time to the next jump is constant and given by $1/\lambda$).
- The stochastic integral with respect to a Poisson process simply sums the values

[1] Note: this is not as restrictive as it may seem: general Levy jump processes can be written as integral with respect to Poisson random measures, a generalization of sums of integrals with respect to Poisson processes.

of the integrand at the jump times:

$$\int_0^T a_t \mathrm{d}J_t = \sum_{n=1}^{J_T} a_{\tau_n}$$

where τ_n is the time at which J_t jumps from $n-1$ to n.

To capture time-varying macroeconomic dynamics, we will allow for a slightly more general process, sometimes called a Cox process, where λ does not need to be a constant, but can be time- and state-dependent (λ_t).

It is important to note that, while a Brownian motion $\mathrm{d}Z_t$ is a martingale, a jump process J_t is not — it is expected to drift up. To get a martingale, we have to "compensate" the jump process by its intensity. In other words, $J_t - \int_0^t \lambda_s \mathrm{d}s$ is a martingale.[2]

The following Itô formulae holds for jump diffusions driven by Brownian and Poisson noise.

Consider geometric jump diffusions X_t, Y_t

$$\frac{\mathrm{d}X_t}{X_{t-}} = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + j_t^X \mathrm{d}J_t, \qquad \frac{\mathrm{d}Y_t}{Y_{t-}} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t + j_t^Y \mathrm{d}J_t,$$

where $\mathrm{d}Z_t$ is a standard Brownian motion and $\mathrm{d}J_t$ is a Poisson process. The following results hold.

Itô's lemma:

$$\mathrm{d}f(X_t) = \left[f'(X_t)(\mu_t^X X_t) + \frac{1}{2}f''(X_t)(\sigma_t^X X_t)^2\right]\mathrm{d}t + \\ f'(X_t)(\sigma_t^X X_t)\mathrm{d}Z_t + (f((1+j_t^X)X_{t-}) - f(X_{t-}))\mathrm{d}J_t$$

Itô's power rule:

[2] More generally, if $X_t = \int_0^t a_s \mathrm{d}J_s$ (and a is "predictable", i.e. a_t uses information only up to right before time t, but does not contain information about potential jumps at time t), then $X_t - \int_0^t a_s \lambda_s ds$ is a martingale.

$$\frac{\mathrm{d}(X_t^\gamma)}{X_{t-}^\gamma} = (\gamma\mu_t^X + \gamma(\gamma-1)(\sigma_t^X)^2)\mathrm{d}t + \gamma\sigma_t^X \mathrm{d}Z_t + ((1+j_t^X)^\gamma - 1)\mathrm{d}J_t.$$

Itô's product rule:

$$\frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X\sigma_t^Y)\mathrm{d}t + (\sigma_t^X + \sigma_t^Y)\mathrm{d}Z_t + (j_t^X + j_t^Y + j_t^X j_t^Y)\mathrm{d}J_t.$$

Itô's quotient rule:

$$\frac{\mathrm{d}(X_t/Y_t)}{X_t/Y_t} = \left[\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right]\mathrm{d}t + (\sigma_t^X - \sigma_t^Y)\mathrm{d}Z_t + \frac{j_t^X - j_t^Y}{1+j_t^Y}\mathrm{d}J_t.$$

Notice that these equations are the same as our earlier rules for geometric Itô processes, but with new terms for the jump process. The new terms in the power, product and quotient rules can be expressed more simply as:

$$1 + j_t^{(X^\gamma)} = (1+j_t^X)^\gamma$$

$$1 + j_t^{XY} = (1+j_t^X)(1+j_t^Y)$$

$$1 + j_t^{X/Y} = \frac{1+j_t^X}{1+j_t^Y}$$

7.2 Model Setup

The model setup follows the structure outlined in Section 4.1 but with CRRA utility and without agents' death. The innovation comes from our postulated price process, which will now include a jump term. That is, the model has the same primitives as before, but we now allow for *self-fulfilling* / *sunspot* jumps.

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions.

Experts. Experts have a CRS technology $y_t^e = a^e k_t^e$. Denote their consumption and investment rate by c_t^e, ι_t^e. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = (\Phi(\iota_t^e) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Still, we have only aggregate risk in the environment. Experts have a CRRA utility function and they each maximize

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} \mathrm{d}t \right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

We let households hold capital to capture fire-sales. Households are more patient than the experts, i.e., $\rho^h \leq \rho^e$. As we have discussed in section 3, assuming that households are more patient than the experts is a modeling trick to ensure that the experts do not hold all the capital in the long run. The households maximize

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} \mathrm{d}t \right].$$

Financial Friction. The financial friction is due to incomplete markets. Although experts are allowed to issue equity, they must hold at least α fraction of their risk. The balance sheets of the two sectors are as following:

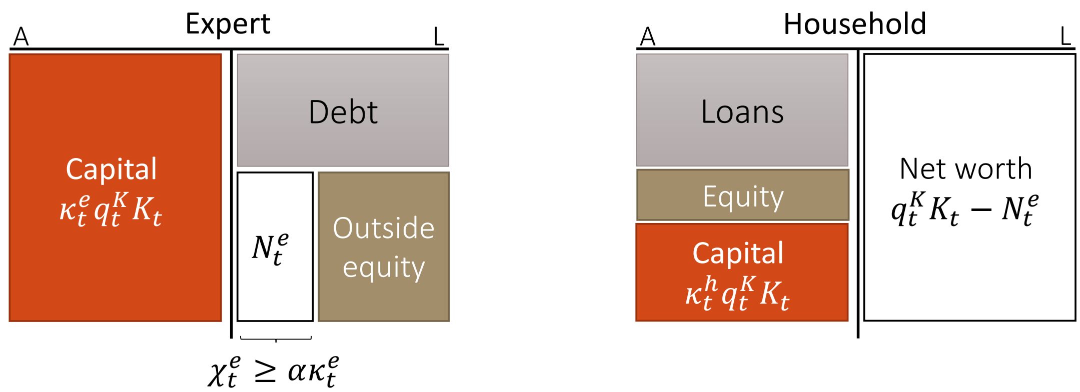

Figure 7.1: Balance sheets of experts and households

The skin-in-the-game constraint can be expressed as $\chi_t^e \geq \alpha\kappa_t^e$, where χ_t^e is the fraction of risk held by experts and κ_t^e is the fraction of capital held by experts.

Unanticipated Run on Experts. Now we investigate the risk of a sudden and unanticipated funding withdrawal, which corresponds to a *bank run* or *sudden stop* on experts. We ask: Can an unanticipated withdrawal of *all* external funding to experts be self-fulfilling?

An *unanticipated crash* occurs if, at some instant, the experts lose their external financing. In terms of the experts' share of aggregate net worth, this leads to the instantaneous transition:

$$\eta_t^e \longrightarrow 0.$$

Once that jump happens, experts are effectively bankrupt and can no longer operate. In normal times (i.e., *absent a run*), the model solution evolves continuously and does not jump to $\eta_t^e = 0$ unless there is a large, coordinated withdrawal. Thus, the event $\eta_t^e \to 0$ reflects a *self-fulfilling* or *sunspot* run rather than a typical shock.

To see when a price drop is sufficient to wipe out experts, note that the model determines the market price of capital $q\left(\eta_t^e\right)$ based on the experts' net worth share. If the price suddenly falls to $q(0)$ when experts lose funding (a "fire-sale" price), then experts' net worth will be fully destroyed if

$$\Big(q(\eta_t^e) - q(0)\Big)\ \underbrace{\Big(\theta_t^{e,K} + \theta_t^{e,OE}\Big)}_{\chi_t^e}\,\eta_t^e\, K_t \quad \geq \quad \eta_t^e\, q(\eta_t^e)\, K_t.$$

Equivalently, one can write

$$q(\eta_t^e)\left(1 \;-\; \frac{\eta_t^e}{\chi^e(\eta_t^e)}\right) \;\geq\; q(0) \quad \Longleftrightarrow \quad q(\eta_t^e)\left(1 \;-\; \frac{1}{\theta_t^{e,K} \;+\; \theta_t^{e,OE}}\right) \;\geq\; q(0),$$

where $\chi^e(\eta_t^e)$ is the fraction of total risk borne by the experts. When leverage and exposure (i.e., $\theta_t^{e,K} + \theta_t^{e,OE}$) are high, the price drop from $q(\eta_t^e)$ to $q(0)$ is large enough to eliminate all expert equity.

There are two types of runs:

1. **Funding supply run (households withdraw).** Households/depositors suddenly cut off credit to experts, causing forced liquidation and a crash in q.
2. **Funding demand run (experts fire-sell).** Even if credit remains available, experts may collectively choose to liquidate and repay debt preemptively, driving q down.

Both cases lead to the same outcome: a sudden jump in η_t^e to zero. The distinction lies in who triggers the liquidation (households or experts themselves).

Vulnerability Region The unanticipated crash leads to a *vulnerability region* for the experts' net worth share η_t^e:

- The price of capital $q\left(\eta_t^e\right)$ is *high* (not very low η_t^e).
- The experts hold a *high* fraction of risk/leverage (not very high η_t^e).

In such states, everyone fears that if others withdraw funding, the resulting price drop will annihilate experts' net worth. That belief can become self-fulfilling, causing the run to occur in equilibrium. Post-run, we set $\eta_t^e = 0$ permanently: the expert sector vanishes, and any residual capital is owned by households or newcomers at $q(0)$.

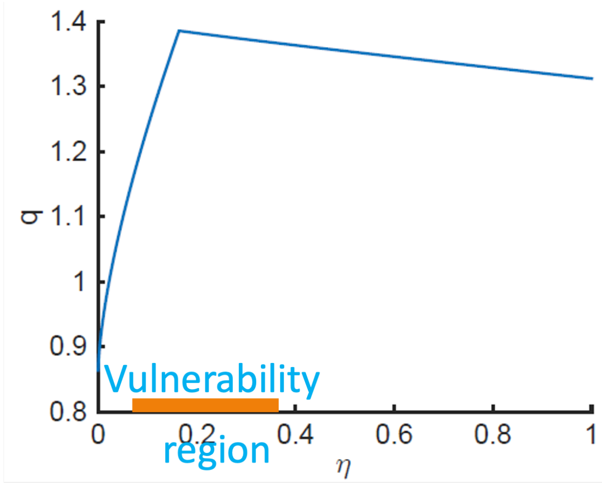

Figure 7.2: Vulnerability region

For numerical solutions, one cannot impose standard boundary conditions of the form $\mu^\eta(0) \geq 0$ or $\sigma^\eta(0) = 0$ because once $\eta^e = 0$, the experts cease to exist and the state is an *absorbing boundary*. The solution algorithm must explicitly allow for the possibility of a jump to zero and the absorbing state thereafter.

Model Advantage. One appealing feature of this framework is that, in the event of a run, the capital price collapses to a single, well-defined value, denoted by $q(0)$. Because experts are forced out of the market at that moment, the equilibrium price afterward is set entirely by the remaining sector (households). This "fire-sale" or "lowest possible" price highlights how a run can effectively *zero out* the experts' net worth, making $\eta_t^e = 0$ an absorbing state.

Modeling Challenges. While the model's ability to pin down a single post-run price $q(0)$ is tractable, it also gives rise to certain challenges. Several of these are discussed in Mendo (2020). We summarize three key issues:

First, experts are wiped out forever. Once the experts' net worth is destroyed by a jump, the baseline model does not let them rebuild. In an OLG structure, all agents face a Poisson death hazard ρ^d; a fraction ζ of newborn agents are designated as experts, so that some expert capacity is gradually reintroduced. Without such an OLG setup, the system simply continues without any expert sector once $\eta^e = 0$.

Second, anticipated runs lead to infinite marginal utility. If a run to $\eta_t^e = 0$ is anticipated, experts effectively foresee a state of infinitely high marginal utility (because losing all wealth is catastrophic). A common device is to impose a transfer, τK, that forces experts into bankruptcy after the run, thereby keeping their utility from blowing up and also ensuring that experts are definitively removed.

Third, bifurcation of experts into defaulters vs. survivors. Large jumps and non-convex default decisions can cause some experts to default while others do not, creating a continuum of outcomes. One modeling trick is to introduce a strong relative-performance penalty so that, if a run occurs, *all* experts default in unison. This prevents partial or selective default, which can greatly simplify the solution.

7.3 Solution Method

7.3.0 Postulate aggregates, price processes and obtain return processes

We introduce jumps by *postulating* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t + j_t^q \mathrm{d}J_t.$$

As before, we can calculate the return rate to capital for both sectors, $r_t^{i,K}(\iota_t^i)$, using Itô's product rule to calculate the capital gains rate (in the absence of purchases or sales). The process is similar to (4.1), but with a new jump term

$$\mathrm{d}r_t^{i,K}(\iota_t^i) = \Bigg[\underbrace{\frac{a^i - \iota_t^i}{q_t}}_{\text{Dividend yield}} + \underbrace{\Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q}_{\mathbb{E}[\text{Capital gain rate}]} = \frac{\mathrm{d}(q_t k_t)}{q_t k_t} \Bigg] \mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t + j_t^q \mathrm{d}J_t. \tag{4.1'}$$

Similarly, the return on defaultable debt is

$$\mathrm{d}r_t^D = r_t \mathrm{d}t + j_t^{r^D} \mathrm{d}J_t.$$

Note that $j_t^{r^D}$ only reflects the default in debt and not the overall change in debt holdings at the time of a jump. We then *postulate* that the SDF ($\xi_t^i = e^{-\rho^i t} u'(c_t^i)$) follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t^{F,i}\mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t - \nu_t^i(\mathrm{d}J_t - \lambda \mathrm{d}t), \tag{4.2'}$$

where $r_t^{F,i}$ is the (shadow) risk-free rate, ς_t^i is the price of Brownian risk, ν_t^i is the price of jump risk and $\lambda \mathrm{d}t$ is the Sunspot arrival rate (which, as discussed above, ensures that the jump process J_t is a martingale). Note that in contrast to Chapter 4, the risk-free rate now depends on i and may vary across agents in the model. This is because the introduction of jumps means that the debt traded in the model is no longer risk-free.

7.3.1 For given SDF processes, derive individual equilibrium conditions

Optimal investment ι. As before, the choice of investment rate is a static and time-separable problem. An agent chooses ι_t^i to maximize her return $r_t^{i,K}(\iota_t^i)$. The first-order condition yields the Tobin's q equation

$$\frac{1}{q_t} = \Phi'(\iota_t^i).$$

This choice applies to all agents with the special functional form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, $\phi\iota_t = q_t - 1$.

Goods market clearing. The goods market clearing condition

$$(A(\kappa) - \iota_t)K_t = \sum_i C_t^i, \tag{7.1}$$

which is the same as condition (4.14) in Chapter 4.

Asset and risk allocation using the martingale approach. To derive the optimal portfolio choice, we can again use the martingale approach, with a slight modification to incorporate the newly added jumps.

Martingale approach with jumps.

Consider a portfolio choice problem in continuous time:

$$\max_{\{c_t, \theta_t^e\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u(c_t) \mathrm{d}t \right]$$

$$\text{s.t.} \quad \frac{\mathrm{d}n_t}{n_t} = -\frac{c_t}{n_t}\mathrm{d}t + \sum_j \theta_t^j \mathrm{d}r_t^j + \text{ labor income/endowment/taxes}$$

$$n_0 \text{ given.}$$

n_t is the net worth of the agent and r_t^j denotes the return of asset j. Let x_t^A be the value of a self-financing trading strategy A where one reinvests all dividends. Define the SDF as $\xi_t^i = e^{-\rho^i t} u'(c_t^i)$. As before, $\xi_t x_t^A$ follows a martingale. Let

$$\frac{\mathrm{d}x_t^A}{x_t^A} = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t + j_t^A \mathrm{d}J_t.$$

Assume that the SDF follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t^{F,i}\mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t - \nu_t^i(\mathrm{d}J_t - \lambda \mathrm{d}t)$$

Using Itô's product rule,

$$\begin{aligned}
\frac{\mathrm{d}(\xi_t^i x_t^A)}{\xi_t^i x_t^A} =& (-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \nu_t^i \lambda)\mathrm{d}t + (\sigma^A - \varsigma_t^i)\mathrm{d}Z_t + \\
& (j_t^A - \nu_t^i - \nu_t^i j_t^A)\mathrm{d}J_t \\
=& (-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \lambda j_t^A - \lambda \nu_t^i j_t^A)\mathrm{d}t + (\sigma^A - \varsigma_t^i)\mathrm{d}Z_t + \\
& (j_t^A - \nu_t^i - \nu_t^i j_t^A)(\mathrm{d}J_t - \lambda \mathrm{d}t).
\end{aligned}$$

Where $(\sigma^A - \varsigma_t^i)\mathrm{d}Z_t + (j_t^A - \nu_t^i - \nu_t^i j_t^A)(\mathrm{d}J_t - \lambda \mathrm{d}t)$ is a martingale, given the inclusion of the compensating $\lambda \mathrm{d}t$ term. Then, since $\xi_t^i x_t^A$ follows a martingale, its drift equals zero, giving us that

$$\mu_t^A + \lambda j_t^A = r_t^{F,i} + \varsigma_t^i \sigma_t^A + \lambda \nu_t^i j_t^A.$$

Where $r_t^{F,i}$ is the (shadow) risk-free rate, ς_t^i is the price of Brownian risk, $\varsigma_t^i \sigma_t^A$ is the required Brownian risk premium, $\lambda \nu_t^i$ is the price of Poisson upside risk if $j_t^A > 0$. For risk-neutral agents $\nu_t^i = 0$. For CRRA utility, $1 - \nu_t^i = (1 + j_t^\omega)^{1-\gamma}(1 + j_t^n)^{-\gamma}$ since SDF is $\xi_t^i = e^{-\rho^i t}(\omega_t^i)^{1-\gamma}(n_t^i)^{-\gamma}$. For log utility, $\nu_t^i = 1 - \frac{1}{1+j_t^n} = \frac{j_t^n}{1+j_t^n}$. For Epstein-Zin, ν_t^i is part of ω_t–process.

We should note that we have three risk assets in the model. $\mathrm{d}r^{e,K}$ is experts' return on capital, $\mathrm{d}r^{h,OE}$ is households' return on outside equity, and $\mathrm{d}r^{h,D}$ is households' return on debt. The debt is risky due to bankruptcy.

For any two self-financing strategies A, B, the martingale approach implies

$$\mu_t^A - \mu_t^B + \lambda(j_t^A - j_t^B) = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \lambda \nu_t^i (j_t^A - j_t^B).$$

Using the martingale approach on expert capital with outside equity issuance (after plugging in households' outside equity choice), we get

$$\begin{aligned}\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - \left[\frac{\chi_t^e}{\kappa_t^e} r_t^{F,e} + (1 - \frac{\chi_t^e}{\kappa_t^e}) r_t^{F,h}\right] + \lambda j_t^q \\ = \left[\frac{\chi_t^e}{\kappa_t^e}\varsigma_t^e + (1 - \frac{\chi_t^e}{\kappa_t^e})\varsigma_t^h\right](\sigma + \sigma^q) + \left[\frac{\chi_t^e}{\kappa_t^e}\nu_t^e + (1 - \frac{\chi_t^e}{\kappa_t^e})\nu_t^h\right]\lambda j_t^q\end{aligned}$$

Similarly, we can derive the household portfolio choice condition by taking the difference between the drift of household capital and defaultable debt

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t^{F,h} + \lambda(j_t^q - j_t^{r^D}) \le \varsigma_t^h(\sigma + \sigma^q) + \nu_t^h \lambda (j_t^q - j_t^{r^D}),$$

This condition holds with equality if $\kappa^e < 1$ (i.e., if households hold a non-zero amount of capital).

Asset and risk allocation using the price-taking social planners problem.

As in Chapter 4, we can also solve for the equilibrium risk and asset allocation using the social planner's problem. In this environment, we can generalize the price-taking planner's theorem to include the choice of jump risk.

Theorem 7.1 (Price-Taking Planner's Theorem)**.** *A social planner that takes prices as given chooses a physical asset allocation, κ_t, Brownian risk allocation, χ_t, and jump risk allocation,*

ζ_t, that coincides with the choices implied by all individuals' portfolio decisions.

The planner's optimization problem is formulated as follows:

$$\max_{\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t} \frac{\mathbb{E}_t[\mathrm{d}r_t^N(\boldsymbol{\kappa}_t)]}{\mathrm{d}t} - \boldsymbol{\varsigma}_t\sigma(\boldsymbol{\chi}_t) - \lambda\boldsymbol{\nu}j(\boldsymbol{\zeta}_t)$$
$$\text{s.t.} \quad F(\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t) \leq 0 \qquad \text{(Financial Frictions)}$$

In this formulation:

- $\boldsymbol{\kappa}_t$ represents the planner's choice of physical asset allocations across agents.
- $\boldsymbol{\chi}_t$ determines the allocation of Brownian risk, influencing the exposure of individual agents to continuous stochastic fluctuations.
- $\boldsymbol{\zeta}_t$ corresponds to the allocation of jump risk, which affects agents' exposure to discontinuous price movements.
- $\boldsymbol{\varsigma}_t = (\varsigma_t^1, \ldots, \varsigma_t^I)$ is a vector of risk price sensitivities for each agent.
- The function $\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1\sigma^N, \ldots, \chi_t^I\sigma^N)$ describes the effect of Brownian risk allocation on expected returns.
- The function $\boldsymbol{j}(\boldsymbol{\zeta}_t) = (\zeta_t^1 j^N, \ldots, \zeta_t^I j^N)$ captures the impact of jump risk allocation.
- The financial friction constraint, $F(\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t) \leq 0$, ensures that allocations are subject to market imperfections and frictions.

For example, if we set $\chi_t = \zeta_t = \kappa_t$ as the financial friction, that means experts can't issue outside equity to offload Brownian or risky debt to offload Jump risk. Alternatively, imposing the constraint $\chi_t \geq \alpha\kappa_t$ represents a skin-in-the-game requirement, where the issuance of outside equity is restricted to a certain limit.

"Invariance" of relative capital demand.

One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households. In other words, the excess market return that experts demand to hold capital remains unaffected.

Subtracting the experts' pricing condition from that of households gives the following inequality:

$$\mu_t^{r^k,e} - \mu_t^{r^k,h} \geq \frac{\chi_t^e}{\kappa_t^e}(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) - \frac{\chi_t^e}{\kappa_t^e}\lambda(1-\nu_t^h)\underbrace{\left(\frac{\partial j_t^D}{\partial \theta_t^{e,K}}(\theta_t^{e,K} - 1) + j_t^q - j_t^D\right)}_{=0}$$

Losses are distributed between experts and households through the use of defaultable debt. Since the losses of experts are capped by their net worth due to limited liability, any additional losses arising from increasing capital holdings, denoted as $\theta_t^{e,K}$, are absorbed by households.

The concept of "invariance" in this setting is dependent on the performance penalty, which approaches infinity as $\tau \to 0$. The presence of this penalty ensures that all experts follow the same default outcome. Without this penalty, experts would bifurcate—some choosing to default after a jump while others would not.

7.3.2 Evolution of state variable η_t

Drift of η_t. We calculate the drift of η_t by changing to the total wealth N_t numeraire. The change of numeraire approach is similar to the case without jumps, with an additional equation for ν_t.

As before, we change the numeraire from consumption goods to total wealth. Consider two assets:

- Asset A: sector = i's portfolio return in terms of total wealth:

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i/N}\right)\mathrm{d}t + \sigma_t^{\eta^i/N}\mathrm{d}Z_t + j_t^{\eta^i/N}\mathrm{d}J_t.$$

Expanding the equation, we get:

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i} - \underbrace{\frac{\rho^d \zeta^i (1-\eta_t^i) - (1-\zeta^i)\eta_t^i}{\eta_t^i}}_{\mu_t^{\text{pop},i} :=} \right) \mathrm{d}t + \sigma_t^{\eta^i} \mathrm{d}Z_t + j_t^{\eta^i} \mathrm{d}J_t.$$

- Asset B: a benchmark asset that everyone can hold, such as a risk-free asset or money, measured in terms of total economy-wide wealth as the numeraire:

$$r_t^{bm} \mathrm{d}t + \sigma_t^{bm} \mathrm{d}Z_t.$$

Apply our martingale asset pricing formula in the total wealth N_t numeraire,

$$\mu_t^A - \mu_t^B + \lambda(j_t^A - j_t^B) = \hat{\varsigma}_t(\sigma_t^A - \sigma_t^B) + \lambda \hat{v}_t (j_t^A - j_t^B).$$

The martingale asset pricing formula gives

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda \left(j_t^{\eta^i} - j_t^{bm} \right) - \mu_t^{\text{pop},i} = (\varsigma_t^i - \sigma_t^N) \left(\sigma_t^{\eta^i} - \sigma_t^{bm} \right) + \lambda \hat{v}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right).$$

Summing across all types weighted by their respective shares in the total economy, we obtain the aggregate condition

$$\begin{aligned} &\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^{bm} + \underbrace{\lambda \sum_{i'}^{I} \eta_t^{i'} j_t^{\eta^{i'}} - \lambda j_t^{bm}}_{=0} - \underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\text{pop},i'}}_{=0} \\ &\qquad = \sum_{i'}^{I} \eta_t^{i'} \hat{\varsigma}_t^{\eta^{i'}} (\sigma_t^{\eta^{i'}} - \sigma_t^{bm}) + \lambda \sum_{i'}^{I} \eta_t^{i'} \hat{v}_t^{\eta^{i'}} (j_t^{\eta^{i'}} - j_t^{bm}). \end{aligned}$$

where capital letters without superscripts denote economy-wide aggregates.

Subtracting this aggregate equation from the individual equation gives the net worth share dynamics

$$\mu_t^{\eta^i} + \lambda j_t^{\eta^i} = \frac{C_t}{N_t} - \frac{C_t^i}{N_t^i} + \hat{\varsigma}_t^i(\sigma^{\eta^i} - \sigma^{bm_t}) - \sum_{i'}^{I} \eta_t^{i'} \hat{\varsigma}_t^{\eta^{i'}} (\sigma_t^{\eta^{i'}} - \sigma_t^{bm})$$
$$+\lambda \hat{v}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right) - \lambda \sum_{i'}^{I} \eta_t^{i'} \hat{v}_t^{\eta^{i'}} (j_t^{\eta^{i'}} - j_t^{bm}) + \mu_t^{\text{pop},i}.$$

For experts

$$\mu_t^{\eta^e} + \lambda j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e \left(\sigma_t^{\eta^e} - \sigma_t^{bm}\right) + (1-\eta_t^e)\hat{\varsigma}_t^h \left(\sigma_t^{\eta^h} - \sigma_t^{bm}\right)$$
$$+ (1-\eta_t^e)\lambda \hat{v}_t^e \left(j_t^{\eta^e} - j_t^{bm} \right) - (1-\eta_t^e)\lambda \hat{v}_t^h \left(j_t^{\eta^h} - j_t^{bm} \right) + \mu_t^{\text{pop},i}$$

In this context, the benchmark asset is risky debt. Since j_t^D is the return on risky debt jump in c-numeraire and j_t^N represents the wealth jump, apply quotient rule for jumps $\sigma_t^{bm} = -\sigma_t^N$ and $j_t^{bm} = \frac{j_t^{r^D} - j_t^N}{1+j_t^N}$.

$$\mu_t^{\eta^e} + \lambda j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e \left(\sigma_t^{\eta^e} + \sigma_t^N\right) + (1-\eta_t^e)\hat{\varsigma}_t^h \left(\sigma_t^{\eta^h} + \sigma_t^N\right) +$$
$$(1-\eta_t^e)\lambda \hat{v}_t^e \left(j_t^{\eta^e} - \frac{j^{r^D} - j^N}{1+j^N} \right) - (1-\eta_t^e)\lambda \hat{v}_t^h \left(j_t^{\eta^h} - \frac{j^{r^D} - j^N}{1+j^N} \right) + \mu_t^{\text{pop},e}$$

Volatility of η_t^i. We can calculate the volatility of η_t^i using Itô's quotient rule. Since $\eta_t^i = N_t^i / N_t$ we have

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = (1-\eta_t^i)\sigma_t^{N^i} - \sum_{\neg i \neq i} \eta_t^{\neg i} \sigma_t^{N^{\neg i}}$$

Jumps in η_t^i. Similarly,

$$j_t^{\eta^i} = \frac{j_t^{N^i} - J_t^N}{1 + j_t^N} = \frac{j_t^{N^i} - \sum_{i'} \eta_t^{i'} J_t^{N^{i'}}}{1 + \sum_{i'} \eta_t^{i'} J_t^{N^{i'}}}$$

For 2 types example,

$$j_t^{\eta^e} = \frac{(1-\eta_t^e)(j_t^{N^e} - j_t^{N^h})}{1+\eta^e j_t^{N^e} + (1-\eta_t^e) j_t^{N^h}}$$

7.3.3 BSDE functions

BSDE functions for CRRA. For CRRA, we generalize the result from earlier lecture by adding jump terms in value function BSDEs. Specifically, we have

$$\xi^i n_t^i = e^{-\rho^i t}(c_t^i)^{-\gamma}\frac{\omega_t^i n_t^i}{\rho^i} = e^{-\rho^i t}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{\rho^i} = \underbrace{\frac{(\omega_t^i n_t^i/K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} K_t^{1-\gamma} e^{-\rho^i t}$$

when transforming $e^{\rho^i t}\xi^i(n_t^i; \boldsymbol{\eta}_t, K_t)$-process into $v^i(\boldsymbol{\eta}_t)$-process, so

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i} = \frac{\mathrm{d}\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}} - \rho^i \mathrm{d}t.$$

By Itô's product rule:

$$\frac{\mathbb{E}_t[\mathrm{d}(\xi_t^i n_t^i)]}{\xi_t^i n_t^i} = \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} - \rho^i + \lambda j_t^{v^i}\right)\mathrm{d}t$$

Recall by consumption optimality for CRRA utility:

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i}\mathrm{d}t + \frac{c_t^i}{n_t^i}\mathrm{d}t \text{ follows a martingale}$$

Hence, $\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = \rho^i - \frac{c_t^i}{n_t^i} - \lambda j_t^{v^i}$

BSDE functions for Epstein-Zin. The SDF in the most general case of Epstein-Zin utility is given by:

$$\xi_t^i = e^{\left(\int_0^t \frac{\partial f}{\partial U}(c_s, V_s^i) ds\right)}\frac{\partial V_t^i}{\partial n_t^i}$$

By envelop condition:

$$\xi_t^i n_t^i = e^{\int_0^t \frac{\partial f}{\partial V}(c_s^i, V_s^i)\mathrm{d}s} \frac{1}{\rho^i}(\omega^i n_t^i)^{1-\gamma} = e^{\int_0^t \frac{\partial f}{\partial V}(c_s^i, V_s^i)\mathrm{d}s} \underbrace{\frac{(\omega_t^i n_t^i / K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} K_t^{1-\gamma}$$

By Itô's product rule:

$$\frac{\mathbb{E}_t[\mathrm{d}(\xi_t^i n_t^i)]}{\xi_t^i n_t^i} = \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \tfrac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} + \tfrac{\partial f}{\partial V}(c_t^i, V_t^i) + \lambda j_t^{v^i}\right)\mathrm{d}t.$$

Similarly, recall by consumption optimality for EZ utility:

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i}\mathrm{d}t + \frac{c_t^i}{n_t^i}\mathrm{d}t \text{ follows a martingale}$$

Hence, $\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = -\frac{\partial f}{\partial V}\left(c_t, V_t^i\right) - \frac{c_t^i}{n_t^i} - \lambda j_t^{v^i}$.

We still have to solve for $\mu_t^{v^i}, \sigma_t^{v^i}$. The numerical solution is the same as in Chapter 4, with minor changes such as adjusting μ_t^{η} and $\mu_t^{v^i}$, and keeping track of the vulnerability region together with all the jump loadings.

Discussions. If we deviate from EIS $=1$, the consumption-wealth ratio of agents will vary with investment opportunities, as these depend on the precise specification of perceived run risk, even under log utility. This variation will, in turn, influence q through goods market clearing. However, if we maintain EIS $=1$ but alter the level of risk aversion, the q-function will only be affected if capital is allocated differently for the same value of η. This is because, in such a case, the average consumption-wealth ratio in the economy remains unchanged, allowing goods market clearing to establish a one-to-one mapping between q and capital allocation. A key question is whether the "invariance" of capital demands holds solely due to the absence of hedging demands or whether it generalizes even in their presence. Without the full set of equations at hand, a reasonable conjecture is that this result lacks robustness, implying that capital allocation and q will still be affected even when EIS $=1$.

7.4 Key Takeaways

In this chapter, we introduced *jumps* into a continuous-time macrofinance model and explored how sudden stops or runs can arise endogenously. Several technical tools were developed in the process. First, we extended the usual Itô's Lemma to handle discontinuous paths driven by Poisson processes or Lévy jumps. We also generalized the martingale approach for portfolio choice and asset pricing, ensuring agents' strategies and pricing conditions fully incorporate both Brownian and jump risk. A "price-taking" social-planner formulation then allowed us to solve for capital and risk allocation across experts and households. Finally, we changed numeraires to economy-wide net worth, carefully adjusting drift and volatility terms so that jumps in wealth shares are properly accounted for.

On the economic side, we saw that *defaultable debt* plays a key role in risk sharing. In extreme downturns, debt can default, offloading part of the experts' downside onto households, which leads to a form of *invariance* in how much capital experts wish to hold. Another striking observation is that there are often *no runs in very bad times*: once the expert sector's net worth is too low, the price can no longer drop enough to wipe them out, so the vulnerability region does not start at $\eta = 0$. Finally, we encountered a *volatility paradox* in the presence of jump risk. Paradoxically, a low-risk environment can breed higher leverage and thus amplify the impact of a jump, ultimately making a system more fragile.

Overall, incorporating *jump processes* in this way offers a tractable lens through which to study self-fulfilling crises, fire-sale prices, and abrupt collapses of intermediaries' balance sheets. By combining these new technical instruments with the economic insights outlined above, we obtain a unified framework for analyzing sudden stops and runs.

7.5 Exercises

7.5.1 Introducing a collateral (borrowing)/leverage constraint

$$-\theta_t^{e,D} \leq \ell \theta_t^{e,K}$$

(a) Show that with log-utility, $q(\eta)$ is not affected by jumps even with the leverage constraint.

(b) Show how jumps affect the risk premium in the "vulnerability region".

(c) Verify whether the occasionally binding leverage constraint lowers the drift μ_t^η for small η values.

(d) Describe conditions under which the leverage constraint leads to a bimodal stationary distribution of η.

(e) Show that with a sufficiently strict leverage constraint, the vulnerability region (when $j_t^q \chi_t \geq \eta_t$) shrinks or even disappears.

(f) Consider a setting in which the debt is fully collateralized, i.e., it is default-free. When entering the vulnerability region, the "worst price" can jump, and hence the debt capacity contracts discontinuously.

To solve this model, one needs an extra loop to determine the fixed point when the vulnerability region starts.

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Part IV

Money

Chapter 8

Simple Money Models

8.1 Introduction / Overview

The 3 Roles of Money.

- **Store of value**
  - fundamental cash flows due to backing
    e.g., commodity standard, exchange rate regime, fiscal backing
  - non-cash-flow benefits: helps overcome *intertemporal* financial frictions
    e.g., OLG (Samuelson), spatial separation (Townsend), uninsured idiosyncratic risk (Bewley)
  - store of value role not exclusive to money: non-monetary assets are substitute stores of values
- **Medium of exchange**
  - helps overcome monetary frictions = frictions in *intratemporal* exchange
  - key monetary friction: double coincidence of wants problem
  - makes money special relative to assets, which cannot serve as substitute media of exchange
- **Unit of account**

- contractual values denominated in monetary unit
- e.g., nominal goods prices (+ commitment to sell at quoted price), nominal debt contracts, nominal labor contracts

The Value of Money and the Price Level.

- Core question of monetary economics: what determines the general level of nominal goods prices?
- Equivalently: what are the determinants of the value of money?
  - price level $\mathcal{P}_t$: price of real goods basket in units of money
  - real value of a single unit of money: $1/\mathcal{P}_t$
- Two aspects:
  1. which economic considerations justify the value of money in a given equilibrium?
  2. determinacy question:
     - does the model have a unique prediction for the value of money / price level?
       ($\sim$ equilibrium uniqueness)
     - more broadly, which economic forces lead to coordination on a specific monetary equilibrium?

Classification of Monetary Theories (see following Chapter 9 for more details)

1. **Backing theories**: value of money derives from fundamental cash flows that back it
   - store of value role (money is just another asset)
   - example: Fiscal Theory of the Price Level (FTPL)
2. **Bubble theories**: money valued because it can be passed on to others
   can be rational expectation if trading money overcomes market frictions:

(a) intertemporal financial frictions (e.g., incomplete markets)
   - emphasizes store of value role
   - examples: Samuelson, Townsend, Bewley, Brunnermeier-Merkel-Sannikov

(b) intratemporal monetary frictions (e.g., cash-in-advance constraint)
   - emphasizes medium of exchange role
   - example: (New) Monetarism

3. **Money as a pure unit of account**
   - value of money derives from role of money as a unit of account
   - not from the value of any monetary assets
   - example: New Keynesianism

Monetary Assets: Credit, Deposits, Cash, Reserves, Government Debt. In first two classes of theories, different assets may play the role of "money":

- Credit can substitute for
  - store of value assets (credit balances to keep track of resource distribution)
  - media of exchange (exchange goods against credit balance)

  imperfect credit prerequisite for bubble theories
- Bank deposits, cash, and central bank reserves all play a role in the payment system as media of exchange
- Government-provided outside money vs. inside money
  - outside money: positive net supply, backed by government fiscal capacity
  - inside money: zero net supply, backed by bank assets
- Cash & reserves (narrow outside money) vs. nom. government liabilities (broad outside money)
  - (primarily) narrow money provides medium of exchange services

- but all nominal government liabilities
  * compete for the same backing real resources
  * serve as a store of value
  * are affected symmetrically by changes in the price level

8.2 A Unified Model of Money

The focus of this section is to establish a tractable model of money that captures the various monetary theories. We begin with a benchmark model of money and government debt, assuming the absence of both intertemporal and intratemporal frictions. In this baseline model, money (and government debt) has value only if it is backed by discounted future primary surpluses (i.e., government surpluses excluding interest payments). The fiscal theory of the price level (FTPL) equation captures this form of backing. In other words, the nominal value of money and government bonds divided by the price level is the expected discounted value of future government primary surpluses.

In the next section, we will introduce financial frictions. Due to incomplete markets, citizens are exposed to idiosyncratic risk that they cannot diversify. Contingent contracts in the form of individual equities or derivatives cannot be traded. Because of these frictions, citizens prefer to hold money and government bonds as safe assets. Money and government bonds serve as a store of value, but they can form a bubble. This occurs when the value of money and government bonds, divided by the price level, exceeds the fundamental value, which is the discounted present value of future primary surpluses.

Then, we introduce monetary frictions in the form of transaction costs. In the limit, they converge to a cash-in-advance constraint. Money, rather than government bonds, provides transaction services, and holders enjoy a convenience yield. Consequently, the interest rate on money is below that of government bonds by Δi_t. A medium-of-exchange bubble theory can emerge.

For tractability we always select a specific equilibrium in this chapter, the monetary steady state. We will return to the determinacy question in the next chapter (Chapter 9).

8.2.1 Baseline Model Setup

Environment. Time is continuous. The economy consists of a continuum of agents $\tilde{i} \in [0,1]$ with logarithmic preferences over consumption $c_t^{\tilde{i}}$. Each agent $\tilde{i}$ operates physical capital $k_t^{\tilde{i}}$, which generates an output flow net of reinvestment and transaction costs of

$$(ak_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}}) k_t^{\tilde{i}})\mathrm{d}t.$$

Transaction costs depend on money velocity $\nu_t^{\tilde{i}}$ and introduce a medium of exchange role for money. We discuss this new feature later.

Capital evolves according to

$$\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta\right) \mathrm{d}t + \tilde{\sigma} \mathrm{d}\tilde{Z}_t^{\tilde{i}} + \mathrm{d}\Delta_t^{k,\tilde{i}}. \tag{8.1}$$

Here, as in previous chapters, $\iota_t^{\tilde{i}}$ is the physical investment rate into the capital stock chosen by agent $\tilde{i}$, $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$, $\phi \geq 0$ captures capital adjustment costs, and δ is the capital depreciation rate. The $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$-term captures *idiosyncratic* capital risk: $\tilde{Z}_t^{\tilde{i}}$ is a Brownian motion specific to agent $\tilde{i}$ and independent across different $\tilde{i}$. So, while agents are ex-ante identical, they will generally be ex-post heterogeneous because they have experienced different idiosyncratic shock histories. Finally, installed capital can be (frictionlessly) traded between agents. The term $\mathrm{d}\Delta_t^{k,\tilde{i}}$ captures these capital trades. In the aggregate, $\int_0^1 k_t^{\tilde{i}} \mathrm{d}\Delta_t^{k,\tilde{i}} \mathrm{d}\tilde{i} = 0$.

There is a government that imposes a proportional output tax on all agents, generating total tax revenues of $\tau_t a K_t \mathrm{d}t$, where, as in previous chapters, $K_t := \int_0^1 k_t^{\tilde{i}} \mathrm{d}\tilde{i}$ denotes aggregate capital. There is an exogenous need for government expenditures that is proportional to aggregate capital, $G_t = \mathcal{G} K_t$. The government issues nominal liabilities. In this model, we do not distinguish between (narrow) money ($\mathcal{M}$) and nominal government bonds ($\mathcal{B}$) and simply refer to the quantity of all outstanding liabilities at time t by $\mathcal{MB}_t \geq 0$. We will separate them later in Section 8.2.5. Note that $\mathcal{MB}_t$ is the *nom-*

inal quantity of government liabilities, i.e., measured in dollars, euros, or whatever is the local currency. In the following, we always use the notation convention that calligraphic letters refer to nominal quantities. The real value of these liabilities is $\mathcal{MB}_t/\mathcal{P}_t$, where $\mathcal{P}_t$ is the nominal price level, the price of output goods in the nominal unit. The government starts with initial nominal liabilities $\mathcal{MB}_0 > 0$ and faces each period the flow budget constraint

$$\underbrace{(\mu_t^{\mathcal{MB}} - i_t^{\mathcal{MB}})}_{=:\breve{\mu}_t^{\mathcal{MB}}}\mathcal{MB}_t + \mathcal{P}_t\underbrace{(\tau_t a - \mathcal{G}_t)}_{=:s_t}K_t = 0, \tag{8.2}$$

where $\mu_t^{\mathcal{MB}}$ is the growth rate of nominal liabilities, $i_t^{\mathcal{MB}}$ is the nominal interest rate paid on these liabilities, and $s_t K_t = (\tau_t a - \mathcal{G})K_t$ are real primary surpluses. Unlike the conventional surplus, which includes interest expenditures, the primary surplus is the difference between government revenues and expenditures other than interest expenditures.

Agent Decision Problem. Each agent $\tilde{i}$ chooses $c_t^{\tilde{i}}$, $\theta_t^{\tilde{i}}$, $\iota_t^{\tilde{i}}$ to maximize expected utility

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t^{\tilde{i}}\mathrm{d}t\right]$$

subject to the net worth evolution

$$\frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}\mathrm{d}t + \mathrm{d}r_t^{\mathcal{MB}} + (1-\theta_t^{\tilde{i}})(\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - \mathrm{d}r_t^{\mathcal{MB}}) \tag{8.3}$$

and the solvency constraint $n_t^{\tilde{i}} \geq 0$. Here, $\mathrm{d}r_t^{\mathcal{MB}}$ denotes the (real) return on government liabilities and $\mathrm{d}r_t^{K,\tilde{i}}(\iota)$ denotes the return on capital operated by agent $\tilde{i}$ conditional on choosing the investment rate ι. Note that capital trading $\mathrm{d}\Delta_t^{k,\tilde{i}}$ is implicit in the portfolio choice $\theta_t^{\tilde{i}}$ and can be backed out ex post.[1]

[1] We remark that it would make sense economically to also impose no short sale constraints for both capital and government liability holdings ($\theta_t^{\tilde{i}} \in [0,1]$) because operating negative quantities of capital is physically impossible and the household may not be in the position to issue nominal liabilities that serve as perfect substitutes to government liabilities. However, it will turn out that short sales never occur in equilibrium, so that imposing such constraints is redundant in this model.

Market Clearing. The final goods market clears if

$$C_t := \int_0^1 c_t^{\tilde{i}} \mathrm{d}\tilde{i} = \int_0^1 (a k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}}) k_t^{\tilde{i}}) \mathrm{d}\tilde{i} - \mathcal{G} K_t = (a - \iota_t - \mathcal{G} - \mathfrak{T}_t) K_t, \qquad (8.4)$$

where ι_t and $\mathfrak{T}_t$ are the (weighted) average investment rate and transaction cost, respectively.[2]

We denote by q_t^K the market price of capital and by $q_t^{\mathcal{MB}} := \frac{\mathcal{MB}_t}{\mathcal{P}_t K_t}$ value of government liabilities per unit of capital in the economy. Total wealth in the economy is then $N_t = (q_t^K + q_t^{\mathcal{MB}}) K_t =: q_t K_t$. The asset market clears[3] if

$$\int_0^1 \theta_t^{\tilde{i}} n_t^{\tilde{i}} \mathrm{d}\tilde{i} = q_t^{\mathcal{MB}} K_t.$$

Letting θ_t be the (weighted) average portfolio weight and dividing this condition by total wealth $N_t = q_t K_t$, we can write asset market clearing equivalently as

$$\theta_t = \frac{q_t^{\mathcal{MB}}}{q_t} =: \vartheta_t,$$

where ϑ_t denotes the fraction of total wealth that takes the form of nominal government liabilities ("nominal wealth share").

Frictions. We will build up the model step by step adding one friction at a time. This results in three model variants with increasing complexity:

1. Frictionless benchmark: $\tilde{\sigma} = 0, \mathfrak{T} \equiv 0$. In this case, money may be valued because it is backed by taxes but having money does not provide any specific benefit to the economy.

2. Adding a financial friction (intertemporal): $\tilde{\sigma} > 0$ and markets are incomplete in the sense that agents cannot write contracts to insure against $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$-shocks. In this case, money may be valued even in the absence of tax backing because nominal government liabilities provide a safe store of value.

[2]Specifically, $\iota_t := \frac{\int_0^1 \iota_t^{\tilde{i}} k_t^{\tilde{i}} \mathrm{d}\tilde{i}}{K_t}$. The definition of $\mathfrak{T}_t$ is analogous.

[3]This equation represents market clearing for government liabilities. Capital market clearing is implied by Walras' law.

3. Adding a monetary friction (intratemporal): transaction costs $\mathfrak{T}_t(\nu_t^{\tilde{i}})$ can be nonzero and are increasing in velocity $\nu_t^{\tilde{i}}$, which depends on agent $\tilde{i}$'s holdings of government liabilities. Transaction costs are a reduced-form device to model the medium of exchange role of money. Our specific approach may be interpreted as transaction costs incurred in an unmodeled supply chain. We provide formal details in Section 8.2.4. There, we also discuss alternative ways of modeling the medium of exchange role.

8.2.2 Frictionless Benchmark

Model Solution

Prices and SDF Processes. Recall that we denote by q_t^K the market price of capital, by $q_t^{\mathcal{MB}}$ the real value of government liabilities per unit of capital, by $q_t = q_t^K + q_t^{\mathcal{MB}}$ total net worth per unit of capital, and by $\vartheta_t = q_t^{\mathcal{MB}}/q_t$ the nominal wealth share. As usual, we postulate that these variables follow Itô processes

$$\mathrm{d}q_t^K/q_t^K = \mu_t^{q,K}\mathrm{d}t, \quad \mathrm{d}q_t^{\mathcal{MB}}/q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}}\mathrm{d}t, \quad \mathrm{d}q_t/q_t = \mu_t^q\mathrm{d}t, \quad \mathrm{d}\vartheta_t/\vartheta_t = \mu_t^\vartheta\mathrm{d}t.$$

Similarly, we postulate that the SDF process of agent $\tilde{i}$, denoted by $\xi_t^{\tilde{i}}$ follows

$$\mathrm{d}\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt$$

with common decay rate r_t. Note that all evolutions are deterministic because there are no shocks in this benchmark model.

Return Processes. The return on capital managed by agent $\tilde{i}$ is

$$\begin{aligned}\mathrm{d}r_t^{K,\tilde{i}}(\iota) &= \left(\frac{a(1-\tau_t)-\iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t \\ &= \left(\frac{a-\mathcal{G}-\iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t\end{aligned}$$

Here, the second line uses the government flow budget constraint (8.2) to eliminate taxes $\tau_t a$.

The real value of one unit of nominal government liabilities is $1/\mathcal{P}_t$. Therefore, the (real) return on government liabilities is

$$\mathrm{d}r_t^{\mathcal{MB}} = \underbrace{\frac{i_t^{\mathcal{MB}}/\mathcal{P}_t}{1/\mathcal{P}_t}\mathrm{d}t}_{\text{dividend yield}} + \underbrace{\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}}_{\text{capital gains}} = i_t^{\mathcal{MB}}\mathrm{d}t + \frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}.$$

Using $1/\mathcal{P}_t = \frac{q_t^{\mathcal{MB}}K_t}{\mathcal{MB}_t}$, which follows from the definition of $q_t^{\mathcal{MB}}$, and applying Itô's lemma, we can write this as

$$\mathrm{d}r_t^{\mathcal{MB}} = \left(i_t^{\mathcal{MB}} + \mu_t^{q,\mathcal{MB}} + \mu_t^K - \mu_t^{\mathcal{MB}}\right)\mathrm{d}t = \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \breve{\mu}_t^{\mathcal{MB}}\right)\mathrm{d}t. \tag{8.5}$$

Optimal Investment and Goods Market Clearing. Like in previous chapters, the investment rate ι enters only the expected capital return $\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota)]$. Assuming that agents hold no negative quantities of capital, all agents agree that it is optimal to choose the investment rate $\iota_t^{\tilde{i}}$ in a way that maximizes the expected capital return $\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}})]$. In particular, $\iota_t^{\tilde{i}} = \iota_t$ for all $\tilde{i}$ and ι_t satisfies the Tobin's Q condition

$$q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi\,\iota_t. \tag{8.6}$$

Due to log utility, the optimal consumption choice of all agents is $c_t^{\tilde{i}} = \rho n_t^{\tilde{i}}$. Aggregating across agents yields $C_t = \rho N_t = \rho q_t K_t$ and substituting this into goods market clearing, equation (8.4), implies (recall that $\mathfrak{T}_t \equiv 0$ in this model)

$$\rho q_t K_t = (a - \mathcal{G} - \iota_t)K_t. \tag{8.7}$$

We can cancel K_t on both sides, solve for q_t and combine this with $q_t^K = (1-\vartheta_t)q_t$ and the optimal investment condition for q_t^K:

$$(1-\vartheta_t)\frac{a-\mathcal{G}-\iota_t}{\rho}=(1-\vartheta_t)q_t=1+\phi\iota_t \qquad \Rightarrow \qquad \iota_t=\frac{(1-\vartheta_t)\breve{a}-\rho}{1-\vartheta_t+\phi\rho}, \quad \breve{a}=:a-\mathcal{G}$$

The previous equation expresses the equilibrium investment rate ι_t as a function of the nominal wealth share ϑ_t. We can substitute this back into $q_t=(a-\mathcal{G}-\iota_t)/\rho$ to express q_t in terms of ϑ_t. Finally, due to $q_t^{\mathcal{MB}}=\vartheta_t q_t$, $q_t^K=(1-\vartheta_t)q_t$, we can then also relate the remaining asset values in the model to ϑ_t:

Proposition 8.1. *In equilibrium, asset prices and the optimal investment rate only depend on the nominal wealth share ϑ_t and are given by*

$$q_t=\frac{1+\phi\breve{a}}{1-\vartheta_t+\phi\rho}, \qquad \iota_t=\frac{(1-\vartheta_t)\breve{a}-\rho}{1-\vartheta_t+\phi\rho},$$
$$q_t^K=(1-\vartheta_t)\frac{1+\phi\breve{a}}{1-\vartheta_t+\phi\rho}, \qquad q_t^{\mathcal{MB}}=\vartheta_t\frac{1+\phi\breve{a}}{1-\vartheta_t+\phi\rho}.$$

Hence, ϑ_t is the key variable in this model. Because, by asset market clearing, $\vartheta_t=\theta_t$, this variable is determined by portfolio choice between capital and bonds.

Portfolio Choice. Agents in this model face a standard portfolio choice problem without constraints or any unconventional features. Therefore, we can directly apply the usual martingale conditions without having to set up a Hamiltonian explicitly. Note that here the martingale conditions reduce to a simple no arbitrage condition because all assets are risk-free:

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t}=r_t=\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}]}{\mathrm{d}t}.$$

Substituting in the return expressions for the two assets yields

$$-\breve{\mu}_t^{\mathcal{MB}}+\mu_t^K+\mu_t^{q,\mathcal{MB}}=\frac{a-\mathcal{G}-\iota_t+q_t^{\mathcal{MB}}\breve{\mu}_t^{\mathcal{MB}}}{q_t^K}+\Phi(\iota_t)-\delta+\mu_t^{q,K}.$$

Aggregating the individual capital evolutions (8.1) leads to $\mu_t^K=\Phi(\iota_t)-\delta$, so that we can cancel out this term on both sides of the equation. In the remaining equation, we make the following two replacements: (i) $a-\mathcal{G}-\iota_t=\rho q_t$ (goods market clearing), (ii)

$q_t^K = (1-\vartheta_t)q_t$, $q_t^{\mathcal{MB}} = \vartheta_t q_t$ (definition of ϑ_t). The equation then simplifies to

$$-\breve{\mu}_t^{\mathcal{MB}} + \mu_t^{q,\mathcal{MB}} = \frac{\rho q_t + \vartheta_t q_t \breve{\mu}_t^{\mathcal{MB}}}{(1-\vartheta_t)q_t} + \mu_t^{q,K}.$$

After canceling q_t on the right-hand side and rearranging, we obtain (the first equality follows from applying Itô's lemma to the definition of ϑ_t, the second from the previous equation)

$$\mu_t^{\vartheta} = (1-\vartheta_t)(\mu_t^{q,\mathcal{MB}} - \mu_t^{q,K}) = \rho + \breve{\mu}_t^{\mathcal{MB}}.$$

The previous condition is a condition on the expected rate of change in ϑ_t that is consistent with portfolio choice and market clearing. Logically, it is a backward equation. Mathematically, this means that it does not have an initial condition but ϑ_0 is a free variable in equilibrium. Economically, this means that the condition is forward-looking. Note that because $\mu_t^{\vartheta} = \mathbb{E}_t[\mathrm{d}\vartheta_t]/\vartheta_t$, we can also write the previous equation as an equation for $\mathbb{E}_t[\mathrm{d}\vartheta_t]$. This is preferable because it turns out that the latter equation remains valid for $\vartheta_t = 0$ (our solution procedure has implicitly assumed $\vartheta_t > 0$ as otherwise the return $\mathrm{d}r_t^{\mathcal{MB}}$ is no longer well-defined).[4]

Proposition 8.2 (Government Liabilities Valuation Equation)**.** *The time path for the nominal wealth share $\vartheta_t \in [0,1]$ satisfies the BSDE*

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{MB}}\right)\vartheta_t dt. \tag{8.8}$$

The government liabilities valuation equation is the key equation in this model. It characterizes the portfolio demand for nominal assets. For interpretation, it is instructive to integrate the equation forward in time and write[5]

$$\vartheta_t = \mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(t'-t)}(-\breve{\mu}_{t'}^{\mathcal{MB}})\vartheta_{t'}\mathrm{d}t'\right] = \mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(t'-t)}\frac{s_{t'}}{q_{t'}}\mathrm{d}t'\right],$$

[4] In the following proposition, we also add the claim that $\vartheta_t \in [0,1]$. This is required by $q_t^K, q_t^{\mathcal{MB}} \geq 0$ and the latter inequalities hold whenever there is free disposal of assets.

[5] To derive this equation, first determine $\mathrm{d}(e^{-\rho t}\vartheta_t)$ by Itô's product rule and then integrate the resulting expression over the interval $[t,T]$. Using $e^{-\rho T}\vartheta_T \to 0$ as $T \to \infty$ (because $\vartheta_T \in [0,1]$ is bounded) and rearranging yields the stated equation. Note that the expectation is here redundant because all paths are deterministic but is kept to align this equation better with extensions in subsequent chapters.

where the second equality uses the government budget constraint (8.2). This equation tells us that, in this model, portfolio demand for government liabilities arises from expectations of future primary surpluses ($s_t K_t$). Surpluses appear in the equation because they summarize the total real cash flows for holders of government liabilities. Such cash flows may take the form of interest payments on existing liabilities ($i_t^{\mathcal{MB}}$) or a reduction of the outstanding stock of nominal liabilities (a negative $\mu_t^{\mathcal{MB}}$). In which form real resources are paid out to government liability holders is irrelevant, only their total quantity (normalized by total wealth in the economy) matters for portfolio demand for government liabilities. This is in analogy to the valuation of a firm that can pay out cash flows to stock holders by either paying a dividend or repurchasing outstanding shares. Only the (expected) total payouts matters for the value of firm equity.

Summary of Model Solution. For any time path for $\vartheta_t \in [0, 1]$ that satisfies the government liabilities valuation equation, we can construct an equilibrium by using the equations in Proposition 8.1, and backing out additional equilibrium quantities from the optimal consumption rule or accounting identities. We leave it as an exercise for the reader to verify that any mathematical solution constructed in this way indeed constitutes a valid equilibrium solution of the model. In general, the solution structure of the government liabilities valuation equation, and therefore the set of possible equilibria, depends on the policy rules assumed for the policy variables $i_t^{\mathcal{MB}}$, $\mu_t^{\mathcal{MB}}$, τ_t, which we have left unspecified thus far. We postpone a discussion of this issue to the next chapter, which links the technical issue of equilibrium uniqueness/multiplicity to (in)determinacy of the price level and value of money. In what follows in this chapter, we sidestep these issues by focusing exclusively on steady-state equilibria.

Steady State Equilibrium

Let us impose a steady state in the government liabilities valuation equation, i.e., assume that $\vartheta_t = \vartheta$ is constant. Proposition 8.1 then implies that also q, q^K, $q^{\mathcal{MB}}$, and ι are constant over time, and so is the consumption-capital ratio $C_t/K_t = \rho q$. Strictly speaking, this steady state is a balanced growth path because the aggregate capital stock K_t grows at a constant rate $g = \Phi(\iota) - \delta$ that may be different from zero.

Imposing $\mathbb{E}_t[\mathrm{d}\vartheta_t] = 0$ in equation (8.8) and restricting attention to $\vartheta > 0$, so that government liabilities are valued in equilibrium, yields the requirement

$$\breve{\mu}^{\mathcal{MB}} = -\rho.$$

This means that a steady state is only possible if the government's policy rules (that we still keep unspecified) are consistent with a constant value for $\breve{\mu}_t^{\mathcal{MB}}$ equal to $-\rho$.[6] By the government budget constraint (8.2) then

$$s = -\breve{\mu}^{\mathcal{MB}} q^{\mathcal{MB}} = \rho q^{\mathcal{MB}} \Rightarrow q^{\mathcal{MB}} = \frac{s}{\rho}.$$

We can use now the equations in Proposition 8.1 to back out the remaining equilibrium quantities of interest as a function of the steady-state primary surplus per unit of capital s:

Corollary 8.1 (Steady State in the Frictionless Benchmark Model)**.** *In any steady state equilibrium the primary surplus-capital ratio s must be constant. For any $s \in \left(0, \breve{a} + \frac{1}{\phi}\right)$, there is a steady state in which both capital and government liabilities have positive value. In this steady state, aggregate consumption and investment are given by*

$$C_t/K_t = \frac{\rho + s + \phi\rho\breve{a}}{1 + \phi\rho}, \qquad \iota = \frac{\breve{a} - s - \rho}{1 + \phi\rho},$$

asset values are given by

$$\vartheta = \frac{s(1 + \phi\rho)}{s + \rho(1 + \phi\breve{a})}, \qquad q^K = \frac{1 + \phi(\breve{a} - s)}{1 + \phi\rho}, \qquad q^{\mathcal{MB}} = \frac{s}{\rho},$$

and the growth rate of nominal government liabilities in excess of interest payments, $\breve{\mu}_t^{\mathcal{MB}} = \mu_t^{\mathcal{MB}} - i_t^{\mathcal{MB}}$, must be constant and satisfy

$$\breve{\mu}^{\mathcal{MB}} = -\rho.$$

[6] This condition essentially says that the dividend yield on the total stock of government liabilities (but not necessarily on an individual unit) must equal $r = \rho$, the equilibrium real rate in this model. If government policy is inconsistent with this condition, capital gains must be different from zero to align the total return with r, so that the nominal wealth share cannot remain constant over time.

The nominal liability growth rate $\mu_t^{\mathcal{MB}}$ and interest rate $i_t^{\mathcal{MB}}$ individually can have arbitrary, possibly non-constant, paths, so long as this condition holds.

We remark that there is also a steady state equilibrium in which $s = 0$ and government liabilities are worthless ($q^{\mathcal{MB}} = 0 \Leftrightarrow \mathcal{P} = \infty$). In this "non-monetary" equilibrium, $\vartheta = 0$, so that equation (8.8) holds with a zero left-hand side regardless of the value of $\breve{\mu}_t^{\mathcal{MB}}$ (which does not even have to be constant over time).

Some Observations from Frictionless Benchmark

The characterization of steady state equilibria in the frictionless benchmark leads to the following observations:

1. The value of money depends on fiscal backing. Specifically, we observe that a positive primary surpluses ($s > 0$) is required for nominal government liabilities to be valued in equilibrium. If holding government liabilities does not generate real cash flows, then private agents are unwilling to hold them and money will be worthless, $q^{\mathcal{MB}} = 0 \Leftrightarrow \mathcal{P} = \infty$. In addition, we observe that the value of government liabilities is increasing in the level of primary surpluses: higher s results in higher $q^{\mathcal{MB}}$ and ϑ.

2. The value of capital assets (q^K), the investment rate (ι), and the growth rate of the economy ($g = \Phi(\iota) - \delta$) are all inversely related to the value of money. This turns out to be a robust prediction from all variants of our money model. We discuss its economic content later once we have introduced the full model with all frictions present.

3. The nominal interest rate paid on money does not matter for the real allocation. Specifically, raising $i^{\mathcal{MB}}$ while maintaining the steady-state condition $\breve{\mu}^{\mathcal{MB}} = -\rho$ raises the growth rate of nominal liabilities, $\mu^{\mathcal{MB}} = i^{\mathcal{MB}} - \rho$, one for one. This affects the inflation rate, $\pi := \mu^{\mathcal{P}} = \mu^{\mathcal{MB}} - g$, but it has no impact on any real variable.

8.2.3 Adding Financial Frictions: Uninsurable Idiosyncratic Risk

We now add uninsurable idiosyncratic capital shocks to the model by assuming $\tilde{\sigma} > 0$. This implies that the idiosyncratic shocks $d\tilde{Z}_t^{\tilde{i}}$ in equation (8.1), which we have ignored thus far, become now relevant.

Model Solution

Prices and SDF Processes. As in the frictionless benchmark, we postulate price processes

$$dq_t^K/q_t^K = \mu_t^{q,K} dt, \quad dq_t^{\mathcal{MB}}/q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}} dt, \quad dq_t/q_t = \mu_t^q dt, \quad d\vartheta_t/\vartheta_t = \mu_t^\vartheta dt.$$

While there are now shocks in the model, prices still evolve deterministically because all shocks are idiosyncratic and wash out in the aggregate. However, as all agents are exposed to their own idiosyncratic shock, the evolution of the SDF $\xi_t^{\tilde{i}}$ for agent $\tilde{i}$ will generally depend on the idiosyncratic shock $d\tilde{Z}_t^{\tilde{i}}$:

$$d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt - \tilde{\varsigma}_t^{\tilde{i}} d\tilde{Z}_t^{\tilde{i}}.$$

Return Processes. The return on capital managed by agent $\tilde{i}$ is now

$$dr_t^{K,\tilde{i}}(\iota) = \left(\frac{a - \mathcal{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K} \breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K} \right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}},$$

which differs from the return expression in the frictionless benchmark only by the inclusion of an additional idiosyncratic risk term $\tilde{\sigma} d\tilde{Z}_t^{\tilde{i}}$. The return on government liabilities is not directly affected by the presence of idiosyncratic capital shocks and therefore still given by the same expression as before:

$$dr_t^{\mathcal{MB}} = \left(\mu_t^{q,\mathcal{MB}} + \Phi(\iota_t) - \delta - \breve{\mu}_t^{\mathcal{MB}} \right) dt.$$

Optimal Investment and Goods Market Clearing. The optimal investment choice is precisely as in the frictionless benchmark and leads to the same Tobin's Q condition:

$$q_t^K = \frac{1}{\Phi(\iota_t)} = 1 + \phi \iota_t.$$

As before, we can combine this with goods market clearing and the definition of ϑ_t to express the equilibrium investment rate ι_t in terms of the nominal wealth share ϑ_t. After performing the same steps as for the previous model, we once again obtain Proposition 8.1.

Portfolio Choice. Once again, agents face a standard portfolio choice problem, so that we can directly apply the usual martingale conditions. These conditions are here given by

$$-\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,\mathcal{MB}} = \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = r_t, \tag{8.9}$$

$$\frac{a - \mathcal{G} - \iota_t}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,K} = \frac{\mathbb{E}_t[\mathrm{d}r_t^K]}{\mathrm{d}t} = r_t + \tilde{\varsigma}_t^{\tilde{i}}\tilde{\sigma}. \tag{8.10}$$

The new element, relative to before, is the presence of the term $\tilde{\varsigma}_t^{\tilde{i}}\tilde{\sigma}$ on the right-hand side of equation (8.10). This term captures an idiosyncratic risk premium that agents require to hold capital. Note that the left-hand side of equation (8.10) does not depend on any $\tilde{i}$-specific terms. Therefore, all agents choose portfolios that result in symmetric risk exposures to idiosyncratic shocks in the sense that the (shadow) price of idiosyncratic risk does not depend on the agent's identity $\tilde{i}$. We will therefore from now on suppress the $\tilde{i}$-superscript and just write $\tilde{\varsigma}_t$.

Next, let us again use $\frac{a-\mathcal{G}-\iota_t}{q_t^K} = \frac{\rho}{1-\vartheta_t}$ and $\frac{q_t^{\mathcal{MB}}}{q_t^K} = \frac{\vartheta_t}{1-\vartheta_t}$ to simplify the left-hand side of equation (8.10) and then subtract that equation from equation (8.9):

$$-\frac{\rho}{1-\vartheta_t} - \frac{1}{1-\vartheta_t}\breve{\mu}_t^{\mathcal{MB}} + \underbrace{\mu_t^{q,\mathcal{MB}} - \mu_t^{q,K}}_{=\frac{\mu_t^{\vartheta}}{1-\vartheta_t}} = -\tilde{\varsigma}_t\tilde{\sigma}.$$

Once again, we can solve this combined portfolio choice condition for the drift of ϑ_t,

$$\mu_t^{\vartheta} = \rho + \breve{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)\tilde{\varsigma}_t\tilde{\sigma}.$$

As in the frictionless benchmark, this equation should be interpreted as a condition on the expected rate of change in ϑ_t that is consistent with both portfolio choice and market clearing. Relative to the analogous equation in the previous subsection, the equation derived here contains an additional term, $(1 - \vartheta_t)\tilde{\varsigma}_t\tilde{\sigma}$, that corresponds to the required risk premium on capital scaled by the share of wealth that is held in capital $(1 - \vartheta_t)$. The implication of this term is that agents are content with a smaller appreciation rate of nominal wealth (relative to total wealth) in equilibrium if either the capital risk premium or the exposure of wealth to capital risk are larger.

The previous equation has been derived under the assumption that government liabilities have a positive value. But, as before, we prefer to rewrite this as an equation for $\mathbb{E}[\mathrm{d}\vartheta_t] = \mu_t^{\vartheta}\vartheta_t\mathrm{d}t$, as this turns out to remain valid even in the special case $\vartheta_t = 0$. Before stating this equation, let us also eliminate the price of idiosyncratic risk $\tilde{\varsigma}_t$ as follows. From $\xi_t^i \propto e^{-\rho t}\frac{1}{c_t^{\tilde{i}}} = e^{-\rho t}\frac{1}{\rho n_t^{\tilde{i}}}$, we know that $\tilde{\varsigma}_t = -\tilde{\sigma}_t^{n,\tilde{i}}$, where $\tilde{\sigma}_t^{n,\tilde{i}}$ denotes the idiosyncratic risk loading of $n_t^{\tilde{i}}$. From the net worth evolution (8.3), we observe further that $\tilde{\sigma}_t^{n,\tilde{i}} = (1 - \theta_t^{\tilde{i}})\tilde{\sigma} = (1 - \vartheta_t)\tilde{\sigma}$, where the last equality follows from asset market clearing and the fact that all agents choose identical portfolios.[7] We can therefore conclude that $\tilde{\varsigma}_t = (1 - \vartheta_t)\tilde{\sigma}$. In sum, the government liabilities valuation equation in this model takes the following form.

Proposition 8.3 (Government Liabilities Valuation Equation)**.** *The time path for the nominal wealth share $\vartheta_t \in [0, 1]$ satisfies the BSDE*

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t dt. \tag{8.11}$$

Once again, this equation is the key equation of the model. It characterizes the portfolio demand for nominal asset. Integrated forward in time, the equation can be

[7] That $\theta_t^{\tilde{i}}$ is the same across all $\tilde{i}$ is, in turn, a consequence of our previous observation that $\varsigma_t^{\tilde{i}}$ does not depend on $\tilde{i}$.

stated equivalently in the form

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \left(-\breve{\mu}_{t'}^{\mathcal{MB}} + (1-\vartheta_{t'})^2 \tilde{\sigma}^2 \right) \vartheta_{t'} \mathrm{d}t' \right].$$

The interpretation is that government liabilities in this model are valued for two distinct reasons. First, because of cash flows from fiscal backing, captured by the term $-\mu_t^{\mathcal{MB}} \vartheta_t = \frac{s}{q_t}$. The economic intuition is here the same as in the frictionless benchmark. Second, government liabilities are also valued because they represent a safe asset that facilitates idiosyncratic risk sharing. This is captured by the second term, $(1-\vartheta_t)^2 \tilde{\sigma}^2 \vartheta_t$. In Chapter 10, we provide a more careful account of this term and link it to the service flows generated from trading government liabilities in response to idiosyncratic shocks. Here, we merely note that the contribution of this safety feature to the nominal wealth share ϑ_t depends positively on the path of the expected future idiosyncratic net worth volatility $\tilde{\sigma}_t^n = (1-\vartheta_t)\tilde{\sigma}$ faced by households in equilibrium.

Steady State Equilibrium

In a steady state with positive nominal wealth share ϑ, the variable $\breve{\mu}_t^{\mathcal{MB}}$ must be constant over time and take on a value such that the right-hand side of equation (8.11) vanishes for some $\vartheta \in (0, 1)$. Possible values for $\breve{\mu}^{\mathcal{MB}}$ that are consistent with this requirement are

$$-\rho < \breve{\mu}^{\mathcal{MB}} < -\rho + \tilde{\sigma}^2. \tag{8.12}$$

For any such $\breve{\mu}^{\mathcal{MB}}$, the equation $\mathbb{E}_t[\mathrm{d}\vartheta_t] = 0$ is a third-order polynomial equation,

$$\left(\rho + \breve{\mu}^{\mathcal{MB}} - (1-\vartheta)^2 \tilde{\sigma}^2 \right) \vartheta = 0$$

with three mathematical solutions:

$$\vartheta = 0, \qquad \vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}, \qquad \vartheta = \frac{\tilde{\sigma} + \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}.$$

The third solution always exceeds 1 and is therefore inconsistent with equilibrium (it would imply a negative value of capital). The remaining two solutions represent valid steady-state equilibria. The first solution corresponds to a non-monetary equilibrium, in which government liabilities are worthless. In the second type of equilibrium, nominal government liabilities have a positive value, which is why we refer to these equilibria as "monetary" steady-state equilibria.

Once we have solved for the steady-state values of ϑ, we can use Proposition 8.1 and the government budget constraint to back out the implied asset prices, investment rate, and primary surplus. The following corollary summarizes all steady-state equilibria in this model.

Corollary 8.2 (Steady State in the Model with Uninsurable Idiosyncratic Risk)**.** *For any constant growth rate of nominal government liabilities in excess of interest payments $\breve{\mu}^{\mathcal{MB}}$ that satisfies condition (8.12), there is a "monetary" steady state in which both capital and government liabilities have positive value. In this steady state,*

$$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}, \qquad q^{\mathcal{MB}} = \frac{(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}})(1 + \phi\breve{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi\rho\tilde{\sigma}},$$

$$q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}(1 + \phi\breve{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi\rho\tilde{\sigma}}, \qquad \iota = \frac{\breve{a}\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} - \tilde{\sigma}\rho}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi\rho\tilde{\sigma}},$$

$$s = -\breve{\mu}^{\mathcal{MB}} \frac{(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}})(1 + \phi\breve{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi\rho\tilde{\sigma}}.$$

In addition, there is a unique "non-monetary" steady state, in which government liabilities are worthless. In this equilibrium,

$$\vartheta = 0, \qquad q^{\mathcal{MB}} = 0,$$

$$q^K = \frac{1 + \phi\breve{a}}{1 + \phi\rho}, \qquad \iota = \frac{\breve{a} - \rho}{1 + \phi\rho},$$

$$s = 0.$$

Observations

Let us revisit the observations from the frictionless benchmark model:

1. The value of money depends on fiscal backing and idiosyncratic risk. Both a

higher primary surplus (s) and higher idiosyncratic risk ($\tilde{\sigma}$) increase $q^{\mathcal{MB}}$ and ϑ. Furthermore, it is now possible for government liabilities to be valued even if the primary surplus is nonpositive ($s \leq 0 \Leftrightarrow \breve{\mu}^{\mathcal{MB}} \geq 0$), provided idiosyncratic risk is sufficiently high so that the upper bound in condition (8.12) is positive. Hence, fiscal backing is no longer necessary for money to be valued.

2. The value of capital assets (q^K), the investment rate (ι), and the growth rate of the economy ($g = \Phi(\iota) - \delta$) are all inversely related to the value of money. This conclusion is exactly the same as in the benchmark model as it does not depend on the portfolio choice and, hence, on the reason for money to be valued.

3. The conclusion that the nominal interest rate paid on money does not matter for the real allocation remains valid if interpreted in the following sense: holding either $\breve{\mu}^{\mathcal{MB}}$ or s fixed, a change in $i^{\mathcal{MB}}$ only changes the growth rate of nominal liabilities and the inflation rate. However, while there was only a single possible value for $\breve{\mu}^{\mathcal{MB}}$ consistent with a steady state in the frictionless benchmark, there is now a continuum of possible values, so we do not necessarily need to hold $\breve{\mu}^{\mathcal{MB}}$ fixed. Instead, we can consider, within the limits of condition (8.12), a different experiment of changing $i^{\mathcal{MB}}$ and $\breve{\mu}^{\mathcal{MB}} = \mu^{\mathcal{MB}} - i^{\mathcal{MB}}$ simultaneously while holding $\mu^{\mathcal{MB}}$ fixed. In this case, a rise in $i^{\mathcal{MB}}$ reduces $\breve{\mu}^{\mathcal{MB}}$, which leads to a portfolio reallocation to government liabilities (higher ϑ) and increases the value of money. This has real implications, e.g., for investment and growth (compare observation 2). The increase in the value of money can be traced to an increase in fiscal backing that must happen in the background: to satisfy its budget constraint with lower $\breve{\mu}^{\mathcal{MB}}$, the government increases primary surpluses (higher s).

Let us further discuss the first observation that money can be valued even in the absence of fiscal backing. For $s \leq 0$, the intrinsic value of government liabilities that arises from the payments made by the government is zero or even negative. However, in the monetary steady state, government liabilities still have positive value. If an asset's value exceeds its intrinsic (or fundamental) value, the asset is said to have a bubble component. In this sense, government liabilities can be a (rational) bubble in this model. The bubble can emerge for sufficiently large idiosyncratic risk, $\tilde{\sigma}^2 > \rho$, so that the upper bound in condition (8.12) is positive. Then, household are willing to trade government liabilities to self-insure against idiosyncratic risk even in the absence of any fundamental cash flows.

When there is a bubble, the government can extract seigniorage revenues by "mining the bubble". It is possible to run a permanent primary deficit, $s < 0$, which is funded by continuously increasing the stock of government liabilities in excess of interest payments, i.e., $\breve{\mu}^{\mathcal{MB}} > 0$. Note that this is possible only within limits as $\breve{\mu}^{\mathcal{MB}}$ must respect the upper bound in condition (8.12) in the monetary steady state. Bubble mining is essentially a tax on holding government liabilities as it dilutes the claim of existing holders of these liabilities to the aggregate bubble. Households react to this tax by tilting their portfolios towards capital, ϑ is falling in $\breve{\mu}^{\mathcal{MB}}$. When the government attempts to extract seigniorage too aggressively, no one is willing to hold government liabilities and only the non-monetary steady state remains.

8.2.4 Adding Monetary Frictions: Transaction Costs

We now complete the model by turning on also the monetary friction. We have not yet specified the transaction costs function $\mathfrak{T}_t$, so we start by providing more details on this part of the model.

Details on the Monetary Friction

Recall that output produced by $\tilde{i}$ net of investment and transaction costs is given by

$$(ak_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}}) k_t^{\tilde{i}})\mathrm{d}t$$

Here, $\nu_t^{\tilde{i}}$ denotes the output velocity of $\tilde{i}$'s government liability holdings, which is defined as

$$\nu_t^{\tilde{i}} := \frac{\mathcal{P}_t a k_t^{\tilde{i}}}{{}_t^{\tilde{i}}} = \frac{1 - \theta_t^{\tilde{i}}}{\theta_t^{\tilde{i}}} \frac{a}{q_t^K}, \tag{8.13}$$

where, in the first equation, ${}_t^{\tilde{i}}$ is the nominal value of $\tilde{i}$'s government liability holdings and $k_t^{\tilde{i}}$ represents the units of capital held by $\tilde{i}$. The second equality uses ${}_t^{\tilde{i}} = \mathcal{P}_t \theta_t^{\tilde{i}} n_t^{\tilde{i}}$ and

$k_t^{\tilde{i}} = \frac{(1-\theta_t^{\tilde{i}})n_t^{\tilde{i}}}{q_t^K}$.

$\mathfrak{T}_t$ is a transaction cost function that is increasing and convex in output velocity. Here, we assume the specific functional form

$$\mathfrak{T}_t(\nu) = \frac{a}{(\mathfrak{z}-1)\,\bar{\nu}}\left[\left(\frac{\nu}{\bar{\nu}}\right)^{\mathfrak{z}-1} - \left(\frac{\nu_t^{\mathrm{eq}}}{\bar{\nu}}\right)^{\mathfrak{z}-1}\right], \tag{8.14}$$

where $\bar{\nu}$ and $\mathfrak{z} > 1$ are parameters and ν_t^{eq} is the average output velocity chosen by all other agents in equilibrium. The term involving ν_t^{eq} is a formal trick that ensures that the aggregate transaction cost in equilibrium is always zero, which makes the model more tractable. In any case, a model of monetary frictions based on transaction costs as here is too crude to take its aggregate resource implications seriously. By eliminating these resource implications, we isolate the incentive distortion generated by the monetary friction, which is what we are ultimately interested in.

The parameter $\bar{\nu}$ controls the average level of velocity in equilibrium and thereby the severity of the monetary friction. Smaller values are associated with more severe monetary frictions as larger real balances of government liabilities need to be held to keep velocity small. If the monetary friction is "too severe", a pathological situation may arise in which no equilibrium with positive capital prices can exist. To exclude this case, we make the following parameter assumption throughout:

$$\bar{\nu} > \frac{\phi a}{1+\phi\check{a}}\rho^{1-1/\mathfrak{z}}. \tag{8.15}$$

As we will see below, the parameter $\mathfrak{z}$ plays the role of the inverse interest elasticity of money demand or, more precisely, demand for government liabilities as a medium of exchange. A notable limit case is when this demand becomes inelastic to rates of return, $\mathfrak{z} \to \infty$. In this limit case, the transaction cost specification is equivalent to a so-called cash-in-advance constraint, which requires households to hold government liability balances that are at least a certain proportion of their output,

$$\mathcal{P}_t a k_t^{\tilde{i}} \leq \bar{\nu}_t^{\tilde{i}}.$$

Remark: Alternative Models of Monetary Frictions

Before turning to the model solution, let us remark on alternative ways of modeling a medium of exchange role of money and explain why we have chosen this specific approach here. Common reduced-form approaches to monetary frictions are (i) transaction costs for expenditures, (ii) cash-in-advance constraints on expenditures, and (iii) money in the utility function.

In approach (i), certain expenditures made by certain agents are subject to transaction costs and these costs can be reduced by carrying larger money balances around. For example, in our model, consumption ($c_t^{\tilde{i}}$) and/or investment expenditures ($\iota_t^{\tilde{i}} k_t^{\tilde{i}}$) of households could be subject to transaction costs. These costs can be interpreted as costs incurred to agents that seek to make payments in other form than by direct settlement with money, e.g., costs of searching for sellers that accept other forms of payments or costs of negotiating credit arrangements. Alternatively, these costs can also capture transaction costs that arise from converting other assets into money when expenditures are made as in the classic Baumol-Tobin model (Baumol, 1952; Tobin, 1956).[8]

Similarly, in the cash-in-advance approach (ii), certain expenditures can only be made if the agent making them has a sufficient amount of money balances available in advance before entering the trade. In our model, this could take the form of a cash-in-advance constraint for consumption and/or investment expenditures made by households, for example $\mathcal{P}(c_t^{\tilde{i}} + \iota_t^{\tilde{i}} k_t^{\tilde{i}}) \leq \bar{v}_t^{\tilde{i}}$, if both types of expenditures require money. Cash-in-advance constraints are similar to transaction costs but replace a "soft" constraint in the form of a cost with a "hard" constraint that renders expenditures without sufficient money balances infeasible.

Finally, approach (iii), simply posits that agents derive utility from holding real money balances, so that money generates a utility service flow on top of its financial return. The money-in-utility specification is not to be taken literally but interpreted as a short-cut for an indirect utility function that would arise from various microfoundations of money demand. In our model, this would amount to replacing the flow

[8] In the Baumol-Tobin model, expenditures have to be made with cash. Carrying a large inventory of cash around avoids costly trips to the bank to withdraw additional cash.

utility $u(c_t^{\tilde{i}}) = \log c_t^{\tilde{i}}$ by a utility function $u(c_t^{\tilde{i}}, {}_t^{\tilde{i}}/\mathcal{P}_t)$ that depends positively on both consumption and real government liability balances ${}_t^{\tilde{i}}/\mathcal{P}_t$ held by this agent.

Our own approach is quite similar to approach (i) and, in the limit $\mathfrak{z} \to \infty$, approach (ii). The main difference is that transaction costs directly enter the production process instead of being tied to final expenditures. While this may seem to be even more ad-hoc than the conventional approaches, it has the formal advantage that the monetary friction only distorts the portfolio choice between monetary and productive assets (capital) but not directly the consumption-savings or physical investment margins. This makes the analysis cleaner and overall more tractable. At the same time, because, in equilibrium, total expenditures equal total production, the economic implications are not all that different in our approach.[9] Indeed, Exercise 8.5.1 asks you to work out a version of our model with a more conventional expenditure-based transaction cost/cash-in-advance constraint. This exercise reveals that both approaches are, in a certain sense, isomorphic in steady-state and make largely identical qualitative predictions with respect to comparative statics.

In addition to these reduced-form approaches, New Monetarist theories of money demand provide a more microfounded approach to the medium-of-exchange role of money based on search frictions and bilateral exchange. We do not discuss these approaches in these lecture notes but refer the interested reader to Williamson and Wright (2010a,b); Lagos et al. (2017) for survey papers and Choi and Rocheteau (2021) for an exposition in continuous time.

Model Solution

Prices, SDF, and Return Processes. All postulated price and SDF processes are precisely as in the previous version of the model,

$$dq_t^K/q_t^K = \mu_t^{q,K}dt, \qquad dq_t^{\mathcal{MB}}/q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}}dt, \qquad dq_t/q_t = \mu_t^q dt,$$

[9] The government is typically not thought to be affected by monetary frictions, so the portion $\mathcal{G}K_t$ of total expenditures would not be subject to these frictions, if frictions were directly tied to expenditures. However, because government expenditures are always a fixed fraction of output in this model, this detail is not important here.

$$d\vartheta_t/\vartheta_t, = \mu_t^{\vartheta} dt, \qquad d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt - \tilde{\varsigma}_t^{\tilde{i}} d\tilde{Z}_t^{\tilde{i}}.$$

The return on government liabilities is also as before as it is not directly affected by transaction costs,

$$dr_t^{\mathcal{MB}} = \left(\mu_t^{q,\mathcal{MB}} + \Phi(\iota_t) - \delta - \breve{\mu}_t^{\mathcal{MB}}\right) dt.$$

In contrast, transaction costs do affect the return on capital via the dividend yield paid to capital holders. Transaction costs themselves depend on velocity chosen by the agent. The return on capital managed by agent $\tilde{i}$ conditional on $\iota_t^{\tilde{i}} = \iota$ and $\nu_t^{\tilde{i}} = \nu$ is

$$\begin{aligned} dr_t^{K,\tilde{i}}(\iota, \nu) &= \left(\frac{a(1-\tau_t) - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K}\right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} \\ &= \left(\frac{a - \mathcal{G} - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K}\right) dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}}. \end{aligned}$$

Optimal Investment and Goods Market Clearing. The optimal investment choice is unaffected by transaction costs and therefore the same as in the previous model variants. Because of our assumption that $\mathfrak{T}_t(\nu_t^{\text{eq}}) = 0$, transaction costs in equilibrium are zero if all agents choose the same velocity $\nu_t^{\tilde{i}} = \nu_t^{\text{eq}}$ (to be verified below). Therefore, transaction costs also do not affect the goods market clearing condition in equilibrium. As a consequence, all derivations from the previous subsections remain valid. Once again, equilibrium investment and asset prices can be expressed in terms of the nominal wealth share ϑ_t. These relationships are as in Proposition 8.1, which also holds in this variant of the model.

Portfolio Choice. Solving for agents' portfolio choice in this model variant requires modifications to our previous approach. The presence of transaction costs means that agent $\tilde{i}$'s net worth evolution is no longer linear in $\theta_t^{\tilde{i}}$ because $\theta_t^{\tilde{i}}$ also enters the return on capital nonlinearly via the velocity $\nu_t^{\tilde{i}}$, compare equation (8.13). It is therefore not possible to use the martingale conditions. Instead, we derive first-order conditions explicitly using the stochastic maximum principle.

The Hamiltonian of an individual household is as follows. For ease of notation, let us in the following suppress $\tilde{i}$-superscripts.

$$H_t = e^{-\rho t}\log c_t - \xi_t c_t + \xi_t n_t \left((1-\theta_t)\frac{\mathbb{E}_t[\mathrm{d}r_t^K(\iota_t, \nu_t)]}{\mathrm{d}t} + \theta_t \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} \right) - \tilde{\varsigma}_t \xi_t n_t (1-\theta_t)\tilde{\sigma}$$

We maximize H_t with respect to θ_t, ν_t subject to the constraint (this is equation (8.13))

$$\theta_t \nu_t = (1-\theta_t)\frac{a}{q_t^K}.$$

Denoting the Lagrange multiplier by $\lambda_t^{\mathcal{MB}}\xi_t n_t$, the first-order conditions are:

$$\theta_t: \qquad \frac{\mathbb{E}_t[\mathrm{d}r_t^K(\iota_t, \nu_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = \tilde{\varsigma}_t\tilde{\sigma} + \lambda_t^{\mathcal{MB}}\left(\nu_t + \frac{a}{q_t^K}\right) \tag{8.16}$$

$$\nu_t: \qquad (1-\theta_t)\frac{\partial \mathbb{E}[\mathrm{d}r_t^K(\iota_t, \nu_t)]/\mathrm{d}t}{\partial \nu_t} + \lambda_t^{\mathcal{MB}}\theta_t = 0 \tag{8.17}$$

Condition (8.16) is similar to the martingale conditions that we have used in previous subsections. We will once again use it to derive a government liabilities valuation equation. To this end, observe that expected returns on capital and government liabilities are

$$\begin{aligned}
\frac{\mathbb{E}_t[\mathrm{d}r_t^K(\iota_t, \nu_t)]}{\mathrm{d}t} &= \overbrace{\frac{a - \mathcal{G} - \iota_t - \mathfrak{T}_t(\nu_t)}{q_t^K}}^{=\rho/(1-\vartheta_t)} + \overbrace{\frac{q_t^{\mathcal{MB}}}{q_t^K}}^{=\vartheta_t/(1-\vartheta_t)}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,K}, \\
\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} &= \qquad - \breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,\mathcal{MB}},
\end{aligned} \tag{8.18}$$

take the difference,

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^K(\iota_t, \nu_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = \frac{\rho}{1-\vartheta_t} + \frac{\breve{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t},$$

and plug this into the first-order condition (8.16):

$$\frac{\rho}{1-\vartheta_t} + \frac{\breve{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \tilde{\varsigma}_t\tilde{\sigma} + \lambda_t^{\mathcal{MB}}\left(\nu_t + \frac{a}{q_t^K}\right).$$

We can further simplify the expressions on the right. As in the previous model variant,

$\tilde{\varsigma}_t = (1-\vartheta_t)\tilde{\sigma}$. In addition, using equation (8.13) to replace a/q_t^K, we obtain

$$\nu_t + \frac{a}{q_t^K} = \nu_t\left(1 + \frac{\theta_t}{1-\theta_t}\right) = \frac{\nu_t}{1-\theta_t} = \frac{\nu_t}{1-\vartheta_t},$$

where the last equality imposes asset market clearing, $\theta_t = \vartheta_t$. Plugging these relationships into the first-order condition and solving for μ_t^ϑ yields

$$\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \lambda_t^{\mathcal{MB}}\nu_t.$$

To formulate the government liabilities valuation equation, we multiply as always by ϑ_t. In addition, let us also introduce a change in notation by defining $\Delta i_t := \lambda_t^{\mathcal{MB}}\nu_t$. While this is pure notation at this stage, we note that Δi_t represents the spread $i_t - i_t^{\mathcal{MB}}$ between the nominal interest rate i_t of a hypothetical nominally risk-free asset that does not provide transaction services and the nominal interest rate $i_t^{\mathcal{MB}}$ paid by the government. In Exercise 8.5.2, you are asked to introduce a zero net supply nominal bond that does not provide transaction services to the model and show that, indeed, $\Delta i_t = i_t - i_t^{\mathcal{MB}}$ in this extended model. Therefore, Δi_t can be interpreted as a liquidity premium earned by holding government liabilities, which reduces its required pecuniary return. With this notation, the government liabilities valuation equation in the full model with all frictions takes the following form.

Proposition 8.4 (Government Liabilities Valuation Equation)**.** *The time path for the nominal wealth share $\vartheta_t \in [0,1]$ satisfies the BSDE*

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \Delta i_t\right)\vartheta_t dt. \tag{8.19}$$

We observe that the monetary friction introduces a third reason for government liabilities to be valued, which enters through the liquidity premium term Δi_t. To characterize this term, let us turn next to the first-order condition for ν_t, equation (8.17). Taking the derivative of the expected capital return in equation (8.18) with respect to ν_t

and using the functional form assumption on $\mathfrak{T}_t(\nu)$ (equation (8.14)) yields

$$\frac{\partial \mathbb{E}[\mathrm{d}r_t^K(\iota_t, \nu_t)]/\mathrm{d}t}{\partial \nu_t} = -\frac{\mathfrak{T}_t'(\nu_t)}{q_t^K} = -\frac{a}{q_t^K}\frac{1}{\bar{\nu}^2}\left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}-2} = -\frac{\vartheta_t}{1-\vartheta_t}\frac{1}{\bar{\nu}}\left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}-1}$$

and plugging this expression and $\theta_t = \vartheta_t$ into the first-order condition (8.17) implies

$$\Delta i_t = \lambda_t^{\mathcal{MB}} \nu_t = \left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}}. \tag{8.20}$$

For interpretation, let us rewrite equation (8.20) by solving for velocity, $\nu_t = (\Delta i_t)^{\frac{1}{\mathfrak{z}}}\,\bar{\nu}$, plugging in the definition of velocity, equation (8.13), and aggregating across all agents. Using the first equality in equation (8.13) to eliminate velocity, we obtain:

$$\mathcal{P}_t Y_t = \underbrace{(\Delta i_t)^{\frac{1}{\mathfrak{z}}}\,\bar{\nu}}_{\nu_t} \mathcal{MB}_t.$$

This shows that a conventional version of the quantity equation of money holds in our model. Here, the quantity of "money" entering this equation is the quantity of all nominal government liabilities, $\mathcal{MB}_t$, and demand for these liabilities as a medium of exchange links their velocity to the opportunity cost of holding them, which is captures by Δi_t.

While the previous equation is instructive, for the purpose of solving the model it is more useful to keep equation (8.20) as an equation for Δi_t and substitute in the second equality in equation (8.13) for velocity. This gives us a second equation that relates ϑ_t and Δi_t, in addition to the government liabilities valuation equation. Because it has the same economic content, we also call the resulting equation the quantity equation.

Proposition 8.5 (Quantity Equation)**.** *In equilibrium, ϑ_t and Δi_t are linked by the quantity equation*

$$\Delta i_t = \left(\frac{1}{\bar{\nu}}\frac{1-\vartheta_t+\phi\rho}{\vartheta_t}\frac{a}{1+\phi\check{a}}\right)^{\mathfrak{z}}. \tag{8.21}$$

In the cash-in-advance limit $\xi \to \infty$, the equation remains valid if interpreted as

$$\Delta i_t \begin{cases} = 0, & \frac{1}{\bar{v}} \frac{1-\vartheta_t+\phi\rho}{\vartheta_t} \frac{a}{1+\phi\check{a}} < 1 \\ \in [0, \infty), & \frac{1}{\bar{v}} \frac{1-\vartheta_t+\phi\rho}{\vartheta_t} \frac{a}{1+\phi\check{a}} = 1 \end{cases}.$$

Equations (8.19) and (8.21) jointly characterize ϑ_t and Δi_t. For $\xi < \infty$, one can substitute the latter into the former to obtain a single equation for the nominal wealth share ϑ_t. However, we prefer to keep them separate, as this will be required in the cash-in-advance case $\xi \to \infty$.

Steady State Equilibrium

Let us once again assume that $\breve{\mu}_t^{\mathcal{MB}} = \breve{\mu}^{\mathcal{MB}}$ is constant and restrict attention to equilibria with constant $\vartheta > 0$. In steady state, the government liabilities valuation equation (8.19) and the quantity equation (8.21) take the forms

$$\rho + \breve{\mu}^{\mathcal{MB}} = (1-\vartheta)^2 \tilde{\sigma}^2 + \Delta i,$$
$$\Delta i = \left(\frac{1}{\bar{v}} \frac{1-\vartheta+\phi\rho}{\vartheta} \frac{a}{1+\phi\check{a}} \right)^{\mathfrak{z}}.$$

Let us first consider the case $\mathfrak{z} < \infty$. In this case, the two equations do not have a closed-form solution, except in the special case of vanishing transaction costs, $\bar{v} \to \infty$, in which case we obtain the same solution as in the previous subsection. However, we can still establish some theoretical properties of the solution:

Proposition 8.6 (Steady State in Full Model, Case $\mathfrak{z} < \infty$)**.** *Let $\mathfrak{z} < \infty$. For any constant $\breve{\mu}^{\mathcal{MB}}$ that satisfies the inequality*

$$\breve{\mu}^{\mathcal{MB}} > -\rho \left(1 - \left(\frac{\phi a}{1+\phi\check{a}} \frac{\rho^{1-1/\mathfrak{z}}}{\bar{v}} \right)^{\mathfrak{z}} \right),$$

there is a unique steady state equilibrium. In this equilibrium, both capital and government liabilities have positive value. The value of government liabilities $q^{\mathcal{MB}}$ and the nominal wealth share ϑ are strictly increasing in both the severity of financial frictions (larger $\tilde{\sigma}$) and monetary *frictions (lower $\bar{v}$), and they are strictly decreasing in the dilution rate of government liabilities, $\breve{\mu}^{\mathcal{MB}}$. Furthermore, a tightening of financial fictions (higher $\tilde{\sigma}$) raises the idiosyncratic risk premium $(1-\vartheta)\tilde{\sigma}$ and lowers the liquidity premium (Δi). In contrast, a tightening of monetary frictions (lower $\bar{v}$) lowers the idiosyncratic risk premium and raises the liquidity premium.*

Note that the condition on $\breve{\mu}^{\mathcal{MB}}$ imposes the same lower bound on $\breve{\mu}^{\mathcal{MB}}$ as condition (8.12) in the model without monetary frictions if $\phi = 0$. The lower bound is strictly tighter in the case $\phi > 0$, but assumption (8.15) ensures that it remains negative, so that $\breve{\mu}^{\mathcal{MB}} \geq 0$ is always sufficient for existence of a steady state. Unlike in condition (8.12) from the previous section, there is no longer any upper bound on $\breve{\mu}^{\mathcal{MB}}$ in the presence of the monetary friction assumed here. The proof of this proposition follows from analyzing the nonlinear equation that results from eliminating Δi from the system of two equations stated above. We provide the formal details in the appendix to this chapter.

Let us next consider the cash-in-advance case $\mathfrak{z} \to \infty$. In this case, there is a closed-form solution for the steady state. To derive it, let us make a case distinction:

1. The cash-in-advance constraint is binding:
   In this case, the quantity equation does not impose a tight restriction on Δi. Instead, it directly determines ϑ[10]

   $$\frac{1}{\bar{v}}\frac{1-\vartheta+\phi\rho}{\vartheta}\frac{a}{1+\phi\breve{a}} = 1 \Rightarrow \vartheta = \frac{\left(1+\phi\rho\right)a}{a+\left(1+\phi\breve{a}\right)\bar{v}}.$$

   In contrast, the liquidity premium Δi adjusts as needed to satisfy the steady-state government liabilities valuation equation for this value of ϑ,

   $$\Delta i = \rho + \breve{\mu}^{\mathcal{MB}} - (1-\vartheta)^2\tilde{\sigma}^2.$$

   We call this case the *medium-of-exchange regime* because here the monetary friction, which models the medium-of-exchange role of money, determines the equilibrium value of government liabilities.

[10] By our parameter assumption (8.15), this value is always smaller than 1, so it corresponds to a valid equilibrium solution.

2. The cash-in-advance constraint is slack:

   In this case, the quantity equation implies $\Delta i = 0$ and does not impose a tight restriction on ϑ. Instead, ϑ solves the government liabilities valuation equation with $\Delta i = 0$, just like in the previous sections

$$\rho + \breve{\mu}^{\mathcal{MB}} = (1-\vartheta)^2\tilde{\sigma}^2 \Rightarrow \vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}.$$

   We call this case the *store-of-value regime* because here the role of money as a store of value, due to fiscal backing and the financial friction, determines the value of government liabilities.

It is easy to see that the two regimes are mutually exclusive, except for the knife-edge case in which the predictions for ϑ are identical in both regimes. In the medium-of-exchange regime, it must be the case that $\Delta i \geq 0$. In terms of ϑ, this means

$$\vartheta \geq \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}},$$

where the right-hand side is the ϑ-value that would occur in the store-of-value regime. Similarly, in the store-of-value regime, it must be the case that $\nu_t \leq \bar{v}$. In terms of ϑ, this means

$$\vartheta \geq \frac{\left(1+\phi\rho\right)a}{a + \left(1+\phi\breve{a}\right)\bar{v}}.$$

Once again, the right-hand side is the value that would occur in the medium-of-exchange regime. In sum, the structure of steady-state equilibria is as follows:

Corollary 8.3 (Steady State in Full Model, Cash-in-advance Case)**.** *Consider the cash-in-advance model ($\mathfrak{z} \to \infty$). For any constant $\breve{\mu}^{\mathcal{B}} > -\rho$, there is a unique steady state. In this steady state, both capital and government liabilities have positive value. The nominal wealth share is given by*

$$\vartheta = \max\left\{ \underbrace{\frac{\left(1+\phi\rho\right)a}{a + \left(1+\phi\breve{a}\right)\bar{v}}}_{\text{medium-of-exchange regime}}, \underbrace{\frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}}_{\text{store-of-value regime}} \right\}.$$

The following table summarizes other equilibrium quantities of interest:

	Medium of Exchange	*Store of Value*
Δi	$\Delta i = \rho + \breve{\mu}^{\mathcal{MB}} - \left(\frac{\bar{v}+\phi\left(\breve{a}\bar{v}-a\rho\right)}{a+\left(1+\phi\breve{a}\right)\bar{v}}\right)^2 \tilde{\sigma}^2$	$\Delta i = 0$
$q^{\mathcal{MB}}$	$q^{\mathcal{MB}} = \frac{a}{\bar{v}}$	$q^{\mathcal{MB}} = \frac{(\tilde{\sigma}-\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}})(1+\phi\breve{a})}{\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}}+\phi\tilde{\sigma}\rho}$
q^K	$q^K = \frac{1+\phi\left(\breve{a}-a\rho/\bar{v}\right)}{1+\phi\rho}$	$q^K = \frac{\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}}(1+\phi\breve{a})}{\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}}+\phi\tilde{\sigma}\rho}$
ι	$\iota = \frac{\breve{a}-\rho(1+a/\bar{v})}{1+\phi\rho}$	$\iota = \frac{\breve{a}\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}}-\tilde{\sigma}\rho}{\sqrt{\rho+\breve{\mu}^{\mathcal{MB}}}+\phi\tilde{\sigma}\rho}$

Comparative Statics. Figure 8.1 plots the asset values $q^{\mathcal{MB}}$ and q^K (top panel) as well as the risk and liquidity premia (bottom panel) as a function of the idiosyncratic shock volatility $\tilde{\sigma}$, which captures the severity of the financial friction. For low idiosyncratic risk, the economy is in the medium-of-exchange regime. In this regime, an increase in idiosyncratic risk raises the idiosyncratic risk premium and lowers the liquidity premium but does not affect the asset values. For high idiosyncratic risk, the economy is in the store-of-value regime, in which the liquidity premium is zero, the idiosyncratic risk premium is constant, and a further increase in idiosyncratic risk raises the value of government liabilities and lowers the value of capital.

Figure 8.2 depicts the same variables but varies the maximum velocity $\bar{v}$, which is inversely related to the severity of the monetary friction. For low maximum velocity, the economy is in the medium-of-exchange regime. In this regime, both asset values and the risk and liquidity premia are locally sensitive to changes in maximum velocity. An increase in $\bar{v}$ raises the value of capital and the idiosyncratic risk premium while it lowers the value of government liabilities and the liquidity premium. Once $\bar{v}$ is sufficiently high, the cash-in-advance constraint no longer binds, so that the economy is in the store-of-value regime. In this regime, the equilibrium is locally insensitive to changes in maximum velocity.

Figures 8.3 and 8.4 depict comparative statics with respect to fiscal backing of government liabilities. Specifically, we measure fiscal backing by $-\breve{\mu}^{\mathcal{B}}$, which is equal to the ratio of primary surpluses to total liabilities, a measure of the real payout yield on government liabilities. The top panel in each figure depicts the nominal wealth share ϑ, while the bottom panel depicts the growth rate of the economy, g, and two real risk-free

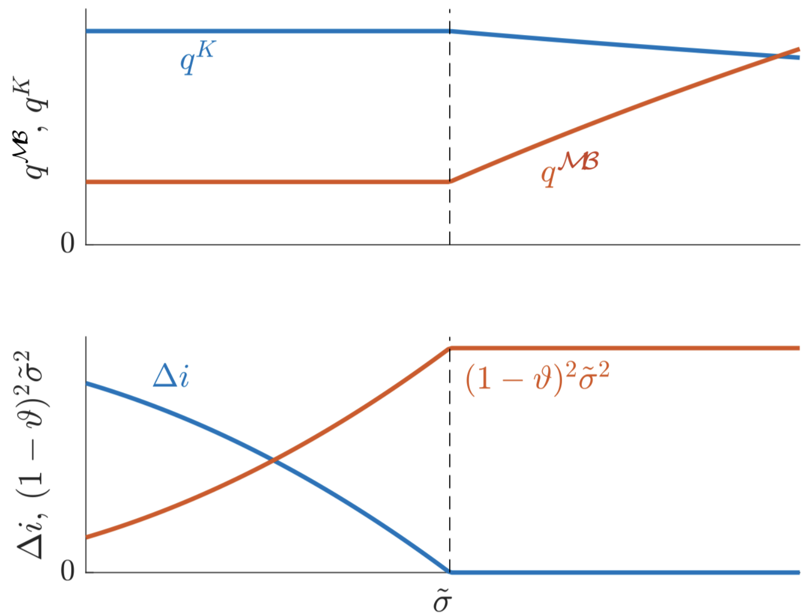


Figure 8.1: Comparative statics with respect to the financial friction ($\tilde{\sigma}$)

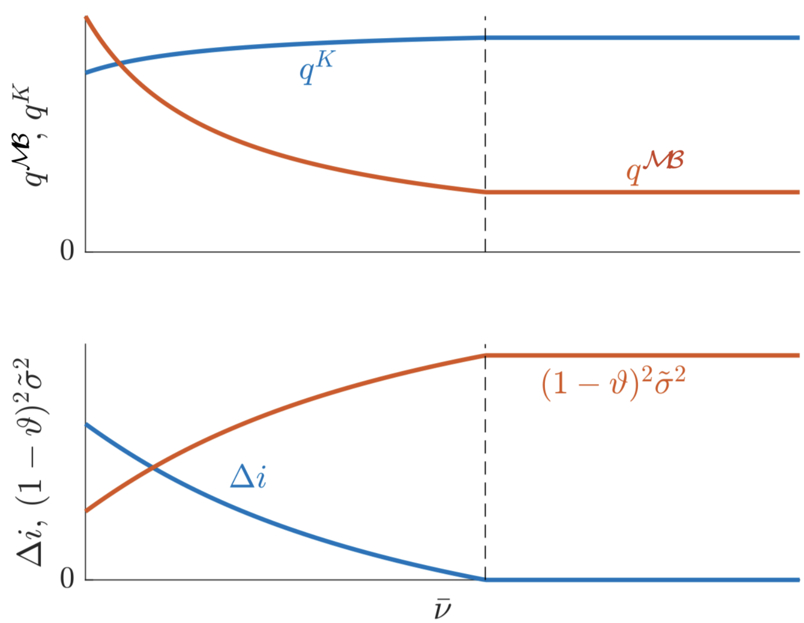


Figure 8.2: Comparative statics with respect to the monetary friction ($\bar{\nu}$)

interest rates, the return on government liabilities $r^{\mathcal{MB}}$, and the return on a hypothetical risk-free bond that does not provide transaction services. The nominal wealth share is locally insensitive to fiscal backing in the medium-of-exchange regime[11] but increasing in fiscal backing in the store-of-value regime, in line with our previous observations. Also in line with previous observations, the growth rate g is inversely related to the nominal wealth share ϑ, so it is decreasing in fiscal backing in the store-of-value regime. The risk-free rate r^f, in contrast, satisfies $r^f = g - \breve{\mu}^{\mathcal{MB}}$, which tends to be increasing in fiscal backing in both regimes, although the aforementioned growth effect in the store-of-value regime may overturn this behavior, which is visible at the right end of both figures. In the medium-of-exchange regime, this risk-free rate is different from the return on government liabilities, which is lower by the liquidity premium Δi. The latter is decreasing in fiscal backing and vanishes at the boundary to the store-of-value regime. In Figure 8.3 this boundary occurs when the real interest rate is below the growth rate, so that rational bubbles are possible in the store-of-value regime, like in the model variant from the previous subsection. However, it can also be the case that $r^f > g$ at the boundary between the two regimes. This happens in Figure 8.4, which is based on the same parameters as Figure 8.3 except that $\bar{v}$ has been reduced to make the monetary friction more severe. In this configuration, bubbles cannot exist in the store-of-value regime (even though $\tilde{\sigma}^2 > \rho$, so the bubble existence condition from the previous subsection is satisfied).

Revising Observations on the Value of Money and Sources of Seigniorage

Consider once again the government liabilities valuation equation in integral form:

$$\vartheta_t = \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(t'-t)} \left(-\breve{\mu}_{t'}^{\mathcal{MB}} + (1-\vartheta_{t'})^2 \tilde{\sigma}^2 + \Delta i_{t'} \right) \vartheta_{t'} \mathrm{d}t' \right].$$

This equation shows that in the full model there are three distinct sources of the value of money:

[11] The fact that ϑ does not depend at all on fiscal backing may appear a bit extreme as it implies that the fiscal authority can extract arbitrarily large seigniorage revenues. Indeed, this result only obtains in the limit case of a money demand that is inelastic to changes in rates of return, $1/\mathfrak{z} = 0$, but not for any positive interest elasticity of money demand $1/\mathfrak{z} > 0$, compare Proposition 8.6.

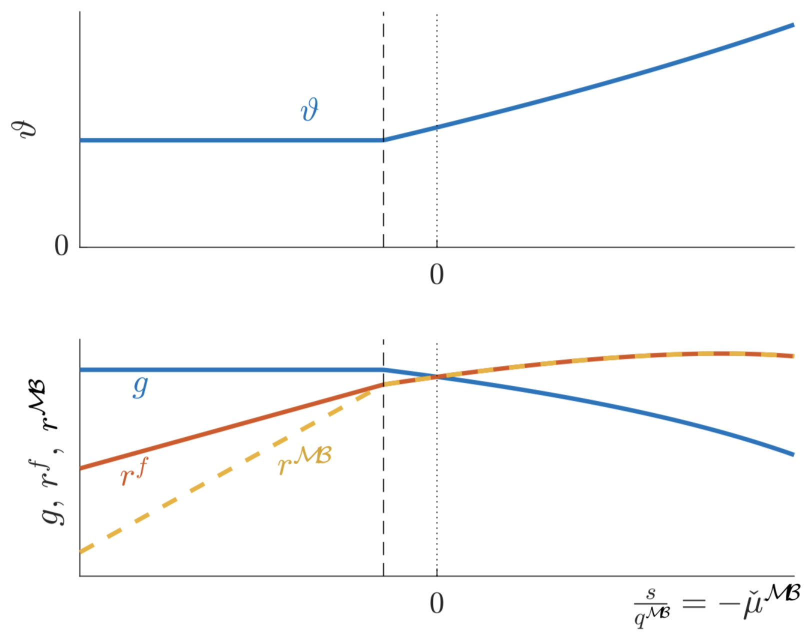


Figure 8.3: Comparative statics with respect to fiscal backing ($s/q^{\mathcal{MB}} = -\breve{\mu}^{\mathcal{MB}}$)

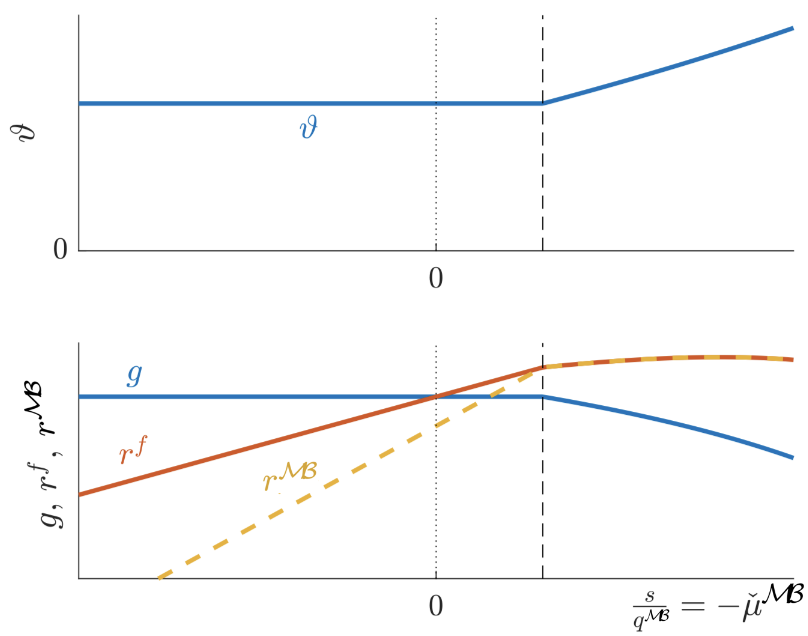


Figure 8.4: Comparative statics with respect to fiscal backing ($s/q^{\mathcal{MB}} = -\breve{\mu}^{\mathcal{MB}}$) for smaller $\bar{v}$

1. cash flows from fiscal backing;
2. risk sharing benefits from money as a safe asset (store of value);
3. transaction benefits from money as a medium of exchange.

Again, fiscal backing may actually be negative ($\breve{\mu}^{\mathcal{MB}} > 0$). In this case, government liabilities may nevertheless be valued if other benefits are sufficiently strong. Then, government liabilities are a (rational) bubble. The government can extract seigniorage revenue from issuing additional liabilities.

Money and Growth: Tobin Effect

We have observed from all three variants of the model that investment and growth depend negatively on portfolio demand for government liabilities (ϑ), regardless of the source for this demand. The intuition here is that government liabilities/money in *positive net supply* crowd out real investment.

By goods market clearing, investment is the difference between goods supply and demand for private and government consumption,

$$\iota_t K_t = Y_t - C_t - \mathcal{G} K_t = a K_t - \rho q_t K_t - \mathcal{G} K_t.$$

An increase in the portfolio demand for government liabilities (ϑ_t) does not affect goods supply but it increases private consumption demand via a wealth effect: for fixed capital price q_t^K, a higher ϑ_t raises total wealth per unit of capital $q_t = q_t^K + q_t^{\mathcal{MB}}$. Market clearing can be achieved by two margins of adjustment: (i) a fall in investment, or (ii) a fall in the capital price (which dampens the increase in wealth q_t). Unless capital is exogenously supplied ($\phi \to \infty$), at least some of the adjustment to equilibrium requires channel (i).[12]

[12]The goods market-centric intuition presented in this paragraph is fairly involved but has the advantage that it is correct for all variants of the model whereas other, seemingly simpler, stories are not. For example: (a) in the frictionless model, one might be tempted to trace the investment effect back to the capital tax, but the same effects arise in other model variants without capital taxes; (b) in model variants with $\phi > 0$, one might argue more directly that a lower capital demand depresses the capital price q_t^K and then ι_t via the Tobin's Q condition, but this logic breaks down for $\phi = 0$.

Our model thus formalizes an argument made by **?** that portfolio choice between monetary and capital assets is a key determinant of real investment. We call this link between demand for monetary assets and investment the *Tobin effect*. Note that the previous explanation of the Tobin effect relies on a wealth effect from government liabilities in positive net supply. The Tobin effect is one feature that distinguishes government-provided outside money from bank-created inside money, which is in zero net supply (see Merkel (2020) for more details).

8.2.5 Separating Money and Government Bonds

In the previous model, money and nominal government bonds are combined in the single government liability $\mathcal{MB}$. A more realistic case is that the government issues money $\mathcal{M}_t$ and nominal bonds $\mathcal{B}_t$ where both serve as a store of value but only the money component of government liabilities is a medium of exchange. In this section, we separate the two forms of government liabilities.

Isomorphism between $\mathcal{MB}$-model and $\mathcal{M}$-and-$\mathcal{B}$-model

It turns out that the model with two types of government liabilities is isomorphic to baseline model with the single $\mathcal{MB}$-liability up to a suitable transformation of parameters (and a small generalization of the transaction cost technology). In this sense, the extended model features the same economics as the model discussed thus far. For this isomorphism between the two models, we need to reinterpret variables as follows:

- We redefine the value of total nominal government liabilities as the sum of the value of money and the value of government bonds:

$$q_t^{\mathcal{MB}} = q_t^{\mathcal{M}} + q_t^{\mathcal{B}}.$$

- The variable ϑ_t retains the interpretation as the share of government liabilities in total wealth and is now defined as:

$$\vartheta_t = \frac{q_t^{\mathcal{M}} + q_t^{\mathcal{B}}}{q_t^{\mathcal{M}} + q_t^{\mathcal{B}} + q_t^K}.$$

- The average dilution rate of nominal government liabilities, denoted by $\breve{\mu}_t^{\mathcal{MB}}$, must now aggregate the dilution effects from both money and bonds. We define it as a weighted average based on the quantities of outstanding money ($\mathcal{M}_t$) and bonds ($\mathcal{B}_t$):

$$\breve{\mu}_t^{\mathcal{MB}} = \frac{\mathcal{M}_t \breve{\mu}_t^{\mathcal{M}} + \mathcal{B}_t \breve{\mu}_t^{\mathcal{B}}}{\mathcal{M}_t + \mathcal{B}_t} =: \vartheta_t^{\mathcal{M}} \breve{\mu}_t^{\mathcal{M}} + \vartheta_t^{\mathcal{B}} \breve{\mu}_t^{\mathcal{B}},$$

  where $\vartheta_t^{\mathcal{M}} := \frac{\mathcal{M}_t}{\mathcal{MB}_t} = \frac{q_t^{\mathcal{M}}}{q_t^{\mathcal{MB}}}$, $\vartheta_t^{\mathcal{B}} := \frac{\mathcal{B}_t}{\mathcal{MB}_t} = \frac{q_t^{\mathcal{B}}}{q_t^{\mathcal{MB}}}$ are the fractions of total government liabilities that are due to money and bonds, respectively.

The main difference between the baseline model with $\mathcal{MB}$-liabilities and the extended bond-and-money framework enters in the treatment of the transaction cost technology. In the extended model, velocity is defined in analogy to equation (8.13) but with m in place of , as only money provides transaction services:

$$v_t^{\tilde{i}} = \frac{\mathcal{P}_t a k_t^{\tilde{i}}}{m_t^i} = \frac{1 - \theta_t^{\tilde{i}}}{\theta_t^{\mathcal{M},\tilde{i}} \theta_t^{\tilde{i}}} \frac{a}{q_t^K},$$

where $\theta_t^{\mathcal{M},\tilde{i}}$ is the share of money holdings by agent $\tilde{i}$ as a fraction of that agent's overall government liability holdings (not as a fraction of total net worth). To be able to map this into the baseline model, in which all all government liabilities provide transaction services, we need to allow in that model for time-varying transaction benefits that reflect changes in the composition of the government liabilities portfolio in the bond-and-money model. We can do so by replacing the constant parameter $\bar{v}$ in the baseline model with a time-varying $\bar{v}_t$. This modification does not invalidate any of the previously derived results, except that we need to add time subscripts to $\bar{v}$ everywhere.

With this small modification and the previously introduced notation in place, we claim that the model with bonds and money is isomorphic to the baseline model with just one type of government liability ($\mathcal{MB}$) if we make the following parameter transformation:

$$\bar{\nu}_t[\mathcal{MB}\text{-model}] = \left(\vartheta_t^{\mathcal{M}}\right)^{1-1/\mathfrak{z}} \bar{\nu}_t[\mathcal{M}\text{-and-}\mathcal{B}\text{-model}].$$

Government Liabilities Valuation and Quantity Equations. Let us use this isomorphism to reformulate the key equations that govern the evolution of ϑ_t in a model with money and bonds under the assumption of a constant $\bar{\nu}$-parameter in the transaction cost technology. In this case, the government liabilities valuation equation becomes

$$\mu_t^{\vartheta} = \rho + \vartheta_t^{\mathcal{M}} \breve{\mu}_t^{\mathcal{M}} + (1 - \vartheta_t^{\mathcal{M}})\breve{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)^2 \tilde{\sigma}^2 - \vartheta_t^{\mathcal{M}} \Delta i_t$$

and the quantity equation becomes

$$\Delta i_t = \left(\frac{1}{\bar{\nu}} \frac{1 - \vartheta_t + \phi\rho}{\vartheta_t^{\mathcal{M}} \vartheta_t} \frac{a}{1 + \phi \breve{a}}\right)^{\mathfrak{z}}.$$

These are two equations in three unknowns, ϑ_t, $\vartheta_t^{\mathcal{M}}$, and Δi_t. To determine the equilibrium, we need to specify additional information on the conduct of fiscal and monetary policy. In the baseline model, the government merely controls τ_t and $\breve{\mu}_t^{\mathcal{MB}} = \mu_t^{\mathcal{MB}} - i_t^{\mathcal{MB}}$, which are linked by the government budget constraint (and $\mu_t^{\mathcal{MB}}$ and $i_t^{\mathcal{MB}}$ individually are only relevant for inflation dynamics). Now, in addition to taxes, which follow from the government budget constraint, the government can choose a subset of $i_t^{\mathcal{M}}$, $i_t^{\mathcal{B}}$, $\mu_t^{\mathcal{M}}$, $\mu_t^{\mathcal{B}}$, and, possibly, the initial condition $\vartheta_0^{\mathcal{M}}$. The dynamics of $\vartheta_t^{\mathcal{M}}$ are given by the forward equation

$$d\vartheta_t^{\mathcal{M}} = (1 - \vartheta_t^{\mathcal{M}})\vartheta_t^{\mathcal{M}}(\mu_t^{\mathcal{M}} - \mu_t^{\mathcal{B}})dt$$

and nominal rates $i_t^{\mathcal{M}}$ and $i_t^{\mathcal{B}}$ are linked by the equation

$$\Delta i_t = i_t^{\mathcal{B}} - i_t^{\mathcal{M}},$$

which the government needs to respect in its policy choice (in particular, it is not feasible to set $i_t^{\mathcal{B}}$ below $i_t^{\mathcal{M}}$). We discuss examples of policy choices below in Section 8.4.

Derivations

We have so far merely claimed that there is an isomorphism between the extended model and the baseline model. Let us now provide some derivations to justify this claim. For the sake of brevity, we focus in the following only on the aspects of the model solution steps that change in the money-and-bond model. All other derivation steps are as in the baseline model and therefore omitted.

Portfolio Choice: $\theta_t^{\tilde{i}}, \theta_t^{\mathcal{M},\tilde{i}}, \nu_t^{\tilde{i}}$-FOC

$$\begin{aligned} H_t =& e^{-\rho t}\log c_t - \xi_t c_t \\ &+ \xi_t n_t \left\{ (1-\theta_t)\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{dt} + \theta_t \underbrace{\left[(1-\theta_t^{\mathcal{M}})\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} + \theta_t^{\mathcal{M}}\frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt} \right]}_{\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt}:=} \right\} \\ &- \xi_t n_t \tilde{\varsigma}_t (1-\theta_t)\tilde{\sigma} \\ &+ \lambda_t^{\mathcal{M}} \xi_t n_t \left[\theta_t \theta_t^{\mathcal{M}} \nu_t - (1-\theta_t)\frac{a}{q_t^K} \right] \end{aligned}$$

First order conditions w.r.t:

$$\begin{aligned} \theta_t^{\tilde{i}}: &\quad \frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = \tilde{\varsigma}\tilde{\sigma} + \lambda_t^{\mathcal{M}}\left(\nu_t\theta_t^{\mathcal{M}} + \frac{a}{q_t^K}\right) \\ \theta_t^{\mathcal{M}\tilde{i}}: &\quad \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{\mathrm{d}t} = \lambda_t^{\mathcal{M}}\nu_t \\ \nu_t^{\tilde{i}}: &\quad (1-\theta_t)\frac{\partial \mathbb{E}[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]/\mathrm{d}t}{\partial \nu_t} + \lambda_t^{\mathcal{M}}\theta_t\theta_t^{\mathcal{M}} = 0 \end{aligned}$$

Recall Return Equation and Take Differences.

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{dt} = \frac{a - \mathcal{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K} + \frac{q_t^{\mathcal{M}}\breve{\mu}_t^{\mathcal{M}} + q_t^{\mathcal{B}}\breve{\mu}_t^{\mathcal{B}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} \tag{1}$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{dt} = \breve{\mu}_t^{\mathcal{B}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{B}}} = i_t^{\mathcal{B}} - \pi_t \tag{2}$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{dt} = \breve{\mu}_t^{\mathcal{M}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{M}}} = i_t^{\mathcal{M}} - \pi_t \tag{3}$$

Take difference (2) and (3):

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{\mathrm{d}t} = \Delta i_t$$

Take weighted sum of (2) and (3):

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = \underbrace{\vartheta_t^{\mathcal{B}}\breve{\mu}_t^{\mathcal{B}} + \vartheta_t^{\mathcal{M}}\breve{\mu}_t^{\mathcal{M}}}_{\breve{\mu}_t^{\mathcal{MB}}} + \vartheta_t^{\mathcal{B}}\breve{\mu}_t^{q^{\mathcal{B}}} + \vartheta_t^{\mathcal{M}}\breve{\mu}_t^{q^{\mathcal{M}}} + \Phi(\iota_t^{\tilde{i}}) - \delta \tag{4}$$

Take difference of (1) and (4)

$$\frac{a - \mathcal{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K} + \frac{1}{1-\vartheta_t}\breve{\mu}_t^{\mathcal{MB}} + \underbrace{\mu_t^{q^K} - \vartheta_t^{\mathcal{B}}\mu_t^{q^{\mathcal{B}}} - \vartheta_t^{\mathcal{M}}\mu_t^{q^{\mathcal{M}}}}_{=-\mu_t^{\vartheta}/(1-\vartheta_t)}$$

Government Liability Valuation Equation. Plug into FOC w.r.t. θ_t:

$$\underbrace{\frac{a - \mathcal{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K}}_{\substack{=\rho/(1-\vartheta_t) \\ \text{by goods-mkt clearing}}} + \frac{1}{1-\vartheta_t}\breve{\mu}_t^{\mathcal{MB}} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \underbrace{\tilde{\varsigma}_t\tilde{\sigma}}_{\substack{=(1-\vartheta_t)\tilde{\sigma}^2 \\ \text{by log utility}}} + \lambda_t^{\mathcal{M}}\underbrace{\left(\theta_t^{\mathcal{M}}\nu_t + \frac{a}{q_t^K}\right)}_{\substack{=\frac{\vartheta_t^{\mathcal{M}}}{1-\vartheta_t}\nu_t \\ \text{by volatility def}}}$$

Plug into FOC w.r.t. $\vartheta_t^{\mathcal{M}}$:

$$\Delta i_t = \lambda_t^{\mathcal{M}}\nu_t$$

Government Liability Valuation Equation:

$$\mu_t^{\vartheta} = \rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \vartheta_t^{\mathcal{M}}\Delta i_t.$$

Fiscal Theory of the Price Level (FTPL). The generalized FTPL identity reflects the present value of seigniorage and fiscal surpluses required to support the initial real value of government liabilities. The expression is:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0\left[\int_0^T e^{-r^f t} s_t K_t \,\mathrm{d}t\right] + \mathbb{E}_0\left[\int_0^T e^{-r^f t}\Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t}\,\mathrm{d}t\right] + \mathbb{E}_0\left[e^{-r^f T}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right],$$

where r^f denotes the international risk-free rate, $s_t K_t$ captures primary surpluses, and the second term reflects the inflation tax revenue associated with money's non-interest-bearing nature.

FTPL-Equation with $\mathcal{B}$ and $\mathcal{M}$. Money valuation equation for log utility $\gamma = 1$:

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t (\underbrace{\rho + \overbrace{g}^{\Phi(\iota)-\delta} - g - (1-\vartheta_t)^2 \tilde{\sigma}^2}_{=r^f - g} + \breve{\mu}_t^{\mathcal{MB}} - \vartheta^{\mathcal{M}} \Delta i_t)$$

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \vartheta_t N_t$$

$$\Rightarrow d\left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right) = \left(r^f - \cancel{g} + \breve{\mu}^{\mathcal{MB}} - \vartheta^{\mathcal{M}} \Delta i + \overset{\frac{d\mathcal{N}_t}{\mathcal{N}_t} = gdt}{\overset{\downarrow}{\cancel{g}}}\right) \left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right) dt$$

Integrate forward:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T e^{-r^f t} \underbrace{(-\breve{\mu}_t^{\mathcal{MB}} + \vartheta_t^{\mathcal{M}} \Delta_i) \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}}_{=sK_t + \frac{\mathcal{M}_t}{\mathcal{P}_t}\Delta_i} \mathrm{d}t + e^{-r^f T} \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right]$$

FTPL Equation: $\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0[\int_0^T e^{-r^f t} s_t K_t \mathrm{d}t] + \mathbb{E}_0[\int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t] + \mathbb{E}_0[e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}]$

FTPL-Equations with $\mathcal{B}$ and $\mathcal{M}$: joint and separately

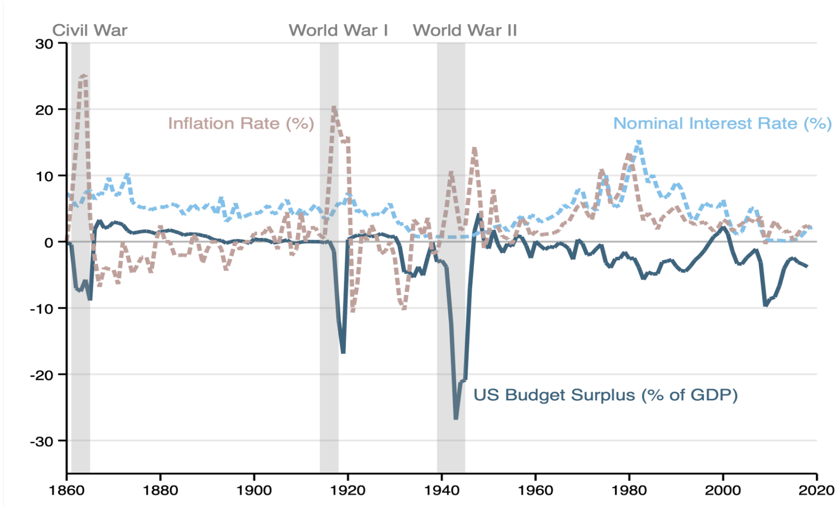
- Alternative way to write FTPL equation

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} s_t K_t \mathrm{d}t + \mathbb{E}_0 \int_0^T e^{-r^f t} \mu_t^{\mathcal{M}} \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{B}_T}{\mathcal{P}_T}$$

- Take difference btw two FTPL equations:

$$\frac{\mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} (\Delta i_t - \mu_t^{\mathcal{M}}) \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{M}_T}{\mathcal{P}_T}$$

(may contain bubble term when take $T \to \infty$)



Source: FRED, MeasuringWorth.com, Mitchell (1908)

8.3 Monetary-Fiscal Connection

Inflation–Fiscal Link.

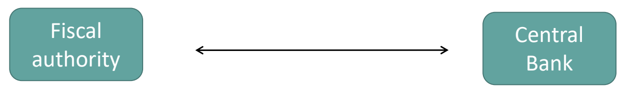
- Friedman (1961):"Inflation is always and everywhere a **monetary phenomenon**"
- Sims (1994): "In a fiat-money economy, inflation is a **fiscal phenomenon**, even more fundamentally than it is a monetary phenomenon".

Two Inflation-Fiscal Connection. There are two conceptually distinct channels through which fiscal policy can be linked to inflation: the **Fiscal Theory of the Price Level (FTPL) channel** and the **short-run aggregate demand channel**.

The **FTPL channel** emphasizes the role of government liabilities in determining the price level. When the government issues additional bonds to finance new fiscal stimulus but does not alter its path of future primary surpluses $s_t K_t$, the real value of total government liabilities must adjust. With more bonds in circulation and no change in the fiscal backing, the value of existing liabilities is diluted. This dilution manifests as an increase in the price level—that is, inflation.

In contrast, the **short-run aggregate demand channel** operates under a different set of assumptions. Suppose the government again issues additional bonds to fund a fiscal stimulus, but now commits to increasing future primary surpluses $s_t K_t$ sufficiently to prevent any dilution of bond values. In this case, the FTPL channel is effectively neutralized: the increased debt is fully backed by future fiscal resources. Whether this stimulus has real effects on output and inflation now depends on the structure of the underlying economic model. In a Ricardian framework, where agents internalize future taxes implied by higher $s_t K_t$, the stimulus may be offset and thus have little impact on current demand. However, in non-Ricardian settings such as New Keynesian models with nominal rigidities and a negative output gap, the fiscal stimulus can boost aggregate demand and output, potentially leading to inflation through conventional demand-side channels.

Fiscal and Monetary Interaction.



Monetary dominance: In a regime of monetary dominance, a tightening of monetary policy leads the fiscal authority to adjust by reducing the fiscal deficit accordingly.

Fiscal dominance: Under fiscal dominance, an increase in interest rates does not induce the fiscal authority to reduce the primary deficit. Instead, the result is often an increase in inflation, as the monetary authority accommodates fiscal imbalances.

Game of chicken: The interaction between monetary and fiscal authorities can be described as a strategic "game of chicken," where each side waits for the other to adjust policy. This is illustrated in the figure below.

For a visual explanation, see YouTube video 4 (starting at minute 4:15).

Sargent and Wallace (1981) Sargent and Wallace (1981) highlight the limits of monetary policy in controlling inflation, even in settings that satisfy traditional monetarist assumptions. In their framework, money $\mathcal{M}$ serves as a medium of exchange, but they assume $\tilde{\sigma} = 0$ to eliminate the possibility of speculative "bubble mining."

Sargent and Wallace famously argue that "even in an economy that satisfies monetarist assumptions [...] monetary policy cannot permanently control [...] inflation." Their analysis considers an environment where the price level $\mathcal{P}_t$ is fully determined by money demand, captured by the relationship $\nu \mathcal{M}_t = \mathcal{P}_t Y_t$. However, in their setting, the fiscal authority is dominant—it sets *deficits* independently of any actions taken by the monetary authority.

A central insight of the Sargent-Wallace framework is the emphasis on seigniorage revenue from money creation. They demonstrate that controlling inflation is not always within the central bank's power. Even when the price level is determined by money demand, the fiscal authority can assert dominance over inflation outcomes in the long run. The key mechanism is that fiscal needs ultimately determine the total present value of seigniorage. If the central bank limits money creation today, it will be forced to create more money in the future to finance ongoing deficits. The required fiscal backing does not disappear—the shortfall must eventually be addressed through future money printing.

This leads to the famous "unpleasant arithmetic" result: tight monetary policy in the present can result in higher inflation later.

To formalize this, suppose that in equilibrium:

1. The payment constraint is always binding.
2. The primary surplus satisfies $s_t = \underline{s}$, with $\underline{s} \leq 0$; that is, the government runs a constant primary deficit as a share of GDP.
3. The money demand parameter satisfies $\nu > \rho$, which holds under log utility.

Under these assumptions, nominal wealth shares must satisfy the following relationships.

From the goods market clearing condition, we obtain:

$$\vartheta_t \cdot \vartheta_t^{\mathcal{M}} = \frac{\rho}{\nu}.$$

Additionally, the value of bond liabilities relative to nominal wealth is determined by:

$$\vartheta_t \cdot \vartheta_t^{\mathcal{B}} = \int_t^{\infty} \rho e^{-\rho(t'-t)} (s_{t'} + \mathscr{s}_{t'})\, dt' = \underbrace{\underline{s}}_{<0} + \int_t^{\infty} \rho e^{-\rho(t'-t)} \mathscr{s}_{t'}\, dt',$$

where $\mathscr{s}_{t'}$ denotes the seigniorage revenue at future dates.

A Fiscally Dominant Regime After T**.** Suppose that at some finite time $T < \infty$, the fiscal authority takes control of the monetary growth rate $\mu_t^{\mathcal{M}}$. From this point onward, fiscal policy dictates the path of seigniorage to maintain a stable debt-to-GDP ratio. Formally, for all $t \geq T$, seigniorage is determined by the rule:

$$\mathscr{s}_t = \hat{\mathscr{s}}(\vartheta_T^{\mathcal{B}}) := -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}},$$

where $\underline{s} \leq 0$ is the constant primary deficit and $\hat{\mathscr{s}}(\cdot)$ captures the required seigniorage to stabilize debt. Although there are limits on feasible seigniorage due to inflationary consequences and money demand constraints, we ignore these limitations here for simplicity.

Before time T, the monetary authority retains independence and chooses a constant rate of money growth $\mu^{\mathcal{M}}$. This choice implies that seigniorage is directly controlled by the monetary authority and given by

$$\mathscr{s}_t = \mu^{\mathcal{M}} q_t^{\mathcal{M}} = \mu^{\mathcal{M}} \cdot \frac{a-g}{\nu} := \mathscr{s},$$

where a is the share of transactions requiring money, g is the growth rate of output, and ν denotes the output velocity of money that governs money demand.

This setting leads to the central result known as the "Unpleasant Arithmetic" proposition. It states that tight monetary policy today—reflected in a lower $\mu^{\mathcal{M}}$ over the interval $[0, T]$—leads to higher inflation in the future. More precisely, the (constant) inflation rate over the interval $[T, \infty)$ is strictly decreasing in the monetary growth rate $\mu^{\mathcal{M}}$ chosen over $[0, T]$. In other words, a more conservative stance by the monetary authority initially will necessitate a more aggressive inflationary policy in the long run due to fiscal constraints.

Why Does the Sargent–Wallace Proposition Hold? The Sargent–Wallace result follows from forward iteration of the government liabilities valuation equation. At time T, the market value of bond liabilities, relative to nominal wealth, must satisfy:

$$\vartheta_T \vartheta_T^{\mathcal{B}} = \vartheta_0 \vartheta_0^{\mathcal{B}} - \int_0^T \rho e^{-\rho t} (\underline{s} + \mathscr{s})\, dt.$$

Here, the term $\mathscr{s} = \mu^{\mathcal{M}} \cdot \frac{a-g}{\nu}$ denotes the constant seigniorage revenue generated over the period $[0, T]$. A lower value of $\mu_t^{\mathcal{M}}$ during this period implies lower seigniorage transfers and therefore less revenue to finance existing debt. As a result, the stock of debt increases more rapidly over time.

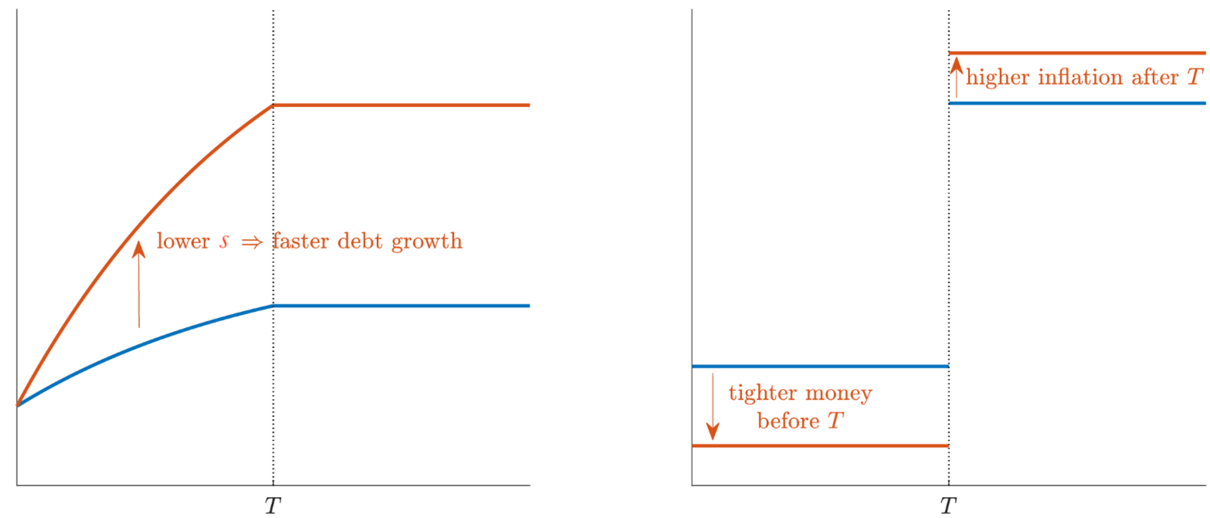
By time T, the higher accumulated debt implies a larger share of nominal wealth must be devoted to bond liabilities, i.e., $\vartheta_T^{\mathcal{B}}$ rises. Since post-T seigniorage is determined by the fiscal rule

$$\mathscr{s}_t = \hat{\mathscr{s}}(\vartheta_T^{\mathcal{B}}) = -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}},$$

which is increasing in $\vartheta_T^{\mathcal{B}}$, the government must raise more seigniorage revenue in the future to cover the interest burden on the higher debt. This is the essence of the Sargent–Wallace argument: tighter money today forces looser monetary policy tomorrow when fiscal policy is dominant, and therefore, inflation must eventually rise.

Illustration of Unpleasant Arithmetic. The figure below visually illustrates the core insight of the Sargent–Wallace "Unpleasant Arithmetic" proposition. It highlights the dynamic link between short-run monetary policy, debt dynamics, and long-run inflation in a fiscally dominant regime.

Real Value of Debt $(\vartheta_t, \vartheta_t^{\mathcal{B}}, q_t^{\mathcal{B}})$ Seigniorage and Inflation $(\mathscr{s}_t, \pi_t)$



On the left panel, we depict the real value of government debt, captured by the product $\vartheta_t \cdot \vartheta_t^{\mathcal{B}}$. The red curve represents a scenario in which the monetary authority adopts a tighter monetary stance over the interval $[0, T]$, meaning a lower money growth rate $\mu^{\mathcal{M}}$. This results in reduced seigniorage revenue during this period, represented by a smaller value of $\mathscr{s}$, and thus faster accumulation of real debt. The gap between the red and blue curves reflects the larger stock of liabilities that the government must finance at time T due to the initially conservative monetary policy.

On the right panel, we illustrate the path of inflation. The blue line corresponds to a more accommodative monetary policy with higher $\mu^{\mathcal{M}}$ before time T, while the red line reflects the tighter policy. Despite achieving lower inflation in the short run, the red path jumps to a higher inflation rate after time T. This increase is necessary because the fiscal authority, once in control after T, must generate more seigniorage to service the higher level of debt. As a result, the inflation rate adjusts upward in the long run.

Together, these panels illustrate the central message of the Sargent–Wallace framework: in the presence of fiscal dominance, tighter monetary policy today can backfire, leading to higher inflation tomorrow. Without sufficient fiscal backing, monetary policy alone cannot ensure long-run price stability.

Monetary Dominance. Consider a regime of monetary dominance, where the monetary authority remains in control of the money supply indefinitely—formally, $T = \infty$. In this case, a natural question arises: does an equilibrium exist?

Suppose that the seigniorage revenue $\mathscr{s}$ is not equal to the difference $\vartheta_0 \vartheta_0^{\mathcal{B}} - \underline{s}$, where $\underline{s} \leq 0$ denotes the constant primary deficit as a share of capital K_t. Then, a constant deficit policy $s_t = \underline{s}$ may no longer support equilibrium. However, this does not imply inconsistency per se—it simply reflects that maintaining a constant deficit may not be a feasible policy under monetary dominance.

Two distinct cases can arise:

1. If seigniorage exceeds the fiscal need, that is, if

$$\mathscr{s} > \vartheta_t \vartheta_t^{\mathcal{B}} - \underline{s},$$

then the government can feasibly maintain the constant deficit $s_t = \underline{s} < 0$. In this case, the fiscal authority passively adjusts by absorbing part of the money supply over time. As a result, the effective money supply is smaller than the nominal money stock $\mathcal{M}_t$. Consequently, inflation is determined by the fiscal authority's actions, since its absorption of money offsets monetary injections. For instance, if the real debt-to-capital ratio is held constant, then the resulting outcomes are observationally equivalent to the benchmark case in which $\mathscr{s} = \vartheta_0 \vartheta_0^{\mathcal{B}} - \underline{s}$.

2. In contrast, if seigniorage is insufficient to support the initial fiscal position, i.e.,

$$\mathscr{s} < \vartheta_t \vartheta_t^{\mathcal{B}} - \underline{s},$$

then the fiscal authority must raise primary surpluses over time in order to avoid default on its nominal bond obligations. In this scenario, the fiscal authority is effectively subject to an intertemporal budget constraint. A simple illustrative case is when the smallest feasible constant primary surplus is given by

$$s = \vartheta_0 \vartheta_0^{\mathcal{B}} - \mathscr{s},$$

ensuring that the government remains solvent over time by adjusting fiscal policy to match available seigniorage revenues.

This regime resembles a situation in which government debt behaves like foreign currency or real debt—very different from the setting envisioned under the Fiscal Theory of the Price Level (FTPL), where fiscal variables drive the price level and monetary policy adjusts passively.

8.4 Monetary Policy

Monetary Policy Implementation.

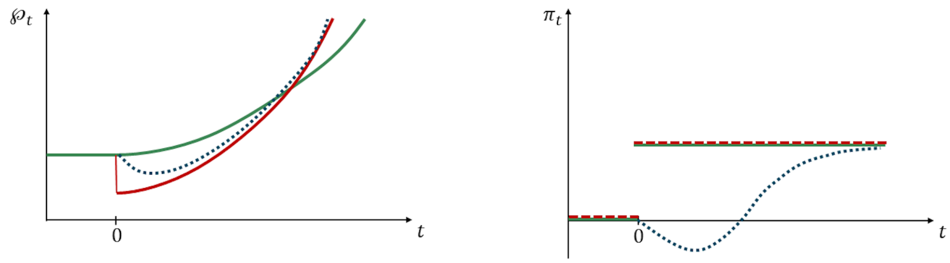
- **Interest on Reserves:**
  - Adjust $i_t^{\mathcal{M}}$, keep $\frac{\mathcal{M}}{\mathcal{M}+\mathcal{B}}$ constant
  - Implement **Neo-Fisherian policy**
- **Open Market Operation:**
  - Keep $i_t^{\mathcal{M}}$ constant, adjust $\frac{\mathcal{M}}{\mathcal{B}+\mathcal{M}}$
  - Implement **Monetarist policy**
    (mixed with some Neo-Fisherian elements since $i^{\mathcal{M}}$ and not $i^{\mathcal{MB}}$ is kept fixed)

8.4.1 Introducing Long-term Government Bonds

- Long-term bond
  - yields fixed coupon interest rate on face value $F^{(i,m)}$
  - Matures at random time with arrival rate $1/m$
  - Nominal price of the bond $P_t^{\mathcal{B}(i,m)}$
  - Nominal value of all bonds outstanding of a certain maturity:

$$\mathcal{B}_t^{(m)} = P_t^{\mathcal{B}(i,m)} F^{(i,m)}$$

  - Nominal value of all bonds $\mathcal{B}_t = \sum_m \mathcal{B}_t^{(m)}$



- Special bonds
  - $\mathcal{B}_t^{(0)}$, note $P_t^{\mathcal{B}(0)} = 1$ (price is independent of i_t since coupon is floating rate)
  - $\mathcal{B}_t^{(\infty)}$: Consol bond

Proposition Maturity composition of $\mathcal{B}^{(m)}$ is irrelevant for real allocation and equilibrium path of ϑ_t ... but it matters for nominal quantities, the price level and inflation.

- Modigliani-Miller intuition (in one sector model) (as s-backing is unchanged)

8.4.2 Sims' Stepping on the Rake: "Bond Reevaluation Effect"

- Unexpected permanent increase in $i_t^{(0)}$ at $t = 0$ for all $t > 0$
  $\Rightarrow$ nominal value $\mathcal{B}_t^{(m>0)}$ of any long-term bond declines
  "Pure *i*-MoPo": keep $\breve{\mu}^{\mathcal{MB}}$ constant, i.e., "debt growth" increases,
  ϑ_t is constant and so is $q^{\mathcal{B}}$ (aside $s_t/q_t^{\mathcal{B}}$ also stays constant)
  - At $t = 0$ on impact: as all $\mathcal{B}_0^{(m>0)}$ decline $\Rightarrow$ $\mathcal{P}_0$ has to jump down
  - For $t > 0$: inflation π_t is higher like in Neo-Fisherian setting
    (with price stickiness like dotted curve)

- In sum, "Stepping on the Rake" only changes inflation (price drop) at $t = 0$.
  . . . only with price stickiness (price drop down is smoothed out).

8.4.3 Quantitative Easing (QE) with T-Bills

- Assume $\mu_t^{\mathcal{M}} = \mu_t^{\mathcal{B}}$ for all t

- At $t = 0$ QE in form of an unexpected swap of $\mathcal{B}^{(0)}$-bonds (T-Bill) for money $\mathcal{M}$ T-Bill QE Proposition T-Bill QE leads to positive price level jump. Suppose $\mathcal{P}_t$ reacts less, so that real balances $\frac{\mathcal{M}_t}{\mathcal{P}_t}$ expand
  $\Rightarrow$ Relaxes CIA constraint and
  $\Rightarrow$ permanently lowers Δi (if CIA was binding beforehand)
  $\Rightarrow$ lowers "money seigniorage"
  $\Rightarrow$ upward jump in the price level (inflation) by

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} s_s K_s ds + \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds + \mathbb{E}_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$$

  The quantity equation (with fixed velocity) $\frac{\mathcal{M}_t}{\mathcal{P}_t} = \frac{C_t}{v}$ would also lead to upward jump of the price level.

8.5 Exercises

8.5.1 Alternative Money Demand Specifications

[to be added]

8.5.2 Illiquid Bonds and Δi_t as an Interest Rate Spread

[to be added]

8.5.3 Nominal Bonds as Safe Assets and Bubble Mining Seigniorage

Consider the following model, which is a variant of the money model covered in Lecture 10. The setup is as on Slide 7 of Lecture 10 with the following modifications:

- We call the liabilities issued by the government "bonds" and denote their nominal quantity by $\mathcal{B}_t$. Bonds are of infinitesimal duration and pay a floating nominal interest rate i_t.

- There are no transaction costs, $\mathfrak{T}_t \equiv 0$.
- There are no government expenditures, $\mathcal{G}_t \equiv 0$.
- Assume that the government holds the nominal bond growth rate $\mu^{\mathcal{B}}$ and the nominal interest rate i constant over time. Treat these two variables as parameters of the model.

In this problem, we focus on monetary steady state equilibria. You may therefore assume from the outset that $q^{\mathcal{B}}$ and q^K are positive and constant over time. Also, assume $\tilde{\sigma}^2 > \rho$.

1. Solve the model by performing the following steps:
   (a) Determine the return expressions for capital and bonds and simplify as much as possible using the assumptions provided above.
   (b) Characterize the optimal consumption, investment, and portfolio choice of households.
   (c) Combine the equations derived in (b) with market clearing to determine the equilibrium values for asset prices (q, q^K, $q^{\mathcal{B}}$), investment (ι), primary surpluses (s), the real risk-free rate, the output growth rate, and the inflation rate.
2. Investigate the potential for the government to extract seigniorage revenues from bubble mining due to the presence of idiosyncratic risk. For your answer, restrict attention to the special case $\phi \to \infty$ without physical investment. In this case, the capital stock grows at the exogenous rate $g = -\delta$.
   (a) Derive a relationship between real seigniorage revenue per unit of K_t, $\mathfrak{s} := -s$, and the policy variables i, $\mu^{\mathcal{B}}$. Explain intuitively why there is a Laffer curve.
   (b) Show that there is a unique maximum $\mathfrak{s}$, let's call it $\bar{\mathfrak{z}}$, and determine how $\bar{\mathfrak{z}}$ is affected by a change in $\tilde{\sigma}$?
   Hint: You can do this without taking any first-order conditions by arguing

that $\mathfrak{s}$ is concave and by investigating how it is affected by a change in $\tilde{\sigma}$ for fixed $\mu^{\mathcal{B}}$ and i.

(c) Seigniorage is often depicted as an "inflation tax". Show that there is a policy choice that extracts seigniorage $\mathfrak{s} = \bar{\mathfrak{z}}$ for arbitrarily large inflation rates. Explain intuitively why high inflation does not necessarily erode the government's ability to extract seigniorage revenue.

8.5.4 Monetary Policy in Model with Bonds and Money

Consider a variant of the model from Lecture 10 with money (nominal quantity $\mathcal{M}_t$, nominal interest rate $i_t^{\mathcal{M}}$), nominal short-term bonds (nominal quantity $\mathcal{B}_t$, nominal interest rate $i_t^{\mathcal{B}}$), idiosyncratic risk and transaction services, but no fiscal policy ($\tau_t = \mathcal{G}_t \equiv 0$). Transaction services are modeled by a cash-in-advance constraint in production,

$$\mathcal{P}_t a k_t^{\tilde{i}} \leq \bar{v} m_t^{\tilde{i}},$$

where $k_t^{\tilde{i}}$ are the capital holdings and $m_t^{\tilde{i}}$ are the nominal money holdings of agent $\tilde{i}$. Importantly, only money relaxes this constraint, bonds do not.

The government starts with initial nominal liabilities $\mathcal{B}_0, \mathcal{M}_0 > 0$, and it sets a constant interest rate on reserves $i^{\mathcal{M}}$ and constant growth rates of nominal liabilities, $\mu^{\mathcal{B}} = \mu^{\mathcal{M}}$, to balance the budget. The nominal rate on bonds, $i_t^{\mathcal{B}}$, is not directly set by the government but left free to clear the bond market.

Throughout, restrict attention to the (unique) monetary steady-state equilibrium. For each of the policy experiments to be analyzed below, you may assume without justification that the economy immediately jumps to a new steady state.

1. Suppose the government unexpectedly announces one of the following policies at $t = 0$. For each of these policies, explain how this affects (i) the equilibrium value of ϑ, (ii) the equilibrium interest rate on bonds, $i^{\mathcal{B}}$, and inflation rate, π, (iii) the initial price level $\mathcal{P}_0$. Assume in all cases that the economy is initially in the "medium of exchange regime" (in which $\Delta i > 0$, analogous to left-hand column

on Slide 32) and the policy change is sufficiently small such that the economy remains in this regime.

- Policy 1: one-time Helicopter drop of money:
  The government prints $\Delta\mathcal{M} > 0$ units of money at $t = 0$ and distributes them lump-sum to households. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
- Policy 2: one-time Helicopter drop of bonds:
  The government prints $\Delta\mathcal{B} > 0$ units of bonds at $t = 0$ and distributes them lump-sum to households. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
- Policy 3: open market operation/bond auction:
  The government sells $\Delta\mathcal{B} > 0$ bonds and uses the proceeds to take money out of circulation. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
- Policy 4: increasing interest on reserves:
  The government raises the reserve rate $i^{\mathcal{M}}$ to a permanently higher level $i^{\mathcal{M}\prime}$. It adjusts the growth rate of nominal liabilities as needed to satisfy its budget constraint.

2. How do the answers to the previous question change if the economy is in the "store of value regime" (in which $\Delta i = 0$)? Which of the policies, if sufficiently aggressive, may move the economy out of this regime into the "medium of exchange regime"?

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Appendix: Omitted Proofs

[to be added]

Chapter 9

Price Level Determination

9.1 Introduction / Overview

One of the core questions of monetary economics is the question of price level determination, i.e., which economic forces determine the general level of nominal goods prices. Equivalently, price level determination is about the determinants of the value of money: if $\mathcal{P}_t$ is the dollar price of the consumption goods basket, then $1/\mathcal{P}_t$ is the real value of a single dollar bill. Note that, because (goods price) inflation is defined as the growth rate of the price level, a theory of price levels necessarily encompasses a theory of inflation.[1]

If money or, more generally, nominal assets are in positive net supply in the economy, then determining the price level is also equivalent to determining the value of the outstanding stock of nominal assets. Indeed, most existing theories of price level determination are valuation theories for the stock of nominal assets. These theories determine the real value of nominal assets from fundamental economic considerations. We obtain a prediction for the price level by multiplying the real value of nominal assets with the outstanding nominal quantities of the assets. For the sake of concreteness, we will in the following always consider a setting in which money is the only nominal as-

[1] If we know the time path of price levels, we can take the time derivative to obtain the time path of inflation rates. The converse it not true. A given time path of inflation rates determines the time path of price levels only up to an integration constant. In this sense, a theory of inflation may not necessarily imply a theory of price levels.

set. But we remark that most of the considerations in this chapter apply more generally. In a money-only model, the primary purpose of the valuation theory is to determine the real value of money balances, $M_t := \mathcal{M}_t/\mathcal{P}_t$. In a second step, we can obtain a price level prediction by multiplying by the (dollar) quantity of money in circulation, i.e., $\mathcal{P}_t = M_t \mathcal{M}_t$.

Broadly speaking, there are three classes of monetary theories that derive the value of money from different considerations:

1. *Backing theories* derive the value of money from the fundamental cash flows that back it. Specifically, money is valued as a store of value because there is an entity, typically the government, that backs the stock of money with real resources or assets. This could take the form of an explicit commitment to exchange money for something else according to a pre-specified exchange rate, as for example under a gold standard or a currency board. But the backing does not have to take the form of an explicit commitment to convert money on demand. It could also arise implicitly if the backing entity expends real resources to redeem outstanding money from circulation.

   The most important backing theory is the *Fiscal Theory of the Price Level* (FTPL). In this theory, the backing entity is the government, the notion of money is a broad one, encompassing all nominal government liabilities, and it is recognized that, by the government budget constraint, the real cash flows backing nominal government liabilities are ultimately the primary fiscal surpluses, that is tax revenues in excess of government expenditures other than those expenditures that are used to service nominal debt.

2. In *bubble theories*, trading money generates certain benefits to its users that help them overcome market frictions. The value of money derives from the value of these benefits as opposed to the value of fundamental cash flows that back the money. We can distinguish two types of bubble theories that differ with regard to the nature of the frictions and the role of money they emphasize:

   (a) The benefits from trading money can alleviate *intertemporal* financial frictions. For instance, in the presence of incomplete market frictions that restrict the set of assets available for moving resources across time or states of nature, trading money may help agents to partially complete the market. These types of bubble theories of money emphasize the *store of value role* of money. In particular, money is not necessarily special. Trading assets with similar payoff characteristics as money could serve the same function.

(b) Money can also more directly facilitate *intratemporal* exchange, for example because certain goods or services need to be paid for immediately with money. Frictions of this type are referred to as monetary frictions. Bubble theories based on monetary frictions emphasize the *medium of exchange role* of money. These theories play a central role in Monetarist and New Monetarist economics.

Note that, unlike for the previous category of bubble theories, the notion of money here is a narrow one. It includes only those assets that provide medium of exchange services. Other assets with similar payoff characteristics such as short-term nominal debt do not serve the same function.

In either instance, these are "bubble" theories because money has small or no intrinsic value that arises from holding it and just consuming the cash flows received from the issuer. Instead, the value of money derives from the benefit in trade with other users of the money. The value is "bubbly" insofar as those benefits only realize if other users value money and they will typically only do so because they themselves believe that another one will value the money down the line when they wish to spend it. In this sense, the value of the money is supported by an infinite chain of beliefs that others will find it useful and therefore pay more than its intrinsic value for it in the future.

Note that the distinction between the two types of categories of bubble theories is useful but not always fully sharp. A pure-type monetary intratemporal friction is a theoretical limit case that is unlikely to exist in practice because all types of exchange are separated in time, at least to some extent. If this is the case, money is always also a store of value.

3. Money can also be a *pure unit of account* if the nominal price level matters because some contracts are written in nominal terms but there is no money or other nominal asset in positive net supply. Unlike in the previous two cases, the price level cannot be determined from a valuation theory of the outstanding money stock if the latter is in zero net supply. And indeed, such a situation typically leads to *price level indeterminacy* because the price level is merely a relative price between goods and some abstract valuation unit that does not have a physical counterpart in positive supply. Nevertheless, it turns out that even in this case the price level or, at least, the inflation rate, can remain determinate (in a certain sense) if policy makers control the nominal interest rate and set interest rates in a particular way, according to the so-called "Taylor principle". Most of the New Keynesian literature relies on the Taylor principle to determine the price level or, at least, the inflation rate.

   Note that the zero net supply assumption is a theoretical limit case that is typically not taken literally. Instead, it is often thought to be an approximation of a situation in which money plays a negligible role, referred to as a "cashless limit".

In the following sections, we discuss each type of theory in the context of our one-sector money model from the previous chapter.

9.2 Price Level Determination in One-sector Money Model when Money Is in Positive Supply

In the last chapter we have focused on situations in which money is in positive net supply. We first use special cases of that model to illustrate backing and bubble theories of money. We provide an illustration of money as a pure unit of account in the next section.

9.2.1 Remark: The Role of Goods Market Clearing and Wealth Effects

If money is in positive net supply, it makes sense to separate the question of valuation of money into two conceptual steps. In a first step, portfolio choice between monetary and real assets determines the relative valuation between the two. We may capture this relative valuation by the share of total wealth in the economy that is due to monetary assets, denoted by ϑ_t. Once we have determined a relative valuation ϑ_t, there is a one-to-one relationship between the real value of money, $\mathcal{M}_t/\mathcal{P}_t$, and total private-sector wealth, N_t, namely $N_t = \vartheta_t^{-1}\mathcal{M}_t/\mathcal{P}_t$. In particular, the absolute level of total wealth is inversely related to the price level. Due to wealth effects, total consumption demand is then typically also inversely related to the price level. E.g., for log utility,

$$C_t = \rho N_t = \frac{\rho}{\vartheta_t}\frac{\mathcal{M}_t}{\mathcal{P}_t}.$$

In a second step, we therefore obtain a prediction for the price level by clearing the goods market.

We remark that the primary difference across monetary theories, at least those that rely on the valuation of a positive stock of nominal assets, lies in the first step, in the source of the *portfolio demand* for monetary assets, which determines ϑ_t. The second step, the link between the price level and consumption demand via *wealth effects* from nominal assets, is common to all theories. In the following, we will therefore discuss how ϑ_t is determined with the understanding that a unique prediction for $\vartheta_t > 0$ will translate into a unique prediction for $\mathcal{P}_t$ via goods market clearing.

9.2.2 Backing Money with Primary Surpluses: The Fiscal Theory of the Price Level

Let us first consider a simplified setting without idiosyncratic risk ($\tilde{\sigma} = 0$) and without transaction costs ($\bar{v} \to \infty$, which implies $\Delta i_t = 0$). Furthermore, suppose that the government sets the tax rate τ_t to a level that generates a positive and constant primary surplus per unit of capital, $s_t = s > 0$.[2] Recall the government budget constraint,

$$\underbrace{(\mu_t^{\mathcal{M}} - i_t)}_{=\breve{\mu}_t^{\mathcal{M}}} \mathcal{M}_t = -\mathcal{P}_t s K_t < 0.$$

This equation tells us that the government uses real resources from taxation to make payments to money holders, either in the form of interest payments (if $i_t > 0$) or by redeeming some of the existing money stock (if $\mu_t^{\mathcal{M}} > 0$). In either case, these payments represent a positive real cash flow to the representative money holder and lead to a positive portfolio demand for money.

Indeed, we can write the government liability valuation equation here as follows:[3]

$$\frac{d\vartheta_t}{dt} = (\rho + \breve{\mu}_t^{\mathcal{M}})\vartheta_t = \rho\vartheta_t - s\frac{1 - \vartheta_t + \phi\rho}{1 + \phi\breve{a}} \tag{9.1}$$

where the last equality uses $\breve{\mu}^{\mathcal{M}} = -\frac{s}{q_t^M} = -\frac{s}{\vartheta_t}\frac{1-\vartheta_t+\phi\rho}{1+\phi\breve{a}}$. Observe that the right-hand side of equation (9.1) vanishes at exactly one value, for

$$\vartheta_t = \vartheta^{ss} := s\frac{1 + \phi\rho}{\rho(1 + \phi\breve{a}) + s},$$

which is a value between 0 and 1.[4] Therefore, if we restrict attention to stationary equilibria, then $\vartheta_t = \vartheta^{ss}$ is the only possible equilibrium solution under the constant tax policy considered here. By following the solution steps from the previous chapter to complete the equilibrium characterization, we could also show that $\vartheta_t = \vartheta^{ss}$ indeed corresponds to a valid equilibrium with a positive demand for money.

Furthermore, the solution $\vartheta_t = \vartheta^{ss}$ is not just the only possibility in a stationary equilibrium but, in fact, the only possibility in any equilibrium. To see this, note that the right-hand side of equation (9.1) is strictly increasing in ϑ_t, so that ϑ^{ss} is an unstable steady state in the forward dynamics described by this equation: if, for a solution, ever

[2] By the government budget constraint, the required tax rate is also constant and given by $\tau = \frac{\mathcal{G}+s}{a}$.

[3] We write the equation for the absolute rate of change $d\vartheta_t/dt$ instead of the relative change μ_t^{ϑ} because the latter is not defined if $\vartheta_t = 0$ but this may actually be a valid equilibrium outcome.

[4] It is clear that $\vartheta^{ss} > 0$ for $s > 0$. To see that $\vartheta^{ss} < 1$, note that $\tau = \frac{\mathcal{G}+s}{a} < 1$ as otherwise agents would want to dispose all capital in the economy. This inequality implies $\breve{a} = a - \mathcal{G} > s$, so that the denominator must be bounded below by $\rho + s(1 + \phi\rho)$.

$\vartheta_t > \vartheta^{ss}$, then also $\vartheta_{t'} > \vartheta^{ss}$ for all $t' \geq t$ and $\vartheta_{t'}$ increases without bounds; if ever $\vartheta_t < \vartheta^{ss}$, then also $\vartheta_{t'} < \vartheta^{ss}$ for all $t' \geq t$ and $\vartheta_{t'}$ decreases without bounds. This means that no solution different from $\vartheta_t = \vartheta^{ss}$ can satisfy $\vartheta_t \in [0,1]$ for all t. But the latter is a necessary requirement for equilibrium as otherwise agents would find it optimal to dispose either their money or their capital holdings.[5]

We have therefore established that fiscal backing by constant (relative to capital) and positive primary surpluses leads to a unique equilibrium prediction for the nominal wealth share ϑ_t and, by the argument in the previous subsection, to a unique equilibrium prediction for the path of nominal prices $\mathcal{P}_t$. This is a simple illustration of the FTPL. We remark that the assumption of a constant s is not essential but has been made for simplicity only. We would have arrived at the same conclusion if s_t was time-varying, except that then ϑ_t would, of course, be time-varying, too. In contrast, the assumption of a positive s is important.[6] For $s \leq 0$, equation (9.1) does not have any positive solutions that remain inside the interval $[0,1]$ at all times. Conceptually, this makes sense: if money is valued solely because of the backing provided by the government, then money should not have any value if the backing cash flows are zero or the government even attempts to make them negative.[7]

9.2.3 Money Bubbles I: Money as a Safe Store of Value

Let us next assume that idiosyncratic risk is positive and sufficiently large, $\tilde{\sigma}^2 > \rho$, but set both primary surpluses and transaction costs to zero, that is $s = 0$ and $\bar{v} \to \infty$. The latter assumptions imply $\breve{\mu}_t^{\mathcal{M}} = 0$ and $\Delta i_t = 0$, so that the government liability valuation equation simplifies to

$$\frac{d\vartheta_t}{dt} = \left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t. \tag{9.2}$$

[5] We do not even need to assume that disposal is an explicit option. For money, holding money indefinitely is as good as disposing it. For capital, agents can dispose of their capital holdings arbitrarily fast by choosing disinvestment $\iota_t \to -1/\phi$.

[6] If surpluses are time-varying, then the relevant assumption would not be that $s_t > 0$ at all dates but rather that the present value of future surpluses going forward must remain positive at all times.

[7] In the former case, $s = 0$, the nominal wealth share and the price level remain determinate but money is not valued, i.e. $\vartheta_t = 0$ and $\mathcal{P}_t = \infty$ for all t. In the latter case, $s < 0$, there is no equilibrium at all, rendering this an infeasible policy specification.

Again, we are looking for solution paths to this equation that remain contained in the interval $[0,1]$ as these, and only these, correspond to valid model equilibria.

We first discuss stationary equilibria and defer an analysis of the structure of all possible equilibria to later. Observe that there are two possible stationary equilibrium solution, namely $\vartheta_t = 0$ and $\vartheta_t = \vartheta^{ss} := \frac{\tilde{\sigma}-\sqrt{\rho}}{\tilde{\sigma}}$.[8] The first solution leads to the *no bubble equilibrium* in which money is not valued. The second solution is the *monetary steady state*, the unique stationary equilibrium in which money has a positive value. In the analysis of the previous chapter, we have implicitly selected this monetary steady state. In this sense, ϑ_t is not uniquely determined by the economic forces present in a rational expectations equilibrium of our model. Instead, additional assumptions on the coordination of equilibrium beliefs are required to select the monetary steady state.

This is a common feature of bubble theories of money that distinguishes them from backing theories like the FTPL. In the latter, simply holding a constant fraction of the money stock generates a positive stream of cash flows from the interest payments and repurchases of money by the issuer. This ensures a positive intrinsic value of money. Here, in contrast, the cash flows received from holding a constant fraction of the money stock are zero. The only way agents can derive benefits from money is by trading it with other agents in the economy. And indeed, if money is valued, then retrading money allows agents to partially share idiosyncratic risk, thereby alleviating the incomplete markets frictions. We analyze and explain the service flows generated from this retrading activity in more detail in Chapter 10. But importantly, an agent will only see a benefit in selling money to another agent in the future if she expects the other agent to value money as well. Whether service flows from retrading materialize therefore depends on social coordination.

Consider next the set of all possible equilibria, that is all solutions to equation (9.2) that remain confined inside the interval $[0,1]$. The right-hand side of (9.2) is negative for all $\vartheta_t \in (0, \vartheta^{ss})$ and positive for all $\vartheta_t \in (\vartheta^{ss}, 1]$. The former observation implies that all solution paths for which $\vartheta_0 \in (0, \vartheta^{ss})$ remain inside this interval throughout, decay over time, and asymptotically approach the no bubble equilibrium, $\vartheta_t \to 0$ as

[8] The equation $d\vartheta_t = 0$ has a third solution, but that solution is always outside of the interval $[0,1]$.

$t \to \infty$. All these solution paths correspond to valid equilibria. The latter observation implies that all solution paths for which $\vartheta_0 \in (\vartheta^{ss}, 1]$ will cross the upper bound 1 in finite time and therefore cannot correspond to valid equilibria. We conclude that there is a continuum of equilibria that can be indexed by $\vartheta_0 \in [0, \vartheta^{ss}]$ and money retains a positive value asymptotically in only one of them, the monetary steady state ($\vartheta_0 = \vartheta^{ss}$). This structure of equilibria makes the monetary steady state the most interesting, and possibly also most plausible, equilibrium to select for two reasons:

First, the monetary steady state is locally unique in the sense that there is no other equilibrium "nearby" and it is the only equilibrium with this property. Specifically, for any initial condition $\vartheta_0 < \vartheta^{ss}$, perturbing the initial condition slightly perturbs the whole equilibrium path only slightly. In contrast, moving slightly away from $\vartheta_0 = \vartheta^{ss}$ leads to an equilibrium path that is vastly different for large t because ϑ_t must asymptotically vanish for any other equilibrium. The monetary steady state is therefore the only equilibrium that is robust to introducing small errors in agents' expectations about the value of money in the far future.[9] Furthermore, local uniqueness also opens the door to introduce another device into the model that somehow prunes the set of equilibria away from the desired locally unique one in order to eliminate all other possibilities and ensure even global uniqueness. We will return to this idea in Section 9.5 by introducing off-equilibrium fiscal backing to make the monetary steady state globally unique.

Second, the monetary steady state is not merely locally unique. It is the only equilibrium consistent with any belief that, with positive probability, money retains a positive value bounded away from zero in the arbitrarily distant future. This follows immediately from the fact that all other equilibria leads to a zero value of money asymptotically with certainty. If agents believe that $\vartheta_T \geq \varepsilon > 0$ for arbitrarily large T has positive probability, then they must expect the monetary steady state to occur.

[9] There are many ways to formalize this idea. A very simple one is the following: select a rational expectations equilibrium and then consider a different economy in which the rational expectations assumption for prices after some terminal time T is replaced by assuming that expectations are the ones in the selected equilibrium plus some small perturbation ε. As we make T large, the resulting equilibrium before time T will converge to the selected rational expectations equilibrium if it is locally unique, i.e. for the monetary steady state in the present model. But it will converge to a different rational expectations equilibrium than the selected one if the selected one is not locally unique.

We close this subsection with a remark on other models of bubbly money as a store of value in the literature. In this model, the service flows that retrading money generates for agents arise from partial insurance against idiosyncratic risk. The same is true in the classic analysis of Bewley (1980), except that the source of uninsurable idiosyncratic risk differs. In Bewley (1980), idiosyncratic risk takes the form of idiosyncratic (labor) income risk whereas here it takes the form of return risk from capital assets. In Samuelson (1958), an overlapping generations friction inhibits trade across generations. Trading bubbly money allows agents to overcome this friction as old households are able to pass on the money to young households in exchange for a real resource transfer. In Townsend (1980), households face deterministic variation in their endowment patterns and would like to trade with each other to smooth consumption, but a borrowing friction prevents intertemporal trade. Trade in bubbly money alleviates the friction by allowing agents some intertemporal trade but it cannot fully restore the allocation that would prevail if credit was available, just like in our model and Bewley (1980)'s model money does not fully eliminate idiosyncratic consumption risk.

9.2.4 Money Bubbles II: Money as a Medium of Exchange

We now isolate the third channel that may generate a positive portfolio demand for money in our model, monetary frictions. Specifically, assume that primary surpluses and idiosyncratic risk are both zero, $s = \tilde{\sigma} = 0$, but that transaction costs are positive, $\bar{v} < \infty$. We also assume that $\bar{v}$ satisfies the inequality (8.15) stated in Chapter 8 to ensure that an equilibrium can exist at all. Under these assumptions, the government liability valuation equation becomes

$$\frac{d\vartheta_t}{dt} = \left(\rho - \Delta i(\vartheta_t)\right) \vartheta_t, \tag{9.3}$$

where $\Delta i_t = \Delta i(\vartheta_t)$ is a function of ϑ_t. It is given by $\Delta i_t = (\nu_t/\bar{v})^{\mathfrak{z}}$ and ν_t satisfies the equation (compare equation (8.21))

$$\nu_t = \frac{1 - \vartheta_t + \phi\rho}{\vartheta_t} \frac{a}{1 + \phi\check{a}}.$$

Note, in particular, that velocity is strictly decreasing in ϑ_t and, hence, $\Delta i(\vartheta_t)$ is a strictly decreasing function.

We are looking, once again, for solution paths to equation (9.3) that remain contained in the interval $[0,1]$. Any such solution path corresponds to a valid model equilibrium. We note that, on a high level, equation (9.3) has the same structure as equation (9.2) from the previous subsection: the expected change in ϑ_t is the product of ϑ_t itself and a term that is strictly increasing in ϑ_t. In addition, that latter term is negative for $\vartheta_t \approx 0$ and positive for $\vartheta_t \approx 1$. This suggests the same solution structure as previously: two steady-state solutions, $\vartheta_t = 0$ and $\vartheta_t = \vartheta^{ss} \in (0,1)$, and a continuum of nonstationary solutions in between that can be indexed by $\vartheta_0 \in (0, \vartheta^{ss})$ and in all of them money is asymptotically worthless, $\vartheta_t \to 0$.

However, this reasoning is not fully correct for the specification of money demand in this model. For the no bubble steady state $\vartheta_t = 0$ to exist, the right-hand side of equation (9.3) must vanish if $\vartheta_t = 0$. It turns out that this is not the case here because $\Delta i(\vartheta_t) \to \infty$ as $\vartheta_t \to 0$ (in fact, even $\Delta i(\vartheta_t)\vartheta_t \to \infty$). As real balances become scarce, agents desperately value additional money to lower their transaction costs. So, in this model, the monetary steady state $\vartheta_t = \vartheta^{ss}$ is in fact globally unique. The monetary friction appears to be more powerful in selecting a unique equilibrium, and hence determining the price level, than the incomplete markets frictions from the previous subsection. A similar conclusion would result from many other (reduced-form) specifications of monetary frictions, including a cash-in-advance constraint on consumption expenditures.

But there is something wrong with the economics of these simple models of monetary frictions when pushed to the extremes of zero real balances. Here, agents are forced to incur arbitrarily high transaction costs in production when velocity rises, making money an essential input to the production process. In models with a cash-in-advance constraint for consumption expenditures and standard utility functions, agents are forced to hold money in order to consume or otherwise they will face an infinite disutility penalty from starvation. Surely, in reality people would find ways to economize on transactions or switch to barter when money becomes sufficiently scarce.

These alternatives keep the money premium Δi_t bounded, so that the logic from the previous paragraph no longer applies. Instead, if we impose any arbitrarily large upper bound on Δi_t, the solution structure of equation (9.3) will indeed be qualitatively identical to the solution structure of equation (9.2). In particular, money is again bubbly and depends on social coordination. Still, the monetary steady state $\vartheta_t = \vartheta^{ss}$ is locally unique and it is the unique equilibrium consistent with a belief that money will retain a positive value bounded away from zero in the arbitrarily distant future with positive probability. In this sense, the monetary friction can again determine the nominal wealth share ϑ_t and the price level $\mathcal{P}_t$ up to the same caveat as for other bubble theories that we need to make a case for selecting the monetary steady state.

Merely bounding Δi_t to conclude that the monetary friction is symmetric to incomplete market frictions with regard to determination of ϑ_t may appear ad-hoc to some readers. Instead, we could formalize the concern from the previous paragraph by augmenting the model as follows: capital holders have access to two technologies, one that produces output at rate $ak_t dt$ but is subject to transaction costs and one that produces output at lower rate $\underline{a}k_t dt$, $\underline{a} < a$ but does not require money to make transactions, and they allocate their capital across the two technologies in any proportion. The reader is invited to work out the details of this alternative model in Exercise 9.6.1 in the special case of the cash-in-advance limit $\mathfrak{z} \to \infty$. In this case, $\Delta i(\vartheta_t)$ takes the following form

$$\Delta i_t = \begin{cases} \frac{a-\underline{a}}{a}\frac{\rho}{\vartheta_t}, & \vartheta_t > \frac{\rho}{\bar{v}} \\ \frac{a-\underline{a}}{a}\bar{v}, & \vartheta_t \leq \frac{\rho}{\bar{v}} \end{cases}.$$

In particular, $\Delta i_t \leq \frac{a-\underline{a}}{a}\bar{v}$ remains bounded as ϑ_t becomes small. The reason is that, as money becomes scarcer, agents dedicate a larger fraction of capital to the technology that does not require monetary transactions instead of bidding up the money premium Δi_t.

9.2.5 Remark: Fiscal Backing and Services from Money Trade at the Same Time

For pedagogical purposes, we have focused in each of the previous subsections exclusively on one of the three forces in the model that might give money value: fiscal backing, idiosyncratic risk, or transaction costs. Here, we briefly discuss what happens if we have positive primary surpluses, $s > 0$, but at the same time also idiosyncratic risk and transaction costs present in the model.[10] In this case, fiscal backing again determines a globally unique equilibrium, i.e., this is another instance of the FTPL. However, the value of money is explained jointly by the backing cash flows and the services that money provides.

Specifically, the government liability valuation equation takes the form

$$\frac{d\vartheta_t}{dt} = \Big(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2 - \Delta i(\vartheta_t)\Big)\,\vartheta_t - s\frac{1-\vartheta_t+\phi\rho}{1+\phi\breve{a}}. \tag{9.4}$$

Analyzing this equation leads to the same conclusions as in Section 9.2.2. There is a unique value $\vartheta_t = \vartheta^{ss} \in [0,1]$ at which the right-hand side vanishes and this value is strictly positive and smaller than 1. Furthermore, the right-hand side of the equation is strictly negative for all $\vartheta_t \in [0, \vartheta^{ss})$ and strictly positive for all $\vartheta_t \in (\vartheta^{ss}, 1]$, so that no other mathematical solution than the steady state $\vartheta_t = \vartheta^{ss}$ can remain confined within the interval $[0,1]$ at all times. Hence, there is at most one equilibrium and it features $\vartheta_t = \vartheta^{ss}$ for all $t \geq 0$. Following the solution steps from the previous chapter, one can recover the remaining equilibrium objects and show that they indeed constitute a valid equilibrium.

So, as in Section 9.2.2, fiscal backing ensures that there is a uniquely determined value of money. But this does not mean that services from trading money do not matter for the ultimate value of ϑ_t. To see this, write equation (9.4) in integral form and let us

[10] We assume, once again, $\bar{\nu} > \frac{1+\phi\rho}{1+\phi a}\frac{a}{\rho^{1/\mathfrak{z}}}$ to ensure that an equilibrium can exist.

also take the limit $\phi \to \infty$, so that total wealth per unit of capital is independent of ϑ_t:[11]

$$\vartheta_0 = \int_0^\infty e^{-\rho t} \frac{\rho s}{\breve{a}} dt + \int_0^\infty e^{-\rho t} \left((1 - \vartheta^{ss})^2 \tilde{\sigma}^2 + \Delta i(\vartheta^{ss}) \right) \vartheta^{ss} dt.$$

We can think about this equation as a decomposition of the portfolio demand for money according to the sources it arises from. The first term captures the contribution of backing whereas the second term captures the contribution of service flows. Evidently, both components affect the value of money positively. In particular, a larger surplus is required to deliver the same value of money if idiosyncratic risk or monetary frictions are less severe.

We observe that the backing by positive primary surpluses plays a dual role here. First, it contributes to the total demand for money. In this respect, it is symmetric to the service flows. Second, it ensures that there is a unique equilibrium. In this respect, it is different from the service flows, whose value depends on social coordination. In Section 9.5 we discuss how to separate these two roles by moving the backing that ensures a unique equilibrium to off-equilibrium contingencies. Then, the demand for money can remain uniquely determined even in the presence of zero or negative primary surpluses along the equilibrium path.

9.3 Price Level and Inflation Determination without Money

Let us now consider a variant of our model without money, i.e., $\mathcal{M}_t = 0$ for all t. For consistency reasons, we then also have to assume that there are no monetary frictions, $\bar{v} \to \infty$, and no primary surpluses or deficits, $\breve{\mu}_t^{\mathcal{M}} = s_t = 0$ for all t. In this case, we obtain $q_t^M = \frac{\mathcal{M}_t}{\mathcal{P}_t K_t} = 0$ regardless of the price level $\mathcal{P}_t > 0$ (and hence $\vartheta_t = 0$). To make nominal prices relevant for something in this economy, suppose that agents still trade nominal debt in zero net supply. This debt pays a real return

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t},$$

[11] The latter operation is not necessary but it makes the following arguments cleaner.

where i_t is the nominal interest rate that we assume is under the control of the government. Maintaining the assumption of no aggregate risk, we have $\frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = -\pi_t dt$, where $\pi_t := \mu_t^{\mathcal{P}}$ denotes the (expected and realized) inflation rate. Consequently, the real return on nominal bonds is $dr_t^{\mathcal{B}} = (i_t - \pi_t)dt$ and no arbitrage requires that the shadow real interest rate in the economy, that is the common decay rate of all individual SDFs ξ_t^i, satisfies the Fisher equation

$$r_t = i_t - \pi_t. \tag{9.5}$$

So long as this equation is satisfied, all agents are willing to hold zero nominal bonds in equilibrium, so that bond market clearing is always ensured.

We can solve the moneyless model by first setting $\vartheta_t = 0$ in the model solution from the last chapter, which fully determines the real allocation independently of nominal quantities such as the price level and inflation. The real allocation is as follows. Aggregate quantities satisfy

$$q_t^K = \frac{1+\phi\breve{a}}{1+\phi\rho}, \quad C_t/K_t = \rho q_t^K, \quad \iota_t = \frac{\breve{a}-\rho}{1+\phi\rho}, \quad dK_t/K_t = (\Phi(\iota_t) - \delta)dt,$$

and individual consumption is $c_t^i = \eta_t^i C_t$, where $d\eta_t^i/\eta_t^i = \tilde{\sigma} d\tilde{Z}_t^i$ and the initial cross-sectional wealth distribution $(\eta_0^i)_{i\in[0,1]}$ is given. In particular, the real interest rate is

$$r_t = r := \rho + \frac{1}{\phi}\log\left(\frac{1+\phi\breve{a}}{1+\phi\rho}\right) - \delta - \tilde{\sigma}^2$$

We next discuss how interest rate policy affects price level and inflation determinacy.

Exogenous interest rates. Consider first a policy specification in which the government sets an exogenous path $\{i_t\}_{t\geq 0}$ for nominal interest rates. Substituting this interest rate path into the Fisher equation and rearranging yields a path of equilibrium inflation rates

$$\pi_t = i_t - r.$$

The presence of a Fisher equation is therefore sufficient to link inflation to a policy path for nominal interest rates, even in the absence of any money. However, the initial price level $\mathcal{P}_0$ is not determined by any force in the model and can take on arbitrary values. Consequently, *there is price level indeterminacy under an exogenous interest rate policy*.

To make matters worse, also the inflation rate is here only determined because we have assumed that there are no aggregate shocks. However, even if there is no fundamental risk, the price level could load on sunspot shocks. Suppose, for example, that

$$d\mathcal{P}_t/\mathcal{P}_t = \mu_t^{\mathcal{P}}dt + \sigma_t^{\mathcal{P}}dZ_t,$$

where dZ_t is a Brownian sunspot shock. The Fisher equation in this model becomes[12]

$$r = i_t - \mu_t^{\mathcal{P}} - (\sigma_t^{\mathcal{P}})^2.$$

For a given path of nominal interest rates, this equation determines $\mu_t^{\mathcal{P}} - (\sigma_t^{\mathcal{P}})^2$, but neither expected inflation $\mu_t^{\mathcal{P}}$ nor price level volatility $\sigma_t^{\mathcal{P}}$ individually.

Wicksellian feedback rules. An alternative to a purely exogenous interest rate path is an interest rate feedback rule that adjusts the interest rate in response to endogenous variables such as inflation. For example, suppose the government follows a price level feedback rule

$$i_t = i_t^0 + \phi_{\mathcal{P}} \log \mathcal{P}_t, \tag{9.6}$$

where i_t^0 is an exogenous intercept path and $\phi_{\mathcal{P}} \log \mathcal{P}_t$ incorporates feedback from the price level to interest rates. The recommendation to incorporate price level feedback into interest rate rules dates back to Wicksell (1898). This type of policy rule is therefore often referred to as a *Wicksellian interest rate rule*.

To analyze the feedback rule, let us, for simplicity, return back to a situation without sunspots. Combining the price level evolution $d\mathcal{P}_t = \pi_t \mathcal{P}_t dt$ with the Fisher equation (9.5) yields

$$d \log \mathcal{P}_t = d\mathcal{P}_t/\mathcal{P}_t = \left(i_t^0 - r + \phi_{\mathcal{P}} \log \mathcal{P}_t\right) dt.$$

This is a linear ODE for $\log \mathcal{P}_t$. If $\phi_{\mathcal{P}} > 0$, there is destablizing feedback in the forward dynamics that lead to all but one solution path to explode. Specifically, the general

[12] Note that there is no risk premium for price level fluctuations because all agents hold a zero quantity of nominal debt, so they are not exposed to this risk in equilibrium.

solution is given by

$$\log \mathcal{P}_t = e^{\phi_{\mathcal{P}} t} \left(\log \mathcal{P}_0 - \log \mathcal{P}_0^*\right) - \int_t^\infty e^{-\phi_{\mathcal{P}}(s-t)}(i_s^0 - r)ds,$$

where

$$\log \mathcal{P}_0^* := -\int_0^\infty e^{-\phi_{\mathcal{P}} t}(i_t^0 - r)dt.$$

If $\mathcal{P}_0 = \mathcal{P}_0^*$, then the first term vanishes and the resulting solution path, denote it by $\{\mathcal{P}_t^*\}_{t\geq 0}$, is bounded, provided $\{i_t^0\}_{t\geq 0}$ is. In contrast, if $\mathcal{P}_0 \neq \mathcal{P}_0^*$, then $\log \mathcal{P}_t$ explodes at an exponential rate to $\pm\infty$. Explosive paths for the price level are rarely seen as an issue (after all, there is also an explosive path at a constant nonzero inflation rate), but here they also imply an explosive path for inflation due to the feedback in the policy rule:

$$\pi_t = i_t^0 - \phi_{\mathcal{P}} \log \mathcal{P}_t^* - r + \phi_{\mathcal{P}} \left(\log \mathcal{P}_0 - \log \mathcal{P}_0^*\right) e^{\phi_{\mathcal{P}} t},$$

which grows or decays at an exponential rate unless $\mathcal{P}_0 = \mathcal{P}_0^*$. While explosive inflation is also not really an issue from the perspective of the model, in which nominal quantities have no impact on real allocations, such equilibrium dynamics are often deemed implausible and therefore ruled out as an additional selection criterion. If we impose this additional selection rule, then the only prediction that remains is $\mathcal{P}_t = \mathcal{P}_t^*$. In other words, a Wicksellian feedback rule can determine the price level even in an economy without money.

We remark that there are two differences between a feedback rule in the formulation (9.6) and an exogenous interest rate rule. First, the feedback rule makes the interest rate path $\{i_t\}_{t\geq 0}$ followed in equilibrium endogenous and, second, it ensures price level determinacy, at least under a rule that equilibria with exploding inflation dynamics are to be disregarded. To separate the roles that interest rates play in equilibrium and in support of equilibrium selection, it sometimes makes sense to construct the policy rule differently as follows. Start out with an exogenous interest rate path $\{\bar{i}_t\}_{t\geq 0}$ to be followed in equilibrium. Due to price level indeterminacy, choosing $i_t = \bar{i}_t$ leads to a (large) set $\mathbb{P}$ of potential price level paths $\{\mathcal{P}_t\}_{t\geq 0}$ consistent with this interest rate path. We may then select any desired path in this set, $\{\mathcal{P}_t^*\}_{t\geq 0} \in \mathbb{P}$, and construct a feedback rule that leads to $i_t = \bar{i}_t$ and $\mathcal{P}_t = \mathcal{P}_t^*$ as the unique equilibrium prediction. To do so, choose $\phi_{\mathcal{P}} > 0$ arbitrary and the intercept of the rule as

$$i_t^0 = \bar{i}_t - \phi_{\mathcal{P}} \log \mathcal{P}_t^*.$$

It is easy to verify that, under this policy rule, $\mathcal{P}_t = \mathcal{P}_t^*$ is indeed the unique solution path with nonexplosive inflation and, in the resulting equilibrium, $i_t = \bar{i}_t$. In particular, this construction shows that there is nothing special about equilibria that can be selected by feedback rules: any equilibrium that arises as a possibility under an exogenous interest rate rule can also be supported as a unique equilibrium under a feedback rule.

Taylor rules. In the contemporary literature, interest rate rules that incorporate feedback from the inflation rate instead of the price level are more common, for example:

$$i_t = i_t^0 + \phi_\pi \pi_t \tag{9.7}$$

with an exogenous intercept i_t^0 and inflation feedback $\phi_\pi \pi_t$. Such rules are often referred to as *Taylor rules* (Taylor, 1993).[13]

A Taylor rule as in equation (9.7) frequently occurs in the discrete-time literature. But it turns out that it provides no advantages for price level or inflation determination relative to an exogenous interest rate rule in a continuous time framework with flexible prices and infinitesimal duration of short-term nominal debt contracts. To see this, note that combining the policy rule (9.7) with the Fisher equation (9.5) yields a static condition for the inflation rate π_t, just like in the case of an exogenous interest rate rule.

This is different to a Wicksellian rule, which relates a rate of change (the interest and inflation rates, which are linked by the Fisher equation) to a level (the price level) and therefore generates a dynamic condition. To emulate this logic with a Taylor rule, we need some source of inertia, which can arise from a longer duration of nominal debt contracts, from price stickiness, or from interest rate smoothing in the policy rule. We

[13] The original "Taylor rule" in Taylor (1993) is more specific in that it suggests a constant intercept, an additional output gap reaction and specific numerical values for the intercept and the reaction coefficients on inflation and the output gap. The contemporary use of the term is broader and refers to a whole class of feedback rules.

remark that the first source of inertia is always present in discrete time models with short-term debt because there the short-term interest rate is fixed for the length of a period.

Under either source of inertia, a Taylor rule of the type (9.7) can be analyzed analogously to a Wicksellian rule (9.6).[14] Because we do not need Taylor rules anywhere in these notes and the analysis is slightly more involved, we omit the formal details here. The key conclusion is that Taylor rules of the type (9.7) can achieve inflation determinacy (under a selection that deems certain explosive paths implausible) if and only if $\phi_\pi > 1$, so that interest rates rise more than one for one with inflation, at least eventually. This condition is called the *Taylor principle*. We note that, due to the lack of any price level component in the rule, the level of prices remains undetermined under such a rule, except if inertia arise from sticky prices, so that the initial price level is an initial condition.

9.4 Fiscal-Monetary Policy Regimes and Determinacy

One conclusion from the previous section is that the design of government policy can affect price level determinacy. Important is here not just how policy is conducted along the equilibrium path but also how policy would be conducted off-equilibrium, as this affects whether alternative paths for the price level can also be valid equilibria. These conclusions are also relevant in settings in which money is in positive supply. In this section, we discuss how certain fiscal and monetary policy specifications can lead to price level indeterminacy even when the nominal wealth share ϑ_t is always positive.

9.4.1 Neutralizing the Store of Value Role for Determinacy with Passive Fiscal Policy

We return to our baseline model with a positive quantity of money outstanding at all times, $\mathcal{M}_t > 0$. To isolate the store of value role of money, we assume that there are

[14] If inertia is created by means of interest rate smoothing, equation (9.7) merely represents the target interest rate, while the actual interest rate is only gradually adjusted towards that target.

no transaction costs ($\bar{v} \to \infty$) but we do allow for the presence of both fiscal policy and idiosyncratic risk. We have shown previously that if the government follows a positive surplus policy, $s > 0$ in all contingencies, then there is a globally unique equilibrium and, in this equilibrium, money is valued at all times, including asymptotically. We have also shown that if the government follows a zero surplus policy, $s = 0$ in all contingencies, and idiosyncratic risk is sufficiently large, $\tilde{\sigma} > \sqrt{\rho}$, then, while there is no global uniqueness anymore, there is still a unique equilibrium in which money is valued asymptotically.

We now perturb the policy space slightly and replace the constant s by a more general linear policy rule[15]

$$s_t = s^0 + \alpha(q_t^M - q^{M,0}), \tag{9.8}$$

where $s^0 \geq 0$ is a constant and $q^{M,0} > 0$ is the unique monetary steady state value for q_t^M under the policy rule $s_t = s^0$. For the latter to exist, we also need to assume $\tilde{\sigma} > \sqrt{\rho}$ in the case $s^0 = 0$, while no such assumption is necessary if $s^0 > 0$.

A policy rule of the type (9.8) with $\alpha > 0$ encodes a fiscal reaction to the level of outstanding government liabilities. In this model, all government liabilities take the form of money. But more generally, such a rule would also commit the government to raise additional primary surpluses in response to an increase in its debt. A rule that conditions surpluses on the level of debt is often associated with the behavior of a "responsible" fiscal authority that ensures that government debt remains sustainable.

We will discuss next that such a behavior might interfere with price level determinacy. Specifically, we revisit the analysis from Sections 9.2.2 and 9.2.3 under the policy rule (9.8). To keep the algebra as simple as possible, we present equations only for the limit case $\phi \to \infty$ without real investment. But the following considerations apply more generally.

Under the assumptions made above, the government liability valuation equation

[15] Strictly speaking, the following policy rule may not be globally feasible: if $\alpha q^{M,0} > s^0$, then this rule requires the government to run a negative primary surplus funded by issuing more money in contingencies in which $q_t^M = 0$, but this is not possible because money is worthless. However, we will disregard this issue in the following discussion.

becomes

$$\frac{d\vartheta_t}{dt} = \left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t - \frac{\rho s^0}{\breve{a}} - \alpha\left(\vartheta_t - \vartheta^0\right)$$
$$= \left(\rho - (1-\vartheta^0)^2\tilde{\sigma}^2 + \left(2 - \vartheta^0 - \vartheta_t\right)\vartheta_t\tilde{\sigma}^2 - \alpha\right)\left(\vartheta_t - \vartheta^0\right). \tag{9.9}$$

Here, the second line follows by using that the right-hand side of the first line becomes zero for $\vartheta_t = \vartheta^0$ to eliminate the s^0-term and rearranging. The solution structure of equation (9.9) depends on the behavior of the first factor on the right. We discuss three cases:

1. The factor is always nonnegative, and strictly positive for all $\vartheta_t > 0$, if

$$\alpha \leq \rho - \left(1 - \vartheta^0\right)^2\tilde{\sigma}^2 = \frac{\rho}{\vartheta^0}\frac{s^0}{\breve{a}} =: \underline{\alpha}.$$

In this case, $\vartheta_t = \vartheta^0$ is an unstable steady state of the ϑ_t-forward dynamics and essentially the same conclusions as in Sections 9.2.2 and 9.2.3 apply: the monetary steady state $\vartheta_t = \vartheta^0$ is the unique equilibrium in which money has asymptotic value and it is the unique equilibrium that is locally unique. In this case, the value of money is determined either due to fiscal backing or due to the retrading service flows it provides as a store of value. In addition, if $\alpha < \underline{\alpha}$, then it is even the globally unique equilibrium.

Note that the condition $\alpha \leq \underline{\alpha}$ is related to the minimum fiscal backing that is provided under the fiscal rule (9.8) in any (off-equilibrium) contingency. Indeed, $\underline{\alpha} = s^0/q^{M,0}$, so $\alpha \leq \underline{\alpha}$ is equivalent to $s_t \geq 0$ regardless of the value of q_t^M. In particular, if $\alpha < \underline{\alpha}$, then the fiscal authority provides positive surplus backing in all contingencies.

2. The first factor on the right-hand side of equation (9.9) is always negative, if

$$\alpha > \rho + \vartheta^0\left(1 - \frac{3}{4}\vartheta^0\right)\tilde{\sigma}^2 =: \overline{\alpha}.$$

In this case, forward dynamics are globally stable. For any initial condition $\vartheta_0 \in (0,1)$, there is a solution path for ϑ_t that remains in $(0,1)$ throughout and this path satisfies $\vartheta_t \to \vartheta^0$ as $t \to \infty$. In particular, all these solutions correspond to valid equilibria. Hence, if primary surpluses are sufficiently reactive to the value of money, then all initial nominal wealth shares $\vartheta_0 \in (0,1)$ can occur in an equilibrium, money is valued in all equilibria, and no equilibrium is even locally unique. In this sense, a sufficiently reactive fiscal policy neutralizes price level determination from fiscal backing or store of value services.

More generally, in the presences of other nominal government liabilities than money, it is the stock of total government liabilities that matters for a rule like (9.8). A high $\alpha > 0$ means that the fiscal authority ensures that government debt remains sustainable and bounded in all contingencies. Such a fiscal policy is called a "passive" fiscal policy.[16] This terminology, due to Leeper (1991), does not refer to a fiscal authority that is inactive. In fact, a passive fiscal policy rule often requires actively adjusting taxes in reaction to changes in the value of government debt. Instead, the terminology means that the fiscal authority "passively" reacts to ensure that debt is always sustainable instead of "actively" following some other policy objective and disregarding the debt evolution.[17]

We see therefore that passive fiscal policy renders the price level indeterminate, even if there is fiscal backing or a sufficient demand for money as a store of value in (any) equilibrium. To understand better why, suppose, for example, that markets coordinate on an equilibrium path in which $\vartheta_0 < \vartheta^0$, and then also $\vartheta_t < \vartheta^0$ for all $t \geq 0$. In this case, the policy rule (9.8) implies $s_t \leq s_t^0$, that is the government lowers tax backing relative to the monetary steady state, thereby validating the private-sector expectation that money has a lower value. Lower tax backing initially also means a higher rate of money issuance to fund the reduction in surpluses, so that the outstanding stock of money grows faster than in the monetary steady state. This raises $q_t^M - q^{M,0}$ and, through the fiscal rule, leads to a gradual increase in primary surpluses. Over time, q_t^M converges to its long-run target value $q^{M,0}$ and surpluses rise to back this value of money.

[16] A fiscal policy that is not passive is referred to as "active". However, this terminology is less useful for determinacy because active policy in this sense includes both the case $\alpha \leq \underline{a}$ discussed previously, which leads to determinacy, and the case $\alpha \in (\underline{a}, \overline{a}]$ discussed below, which does not.

[17] footnote about terminology to be added...

A similar situation, but with inverse signs, arises if markets coordinate on ϑ_0 that initially exceeds the monetary steady state. In this case, the fiscal authority raises additional surpluses to retire some of the outstanding money in a transition back to the steady state. We see therefore that a passive fiscal policy leads to indeterminacy because it commits itself to adjust surpluses to validate whatever real value of money the private sector coordinates on.

3. In the remaining case, $\alpha \in (\underline{\alpha}, \overline{\alpha}]$, the first factor on the right of equation (9.9) changes sign, so that the forward dynamics are neither globally stable nor globally unstable on the interval $(0, 1)$. Then, there are multiple possible equilibrium solutions in which money is asymptotically valued. However, not all values for ϑ_0 can be supported by some equilibrium, fiscal policy is not fully passive.

   Inside the region $\alpha \in (\underline{\alpha}, \overline{\alpha}]$, it is an important distinction whether the monetary steady state $\vartheta_t = \vartheta^0$ is locally unique or locally non-unique. We may associate the former outcome with a "locally active" and the latter with a "locally passive" fiscal policy. By equation (9.9), forward dynamics are locally unstable and hence the steady state locally unique if and only if the first factor on the right-hand side is positive in a neighborhood of $\vartheta_t = \vartheta^0$. This is the case precisely when

$$\alpha < \hat{\alpha} := \rho - (1 - \vartheta^0)^2 \tilde{\sigma}^2 + 2(1 - \vartheta^0)\vartheta^0 \tilde{\sigma}^2 = \rho + \left(3\vartheta^0 - 1\right)(1 - \vartheta^0)\tilde{\sigma}^2.$$

   If this inequality is violated, then there are other equilibria arbitrarily close to the monetary state state.

   Note that $\underline{\alpha} \leq \hat{\alpha} \leq \overline{\alpha}$ and the three thresholds coincide if markets are complete due to $\tilde{\sigma} = 0$.

We remark that even in cases 2. and 3., when the store of value role of money does not contribute to (global) price level determinacy, it always still matters for the value of money and the total level of tax backing along the selected equilibrium path. In addition, if $\tilde{\sigma} > 0$, then this also matters for the real allocation because Ricardian equivalence does not hold. Specifically, the amount of idiosyncratic risk that agents are exposed to depends on the level of ϑ_t.

9.4.2 Neutralizing the Medium of Exchange Role for Determinacy with Interest Rate Policy

Next, suppose that prices are determined by the medium of exchange role of money. Specifically, to isolate this channels for price level determinacy, let us make the following assumptions:

(i) there are both government bonds at quantity $\mathcal{B}_t$ and money at quantity $\mathcal{M}_t$;

(ii) $\tilde{\sigma} = 0$ and $\phi \to \infty$;

(iii) the fiscal authority follows a passive fiscal policy rule of the form

$$s_t + \breve{\mu}_t^{\mathcal{M}} q_t^M = \alpha q_t^B, \qquad \alpha > \rho$$

that stabilizes the value of government bonds around zero.[18]

Assumption (i) separates the store of value role of government liabilities from the medium of exchange role by introducing a pure store of value asset, bonds, that does not enter the transaction technology. Assumption (ii) ensures Ricardian equivalence with respect to government bonds, so that the quantity of store of value assets does not affect real allocations. Finally, assumption (iii) ensures that the store of value role plays no role for price level determination as discussed in the previous subsection.

Price level determination from the medium of exchange role of money. Even in such an environment, the price level may still be determined from the medium of exchange role of money if the monetary authority sets an exogenous path $\{\mathcal{M}_t\}_{t\geq 0}$ for the money supply.[19] To see this, note that portfolio dynamics can be characterized by

[18] $s_t + \breve{\mu}_t^{\mathcal{M}} q_t^M$ is the sum of the primary fiscal surplus and seigniorage from money dilution. For the results below, we could also have assumed that s_t follows a rule which stabilizes the total value of all government liabilities as in the previous subsection. But the rule chosen here leads to a cleaner separation between money and bonds.

[19] This can be implemented by exchanging money for bonds. Here, we allow $\mathcal{B}_t$ to become negative. We interpret negative $\mathcal{B}_t$ as a situation in which the government is a net holder of nominal bonds issued by private agents.

two valuation equations, one for bonds and one for money:

$$d(\vartheta_t^B \vartheta_t) = (\rho - \alpha)\vartheta_t^B \vartheta_t, \tag{9.10}$$

$$d(\vartheta_t^M \vartheta_t) = \left(\rho - \Delta i(\vartheta_t^M \vartheta_t) + \breve{\mu}_t^{\mathcal{M}}\right) \vartheta_t^M \vartheta_t dt, \tag{9.11}$$

where $\Delta i(\vartheta_t^M \vartheta_t)$ is nonnegative and strictly decreasing in $\vartheta_t^M \vartheta_t$ as in Section 9.2.4.

Equation (9.10) behaves like equation (9.9) studied in the last subsection but for the valuation of bonds only instead of the valuation of all nominal assets. For $\alpha > \rho$, $\vartheta_t^B \vartheta_t = 0$ is a globally stable steady state in the forward evolution, so any initial value $\vartheta_0^B \vartheta_0$ is consistent with a solution to this equation and any solution will in the long run converge to zero (that is all bonds are repaid asymptotically).

Equation (9.11), in contrast, behaves like equation (9.3) studied in Section 9.2.4. In particular, for constant $\breve{\mu}^{\mathcal{M}} > -\rho$, the analysis from that section carries over and we can conclude that there is a unique solution path for $\vartheta_t^M \vartheta_t$, which happens to be a steady-state solution. More generally, if $\breve{\mu}_t^{\mathcal{M}}$ is time-varying but fixed (and remains bounded away from $-\rho$), there is generally no steady-state solution but there is still a unique path for $\vartheta_t^M \vartheta_t$. Note that an exogenously fixed path for the money supply implies a fixed path for the money growth rate $\mu_t^{\mathcal{M}}$. To have also $\breve{\mu}_t^{\mathcal{M}}$ exogenously fixed, we need to assume, in addition, that $i_t^{\mathcal{M}}$ is exogenously fixed, either by technological constraints (e.g., the interest rate on cash is zero) or because it is set by policy (e.g., interest paid on reserves).

In sum, under the fiscal policy arrangement studied here there are many possible equilibrium paths for the total nominal wealth share ϑ_t. But the equilibrium path of the money wealth share $\vartheta_t^M \vartheta_t$ is uniquely determined, if policy controls the path of the money supply (and $i_t^{\mathcal{M}}$ is also exogenously fixed). This is sufficient for price level determination by the same argument as in Section 9.2.1: goods market clearing requires

$$\breve{a}K_t = C_t = \frac{\rho}{\vartheta_t^M \vartheta_t} \frac{\mathcal{M}_t}{\mathcal{P}_t}$$

and because the paths for $\mathcal{M}_t$ and $\vartheta_t^M \vartheta_t$ are uniquely determined, so must be the path for $\mathcal{P}_t$.

Indeterminacy under an interest rate policy. The determinacy from the medium of exchange role of money can be switched off in a policy regime that sets the interest rate differential Δi_t and supplies money elastically to implement this differential in equilibrium. This regime is traditionally associated with interest rate policy: the traditional view is that $i_t^{\mathcal{M}} = 0$ is the rate of return on physical cash and interest rate policy seeks to control the rate $i_t^{\mathcal{B}}$ on bonds. Adjusting this nominal interest rate is then equivalent to adjusting the interest rate differential Δi_t. Note, however, that the modern, post-financial crisis, implementation of interest rate policy by paying interest on central bank reserves is closer to varying $i_t^{\mathcal{M}}$, interpreted as the reserve rate.

To analyze such a policy formally, let us suppose that $i_t^{\mathcal{M}}$ is exogenously fixed, by physical constraints or interest on reserve policy, and that, in addition, the government is committed to vary $\mathcal{M}_t$ with open market operations to implement an exogenous target path for Δi_t or, equivalently, the bond rate $i_t^{\mathcal{B}}$. Denote by $\overline{\Delta i}_t$ the desired target path for Δi_t.

Under these assumptions, equations (9.10) and (9.11) remain valid equilibrium conditions. In particular, our previous conclusion that equation (9.10) renders the bond wealth share $\vartheta_t^B \vartheta_t$ indeterminate remains valid. What changes is how we have to read equation (9.11). Now, $\Delta i_t = \overline{\Delta i}_t$ is exogenously fixed, so that the money wealth share $\vartheta_t^M \vartheta_t$ adjusts to satisfy the equation

$$\Delta i(\vartheta_t^M \vartheta_t) = \overline{\Delta i}_t,$$

which has a unique solution for each t. Equation (9.11) then tells us the value for $\breve{\mu}_t^{\mathcal{M}}$ that is required to make the money portfolio choice consistent with the government's target for Δi_t. Because $i_t^{\mathcal{M}}$ is fixed, this determines the money growth rate $\mu_t^{\mathcal{M}}$.

In sum, the equilibrium conditions now determine the path for the money wealth share, $\vartheta_t^M \vartheta_t$, and the path for the money growth rate, $\mu_t^{\mathcal{M}}$. But the path for the price level is not determined because the initial money stock, $\mathcal{M}_0$ is endogenous and indeterminate. Whatever the price level $\mathcal{P}_0 \in (0, \infty)$, the government commits to supply the quantity of money $\mathcal{M}_0$ required to hit its interest rate target, that is

$$\mathcal{M}_0 = \vartheta_0^M \vartheta_0 \frac{\breve{a} K_0}{\rho} \mathcal{P}_0.$$

The observation that an exogenous interest rate policy leaves the price level indeterminate even if there is a well-defined medium-of-exchange demand for money is due to Sargent and Wallace (1975). The situation here is reminiscent of the one we have encountered in Section 9.3 in the context of the model without money. Indeed, in the policy regime studied here in which fiscal policy neutralizes price level determination from the store of value role and interest rate policy neutralizes price level determination from the medium of exchange role, we essentially end up with an environment that behaves like the cashless model studied there.

9.5 Off-equilibrium Backing: The Fiscal Theory of the Price Level with a Bubble

This part is still missing. Until it has been added, readers are advised to read the paper Brunnermeier et al. (2020) as a substitute.

9.6 Exercises

9.6.1 Upper Bound on Liquidity Premium when Transactions Can Be Avoided

[to be completed]

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Chapter 10

Safe Assets

This chapter is based on Brunnermeier et al. (2024).

10.1 Introduction / Overview

What is a safe asset, and why does it matter? Why does it tend to have a negative β? Why can governments such as those of the United States and Japan issue large amounts of debt at low interest rates, and even run persistent primary deficits, without triggering default concerns? This chapter develops a theory of safe assets that sheds light on these questions and introduces the concept of a *Debt Laffer Curve*.

We define a safe asset through the *Good Friend Analogy*: it is an asset that is valuable and tradable precisely when it is needed most. In an environment where individuals face uninsurable idiosyncratic risk, government bonds provide a means of self-insurance. Their ability to be retraded after adverse shocks generates a flow of services distinct from the asset's cash return. This chapter introduces a two-part asset pricing framework that captures this feature,

$$price_t = \mathbb{E}_t[PV_{r^{**}}[cash\ flows]] + \mathbb{E}_t[PV_{r^{**}}[service\ flows]],$$

which separates the present value of cash flows from the present value of service flows. Each term is discounted at an endogenous rate r^{**} that reflects preferences in an incomplete market setting.

Because of this service value, safe assets appreciate during recessions and perform well in bad states of the world. This makes their β non-positive, in contrast to typical risky assets such as equity, which depreciate during downturns. The resulting low required cash return allows governments to issue debt cheaply and confers an *exorbitant privilege* on advanced sovereigns.

When the real return on safe assets falls below the growth rate of the economy, the government can issue debt on a potentially permanent basis. This resembles a Ponzi scheme, yet remains sustainable and consistent with individual transversality conditions. In our framework, this sustainability arises from a bubble component embedded in asset prices. While private issuers may offer similar service flows, only the government—backed by taxation and regulation—can credibly support such a bubble.

Importantly, safe assets do not need to be bubbly, but bubbles can reinforce their safe asset properties. The value of the service flow rises with the market value of the asset, so attaching a bubble can turn an otherwise risky asset into one that behaves safely. In such cases, the safe asset is safe because it is perceived to be so—a phenomenon we refer to as the *Safe Asset Tautology*. This status can be lost if the bubble bursts.

Governments may attempt to extract revenue from the bubble by accelerating bond issuance and tolerating higher inflation, thereby reducing the real return on safe asset holdings. This process, which we call *bubble mining*, resembles a tax on self-insurance and leads to a Debt Laffer Curve: beyond a certain point, more aggressive issuance reduces the overall value of the debt and hence fiscal revenue. Our quantitative analysis shows that this form of seigniorage is only effective when the safe asset has a sufficiently negative β.

10.2 Model Setup

Environment. The model is set in continuous time with an infinite horizon. There is a continuum of households indexed by $i \in [0, 1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

with discount rate $\rho > 0$.

Each agent operates one firm that produces an output flow $a_t k_t^i dt$, where k_t^i is the capital input chosen by the firm and a_t is an exogenous productivity process that is common for all agents. Capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_t^{k,i}, \tag{10.1}$$

where Φ is an increasing concave function that captures adjustment costs in capital accumulation, ι_t^i the investment rate (in output goods) per unit of capital, δ is the depreciation rate, $\tilde{\sigma}_t d\tilde{Z}_t^i$ represents idiosyncratic Brownian shocks, and $d\Delta_t^{k,i}$ represents firm i's market transactions in physical capital. Brownian motions $\tilde{Z}^i$ are agent-specific and i.i.d. across agents. The levels of idiosyncratic risk $\tilde{\sigma}_t$ and productivity a_t are exogenous processes.

To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1+\phi\iota\right)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.[1]

Each agent i can reduce idiosyncratic risk exposure by selling equity to other agents. Outside equity claims on i's capital k^i have the same aggregate and idiosyncratic risk as capital itself, but may pay a lower expected return, reflecting an insider premium that i earns for managing the capital stock. Agents can hold a diversified equity portfolio and thereby eliminate idiosyncratic risk.

In addition to households, there is a government that issues nominal government bonds, funds government spending, and imposes taxes on firms. Outstanding nominal government debt has a face value of $\mathcal{B}_t$ and pays nominal interest i_t. The face value follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ with growth rate $\mu_t^{\mathcal{B}}$. There is an exogenous need for real spending $\mathcal{G} K_t dt$, where $K_t := \int k_t^i di$ is the aggregate capital stock and $\mathcal{G}$ is a model parameter. The government can finance this spending by setting a proportional tax τ_t (subsidy if negative) on firms' output and by adjusting the bond issuance

[1] This function is defined for all $\iota \geq \underline{\iota} := -1/\phi$. We set $\Phi(\iota) := -\infty$ for $\iota < \underline{\iota}$.

rate $\mu_t^{\mathcal{B}}$ (repurchasing bonds if negative). We assume that $\mu_t^{\mathcal{B}}$ and i_t are exogenous processes while taxes τ_t adjust to satisfy the nominal government budget constraint

$$i_t \mathcal{B}_t + \mathcal{P}_t \mathcal{G} K_t = \mu_t^{\mathcal{B}} \mathcal{B}_t + \mathcal{P}_t \tau_t a_t K_t, \tag{10.2}$$

where $\mathcal{P}_t$ denotes the price level.

We assume that the exogenous processes a_t, $\tilde{\sigma}_t$, $\mu_t^{\mathcal{B}}$, and i_t follow a joint Markov diffusion process that is driven by some Brownian motion Z_t, which captures aggregate risk and is independent of all idiosyncratic Brownian motions $\tilde{Z}_t^i$.

Finally, the aggregate resource constraint is

$$C_t + \mathcal{G} K_t + \iota_t K_t = a_t K_t, \tag{10.3}$$

where $C_t := \int c_t^i di$ is aggregate consumption and $\iota_t = \int \iota_t^i k_t^i / K_t di$ is the average investment rate.[2]

Financial Frictions. The key friction in the model is that agents are unable to share idiosyncratic risk perfectly. Specifically, we assume that agents face a skin-in-the-game constraint and must retain at least a fraction $\bar{\chi} \in (0, 1]$ of their capital in undiversified form. As a consequence, agents have to bear the residual idiosyncratic risk of at least $\bar{\chi} \tilde{\sigma}_t d\tilde{Z}_t^i$ per unit of capital in their firms.

Besides this limit on idiosyncratic risk sharing, there are no further financial frictions. Agents are allowed to trade physical capital and contingent claims on aggregate risk subject to standard no Ponzi conditions.

Household Problem. We formulate the household problem as a standard consumption-portfolio-choice problem that does not make explicit reference to the capital trading process $d\Delta_t^{k,i}$ as a choice variable. For this purpose, denote by n_t^i the net worth of household i and let $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ be the fraction of net worth invested into capital, own outside equity, and the diversified portfolio of equity, respectively. The own out-

[2] To ensure that this constraint can be satisfied given our assumptions on government spending and investment, we assume that the productivity process satisfies $a_t > \mathcal{G} + \underline{\iota}$ at all times. Here, $\underline{\iota}$ is as in footnote 1.

side equity share $\theta_t^{E,i}$ is typically negative as this asset is issued by the household. The remaining fraction of net worth is invested in bonds. Net worth evolves according to

$$\begin{aligned}\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt \\ + \underbrace{dr_t^{\mathcal{B}} + \theta_t^{K,i}\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right) + \theta_t^{E,i}\left(dr_t^{E,i} - dr_t^{\mathcal{B}}\right) + \theta_t^{\bar{E},i}\left(d\bar{r}_t^{E} - dr_t^{\mathcal{B}}\right)}_{=:dr_t^{n,i}}.\end{aligned} \tag{10.4}$$

where $dr_t^{\mathcal{B}}$, $dr_t^{K,i}(\cdot)$, $dr_t^{E,i}$ and $d\bar{r}_t^E$ denote the returns on bonds, capital, own outside equity and the diversified equity portfolio, respectively. These returns depend on the evolution of market prices, which individuals take as given.[3] We provide explicit expressions for them below.

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio shares $\theta_t^{k,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ to maximize utility V_0^i subject to (10.4), the skin-in-the-game constraint

$$-\theta_t^{E,i} \leq (1-\bar{\chi})\theta_t^{K,i}, \tag{10.5}$$

and a solvency constraint $n_t^i \geq 0$ that rules out Ponzi schemes.

Prices and Returns. Here we formalize how market prices determine returns in (10.4). We denote by q_t^K the market price of a single unit of physical capital. Recall that $\mathcal{P}_t$ denotes the nominal price level, so that the real value of a single unit of bonds is $1/\mathcal{P}_t$. It is more convenient to work with $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$, which is the ratio of the real value of government debt to total capital in the economy.[4]

[3]For the capital return, individuals take the function $\iota^i \mapsto dr_t^{K,i}(\iota^i)$ as given but do internalize how their own investment choice affects the capital return.

[4]This is a normalized version of the inverse price level $1/\mathcal{P}_t$. While the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, q_t^B turns out to be stationary.

With these definitions, the return on bonds is[5]

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d\left(1/\mathcal{P}_t\right)}{1/\mathcal{P}_t} = i_t dt + \frac{d\left(q_t^B K_t/\mathcal{B}_t\right)}{q_t^B K_t/\mathcal{B}_t} = \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - \underbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}_{=:\check{\mu}_t^{\mathcal{B}}} dt \tag{10.6}$$

and the return on agent i's capital is

$$dr_t^{K,i}\left(\iota_t^i\right) = \frac{\left(1-\tau_t\right)a_t - \iota_t^i}{q_t^K}dt + \frac{d(q_t^K \tilde{k}_t^i)}{q_t^K \tilde{k}_t^i},$$

where $\tilde{k}_t^i$ follows the same evolution (10.1) but with trades $d\Delta_t^{k,i}$ set to 0.

In addition to bonds and capital, there are outside equity claims in zero net supply. Outside equity claims issued by household i have the same risk characteristics as the capital return $dr_t^{K,i}$ but may have a different expected return. A return differential may exists because the inside equity holder i requires a compensation for bearing idiosyncratic risk whereas outside equity holders can eliminate this risk by diversification. Specifically,

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \left(dr_t^{K,i} - \mathbb{E}_t[dr_t^{K,i}]\right),$$

where the expected return component $\mathbb{E}_t[dr_t^{E,i}]$ is determined in equilibrium. In equilibrium, all agents optimally hold a perfectly diversified equity portfolio. The return on that portfolio is

$$d\bar{r}_t^E = \int dr_t^{E,i} di.$$

Equilibrium. A competitive equilibrium, given an exogenous government policy and initial conditions, is defined in the usual way as a set of allocations and prices such that all households maximize utility and all markets clear. Here, prices and aggregate variables may depend only on aggregate exogenous histories, that is histories of the aggregate exogenous shocks dZ_t. In contrast, individual outcomes for household i can depend on both aggregate and individual idiosyncratic histories, that is joint histories of the shocks dZ_t and $d\tilde{Z}_t^i$.

Formally, we define equilibrium as follows. For ease of exposition, we restrict atten-

[5] The last equality uses $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$.

tion here to a *symmetric* equilibrium in which the expected return $\mathbb{E}_t[dr_t^{E,i}]$ of all outside equity claims is the same and in which all agents make identical choices for scaled consumption $\hat{c}_t^i := c_t^i / n_t^i$, the investment rate ι_t^i, and portfolio weights $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$.[6] This does not mean that all agents are identical because they can differ with regard to the level of their net worth n_t^i.

Definition 10.1. *Let $K_0 > 0$ be the initial level of capital and let a_t, $\tilde{\sigma}_t$, $\check{\mu}_t^{\mathcal{B}}$ be exogenous processes adapted to the filtration generated by Z. A* symmetric competitive equilibrium *consists of processes for prices $\left(q_t^B, q_t^K, \mathbb{E}_t[dr_t^E]\right)_{t\geq 0}$, scaled consumption $(\hat{c}_t)_{t\geq 0}$, investment rates $(\iota_t)_{t\geq 0}$, portfolio weights $(\theta_t^K, \theta_t^E, \theta_t^{\bar{E}})_{t\geq 0}$, taxes $(\tau_t)_{t\geq 0}$, and aggregate capital $(K_t)_{t\geq 0}$, all adapted to the filtration generated by Z, such that*

1. *Aggregate capital is consistent with the initial condition and satisfies*[7]

$$dK_t = \left(\Phi\left(\iota_t\right) - \delta\right) K_t dt$$

2. *Taxes satisfy the government budget constraint*[8]

$$\tau_t a_t K_t + \check{\mu}_t^{\mathcal{B}} q_t^B K_t = \mathcal{G} K_t$$

3. *For each household $i \in [0,1]$, $c_t^i = \hat{c}_t n_t^i$, $\iota_t^i = \iota_t$, $\theta_t^{K,i} = \theta_t^K$, $\theta_t^{E,i} = \theta_t^E$, $\theta_t^{\bar{E},i} = \theta_t^{\bar{E}}$ solves i's optimization problem (described previously) for arbitrary n_0^i under the assumption that $\mathbb{E}[dr_t^{E,j}] = \mathbb{E}[dr_t^E]$ for all $j \in [0,1]$.*

4. *All markets clear:*

   - *goods market clearing: equation* (10.3) *holds;*
   - *asset market clearing:*[9]

$$\theta_t^K = \frac{q_t^K K_t}{(q_t^K + q_t^B) K_t}, \qquad \theta_t^E + \theta_t^{\bar{E}} = 0.$$

[6] The restriction to symmetric equilibria is without loss of generality. In our environment, any equilibrium must be symmetric because agents face identical investment opportunities and utility is isoelastic.

[7] This equation follows immediately from the individual capital evolutions (10.1) and the fact that idiosyncratic shocks and trading average out.

[8] This equation follows from (10.2) by dividing by $\mathcal{P}_t$ and combining $\mu_t^{\mathcal{B}}$ and i_t.

[9] These are for the capital and equity markets. The bond market clears by Walras' law.

We remark here that we have eliminated nominal quantities from this equilibrium definition by not explicitly adding $\mathcal{B}_0 > 0$ to the initial conditions, by using the scaled bond value q_t^B instead of the nominal price level $\mathcal{P}_t$, and by exogenously specifying government policy in terms of the differential $\check{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$. This is convenient because $\mathcal{P}_t$ and $\mathcal{B}_t$ do not enter any decision problem directly. If one is interested in nominal variables, one can easily recover them ex post from any given equilibrium together with a specification for $\mathcal{B}_0$ and for either $\mu_t^{\mathcal{B}}$ or i_t.

We also remark that we have only defined the equilibrium in terms aggregate variables. However, for any given equilibrium and any given initial net worth distribution $(n_0^i)_{i\in[0,1]}$ consistent with it, i.e. $\int n_0^i di = (q_0^B + q_0^K)K_0$, we can recover individual variables from condition 3 in Definition 10.1 as this condition requires that choices are optimal for arbitrary initial net worth.[10]

10.3 Solution Method

Optimal Consumption and Investment Choice. Like in previous chapters, the investment rate ι enters only the expected capital return $\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota)]$. Assuming that agents hold no negative quantities of capital, all agents agree that it is optimal to choose the investment rate $\iota_t^{\tilde{i}}$ in a way that maximizes the expected capital return $\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}})]$. In particular, $\iota_t^{\tilde{i}} = \iota_t$ for all $\tilde{i}$ and ι_t satisfies the Tobin's Q condition

$$q_t^K = 1 + \phi\iota_t. \tag{10.7}$$

Due to log utility, the optimal consumption choice of all agents is $c_t^{\tilde{i}} = \rho n_t^{\tilde{i}}$. Integrating the optimal consumption condition across all households i yields

$$C_t = \int c_t^i di = \rho \int n_t^i di = \rho(q_t^B + q_t^K)K_t,$$

where the last equality follows from the fact that aggregate net worth consists precisely

[10]The individual variables are, of course, dependent on histories of the idiosyncratic shocks $d\tilde{Z}_t^i$ as well, that is these processes for agent i are adapted to the filtration generated by the pair $(Z, \tilde{Z}^i)$.

of all capital and bond wealth combined.[11]

We next use $q_t^B + q_t^K = q_t^K/(1-\vartheta_t)$ by the definition of ϑ_t to replace the right-hand side of the previous equation and get

$$C_t = \frac{\rho}{1-\vartheta_t} q_t^K K_t.$$

Substituting this into goods market clearing (10.3) and using equation (10.7) to eliminate q_t^K yields the equation

$$\frac{\rho}{1-\vartheta_t}\left(1+\phi\iota_t\right) + \mathcal{G} + \iota_t = a_t.$$

Solving this linear equation yields the expression for ι_t. Substituting this result into equation (10.7) recovers the solution for q_t^K. Finally, using the identity $q_t^B = \frac{\vartheta_t}{1-\vartheta_t} q_t^K$, which follows directly from the definition of ϑ_t, we obtain the expression for q_t^B.

Proposition 10.1. *In any equilibrium, the investment rate, (scaled) value of government bond, and price of physical capital are given by*

$$\iota_t = \frac{(1-\vartheta_t)\check{a}_t - \rho}{1-\vartheta_t+\phi\rho}, \tag{10.8}$$

$$q_t^B = \vartheta_t \frac{1+\phi\check{a}_t}{1-\vartheta_t+\phi\rho}, \tag{10.9}$$

$$q_t^K = (1-\vartheta_t)\,\frac{1+\phi\check{a}_t}{1-\vartheta_t+\phi\rho}, \tag{10.10}$$

where $\check{a}_t := a_t - \mathcal{G}$.

These equations determine the equilibrium uniquely as a function of the exogenous process a_t and the (endogenous) bond wealth share ϑ_t. To fully characterize the equilibrium, we thus only need to determine ϑ_t.

ϑ_t can be thought of as a relative price between capital assets (including equity which is a claim to capital) and government bonds. It is determined by households'

[11] Note that, in our formulation, taxes are effectively imposed on capital holdings such that the present value of tax liabilities of households is implicitly capitalized in capital valuations. Also note that outside equity claims are in zero net supply and thus do not contribute to aggregate net worth.

portfolio choice and asset market clearing. We use the martingale method to derive optimal choice conditions for the household problem.

Optimal Portfolio Choice. Recall that under the martingale method, the excess expected return of the risky asset A to the risky asset B is given by

$$\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \tilde{\varsigma}_t^i(\tilde{\sigma}_t^A - \tilde{\sigma}_t^B),$$

where the excess return is determined by the differences in their exposures to the underlying Brownian motions.

There are four alternative derivations for the expected excess return relative to the bond return, each based on a different numeraire or asset definition.

Under the consumption numeraire:

(i) The expected excess return of capital relative to the bond return. This expression becomes more involved when $\sigma_t^{\mathcal{B}} \neq 0$, as seigniorage becomes stochastic and lowers capital taxes, complicating the return on capital. The expression takes the following form:

$$\mathrm{d}r_t^{K,\tilde{i}} = \left(\frac{a_t - \mu_t^{\mathcal{G}} - \iota_t^{\tilde{i}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\left(\check{\mu}_t^{\mathcal{B}} + (\sigma + \sigma_t^{q^{\mathcal{B}}} - \sigma_t^{\mathcal{B}})\sigma_t^{\mathcal{B}}\right)\right)\mathrm{d}t$$
$$+ \left(\sigma + \sigma_t^{q^K} + \frac{q_t^{\mathcal{B}}\sigma_t^{\mathcal{B}} - \sigma_t^{\mathcal{G}}}{q_t^K}\right)\mathrm{d}Z_t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}}.$$

(ii) The expected excess return of individual net worth (i.e., the portfolio return) relative to the bond return.

Under the **total net worth numeraire**:

(iii) The expected excess return of capital relative to the bond return.

(iv) The **expected excess return of individual net worth (i.e., the net worth share)** relative to the bond return.

This last expression remains tractable even when $\sigma_t^{\mathcal{B}} \neq 0$, and for this reason, we adopt this derivation as our baseline going forward.

We define the drift and volatility of the value of a single bond or coin expressed in the N_t-numeraire as

$$\frac{\mathrm{d}(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} = \mu_t^{\vartheta/\mathcal{B}}\,\mathrm{d}t + \sigma_t^{\vartheta/\mathcal{B}}\,\mathrm{d}Z_t.$$

Applying Itô's quotient rule to the ratio $\vartheta_t/\mathcal{B}_t$, we obtain

$$\begin{aligned}\frac{\mathrm{d}(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} &= \left(\mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})\right)\mathrm{d}t + \left(\sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}}\right)\mathrm{d}Z_t \\ &= (\mu_t^{\vartheta} - \mu_t^{\mathcal{B}})\mathrm{d}t + \sigma_t^{\vartheta}\mathrm{d}Z_t,\end{aligned}$$

because the outstanding nominal government debt follows the continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}}\mathcal{B}_t dt$ with no volatility. This result implies that the drift and volatility of the value of the bond/coin under the N_t-numeraire are given by

$$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}}, \quad \sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta}.$$

By the asset pricing equation under the martingale method, the expected return of individual net worth satisfies

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\tilde{\eta}^{\tilde{i}}}]}{\mathrm{d}t} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right) + (\varsigma_t - \sigma_t^N)\cdot 0 + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}.$$

Similarly, the expected return of a bond or coin under the N_t-numeraire is

$$\frac{\mathbb{E}[\mathrm{d}r_t^{\vartheta/\mathcal{B}}]}{\mathrm{d}t} = i_t + \mu_t^{\vartheta/\mathcal{B}} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^{\mathcal{B}}} - \sigma\sigma_t^{q^K+q^{\mathcal{B}}} + \varsigma_t\sigma_t^{q^K+q^{\mathcal{B}}}\right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}}}_{\text{price of risk in } N_t\text{-numeraire}}.$$

The advantage of this martingale method is that when we subtract the bond return from the portfolio return, all terms associated with the risk-free rate in the N_t-numeraire cancel, leaving only the risk premium terms:

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/\mathcal{B}} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/\mathcal{B}} + \tilde{\varsigma}_t(1-\theta_t)\bar{\chi}\tilde{\sigma}.$$

We then apply the capital market clearing condition $1 - \theta_t = 1 - \vartheta_t$, and substitute in the expressions for the prices of risk under log utility. Under log utility, we have $\check{\rho}_t = \rho$, $\varsigma_t = \sigma_t^N$, and $\tilde{\varsigma}_t = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t$, which simplifies the expression to:

$$\rho - i_t - \mu_t^{\vartheta/\mathcal{B}} = (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2.$$

Recall the change-of-numeraire adjustment formulas:

$$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}}, \quad \sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta}.$$

Substituting into the asset pricing equation under the martingale method, we obtain:

$$\rho = (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2 + \mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}}.$$

Multiplying both sides by ϑ, and expressing ϑ as a function of $\tilde{\sigma}^2$, yields:

$$\rho\, \vartheta(\tilde{\sigma}^2) = (1 - \vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2\, \vartheta(\tilde{\sigma}^2) + \mu_t^{\vartheta}\, \vartheta(\tilde{\sigma}^2) - \check{\mu}_t^{\mathcal{B}}\, \vartheta(\tilde{\sigma}^2).$$

To compute the drift of $\vartheta(\tilde{\sigma}^2)$, we apply Itô's Lemma:

$$\mu_t^{\vartheta}\, \vartheta_t = \vartheta'(\tilde{\sigma}^2)\, \mu_t^{\tilde{\sigma}}\, \tilde{\sigma}_t + \frac{1}{2}\, \vartheta''(\tilde{\sigma}^2)\, (\sigma^{\tilde{\sigma}}\, \tilde{\sigma}_t)^2.$$

Next, we equate the drift expressions and include the time derivative in the dynamic case. The resulting equation becomes:

$$\rho\, \vartheta(\tilde{\sigma}^2) = \Big[(1 - \vartheta(\tilde{\sigma}^2))^2\, \bar{\chi}^2\, \tilde{\sigma}^2 - \check{\mu}_t^{\mathcal{B}}\Big]\, \vartheta(\tilde{\sigma}^2) - \psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)\, \vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2}\, \vartheta''(\tilde{\sigma}^2),$$

or in time-dependent form:

$$\rho\, \vartheta_t(\tilde{\sigma}^2) = \partial_t \vartheta_t(\tilde{\sigma}^2) + \underbrace{\Big[(1 - \vartheta(\tilde{\sigma}^2))^2\, \bar{\chi}^2\, \tilde{\sigma}^2\, \vartheta(\tilde{\sigma}^2) - \check{\mu}^{\mathcal{B}}\, \vartheta(\tilde{\sigma}^2)\Big]}_{\boldsymbol{u\vartheta}} - \underbrace{\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)\, \vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2}\, \vartheta''(\tilde{\sigma}^2)}_{\mathbf{M}\boldsymbol{\vartheta}}.$$

To solve for $\vartheta(\tilde{\sigma}^2)$, one can use a function iteration method as demonstrated in the previous chapters.

We temporarily depart from log utility and consider CRRA utility as a model extension, both to enrich the analysis for interested readers and to illustrate the numerical solution method in greater depth. For CRRA utility, the analysis requires de-scaling individual value functions to v^i. In this setting (with a single sector), equilibrium is characterized by solving two partial differential equations for $v(\tilde{\sigma}^2)$ and $\vartheta(\tilde{\sigma}^2)$, which jointly determine the endogenous distribution of risk and asset prices.

We begin by recalling the CRRA value function. For an agent $\tilde{i}$, the value function takes the form:

$$V_t^{\tilde{i}} = \frac{1}{\rho}\frac{(\omega_t^i \tilde{n}_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} = \underbrace{\frac{1}{\rho}\left(\frac{\omega_t^i n_t^i}{K_t}\right)^{1-\gamma}}_{v_t^i :=} \underbrace{\left(\frac{\tilde{n}_t^{\tilde{i}}}{n_t^i}\right)^{1-\gamma}}_{(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}} \cdot \frac{K_t^{1-\gamma}}{1-\gamma}.$$

Next, we recall the envelope condition for the value function:

$$\begin{aligned}
\frac{\partial V_t^{\tilde{i}}}{\partial \tilde{n}_t^{\tilde{i}}} &= \underbrace{\frac{1}{\rho}(\omega_t^i)^{1-\gamma}}_{=v_t^i\left(\frac{K_t}{n_t^i}\right)^{1-\gamma}} \cdot \underbrace{(\tilde{n}_t^{\tilde{i}})^{-\gamma}}_{=(\tilde{\eta}_t^{\tilde{i}})^{-\gamma}(n_t^i)^{-\gamma}} = (c_t^{\tilde{i}})^{-\gamma} = \frac{\partial u}{\partial c_t^{\tilde{i}}} \\
&= v_t^i K_t^{1-\gamma}(n_t^i)^{-1}(\tilde{\eta}_t^{\tilde{i}})^{-\gamma} \\
&= v_t^i K_t^{-\gamma}(q_t^{\mathcal{B}} + q_t^K)^{-1}(\tilde{\eta}_t^{\tilde{i}})^{-\gamma} \quad \text{(after noting that } n_t^i = N_t = (q_t^{\mathcal{B}} + q_t^K)K_t).
\end{aligned}$$

Turning to the pricing of risk, for the aggregate price of risk (noting that $\sigma = 0$, i.e., capital has no aggregate risk), we use:

$$\sigma_t^v - \gamma\not\sigma - \sigma_t^{q^{\mathcal{B}}+q^K} = -\gamma\sigma_t^{c^{\tilde{i}}} = -\varsigma_t.$$

For the idiosyncratic price of risk, we recall that $\sigma_t^{\tilde{\eta}^{\tilde{i}}} = \sigma_t^{\tilde{n}^{\tilde{i}}}$, and obtain:

$$\tilde{\varsigma}_t^{\tilde{i}} = \gamma\tilde{\sigma}_t^{n^{\tilde{i}}} = \gamma(1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t.$$

Combining the capital market clearing: $1-\theta = 1-\vartheta$, we get

$$\check{\rho}_t - \mu_t^{\frac{\vartheta}{\mathcal{B}}} = (\sigma_t^v - \cancel{(\gamma-1)\sigma})\sigma_t^{\frac{\vartheta}{\mathcal{B}}} + \gamma(1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}^2$$

Next, we determine the BSDE for v_t^i. By Itô's product rule, the value function is given by

$$\begin{aligned}\frac{\mathrm{d}V_t^{\tilde{i}}}{V_t^{\tilde{i}}} &= \frac{\mathrm{d}\left(v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}K_t^{1-\gamma}\right)}{v_t^i(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}K_t^{1-\gamma}} \\ &= \left[\mu_t^v + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)(\cancel{\sigma^2} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2) + \cancel{(1-\gamma)\sigma\sigma_t^v}\right]\mathrm{d}t + \text{volatility}\end{aligned}$$

Recall by consumption optimality, $\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho \mathrm{d}t + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$ follows a martingale. Hence, drift above $= \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$. Equate drift terms to obtain BSDE:

$$\mu_t^v + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)(\cancel{\sigma^2} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2) + \cancel{(1-\gamma)\sigma\sigma_t^v} = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

Now, we derive two PDEs of $v(\tilde{\sigma}^2)$ and $\vartheta(\tilde{\sigma}^2)$ for the numerical model solution. We begin by generalizing the PDE for ϑ, now allowing for $\gamma \neq 1$ and including aggregate risk via σ_t^v. Recall that the drift and volatility of $\vartheta/\mathcal{B}$ satisfy:

$$\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}}, \quad \sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta}.$$

Substituting these expressions into the asset pricing equation gives:

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/\mathcal{B}} = (\sigma_t^v - \cancel{(\gamma-1)\sigma})\sigma_t^{\vartheta/\mathcal{B}} + \gamma(1-\theta_t)^2\bar{\chi}^2\tilde{\sigma}^2.$$

This implies:

$$\check{\rho}_t = \mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}} + (\sigma_t^v - \cancel{(\gamma-1)\sigma})\sigma_t^{\vartheta} + \gamma(1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}^2. \tag{10.11}$$

Multiplying through by ϑ and expressing all variables as functions of $\tilde{\sigma}^2$, we obtain:

$$\check{\rho}_t(\tilde{\sigma}^2)\,\vartheta(\tilde{\sigma}^2) = \left[\mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}} + \sigma_t^{v}(\tilde{\sigma}^2)\sigma_t^{\vartheta} + \gamma(1-\vartheta(\tilde{\sigma}^2))^2\bar{\chi}^2\tilde{\sigma}^2\right]\vartheta(\tilde{\sigma}^2).$$

To characterize the dynamics of $\vartheta(\tilde{\sigma}^2)$, we apply Itô's Lemma. The drift and volatility of ϑ are given by:

$$\mu_t^{\vartheta}\vartheta_t = \vartheta'(\tilde{\sigma}^2)\mu_t^{\tilde{\sigma}}\tilde{\sigma}_t + \frac{1}{2}\vartheta''(\tilde{\sigma}^2)(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2, \quad \sigma_t^{\vartheta}\vartheta_t = -\vartheta'(\tilde{\sigma}^2)\sigma^{\tilde{\sigma}}\tilde{\sigma}_t.$$

We equate drift terms and include the time derivative, leading to the PDE:

$$\begin{aligned}\check{\rho}(\tilde{\sigma}^2)\,\vartheta(\tilde{\sigma}^2) = {} & \left[\gamma(1-\vartheta(\tilde{\sigma}^2))^2\bar{\chi}^2\tilde{\sigma}^2 - \check{\mu}_t^{\mathcal{B}}\right]\vartheta(\tilde{\sigma}^2) - \psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)\vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2}\vartheta''(\tilde{\sigma}^2) \\ & - \sigma_t^{v}(\tilde{\sigma}^2)\,\sigma^{\tilde{\sigma}}\tilde{\sigma}_t\,\vartheta'(\tilde{\sigma}^2).\end{aligned}$$

Or, in full dynamic form:

$$\begin{aligned}\check{\rho}_t(\tilde{\sigma}^2)\,\vartheta_t(\tilde{\sigma}^2) = \partial_t\vartheta_t(\tilde{\sigma}^2) + & \underbrace{\left[\gamma(1-\vartheta(\tilde{\sigma}^2))^2\bar{\chi}^2\tilde{\sigma}^2 - \check{\mu}_t^{\mathcal{B}}\right]\vartheta(\tilde{\sigma}^2)}_{\boldsymbol{u\vartheta}} \\ & + \underbrace{\left[-\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) - \sigma^{\tilde{\sigma}}\tilde{\sigma}_t\sigma_t^{v}(\tilde{\sigma}^2)\right]\vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2}\vartheta''(\tilde{\sigma}^2)}_{\mathbf{M}\boldsymbol{\vartheta}}.\end{aligned}$$

Next, we derive the PDE for $v(\tilde{\sigma}^2)$. Starting from the backward stochastic differential equation:

$$\mu_t^{v} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\cancel{\sigma^2} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2\right) + \cancel{(1-\gamma)\sigma\sigma_t^{v}} = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}},$$

we apply Itô's Lemma to $v(\tilde{\sigma}^2)$ to obtain:

$$dv(\tilde{\sigma}^2) = \underbrace{\left(-\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)v'(\tilde{\sigma}^2) + \frac{1}{2}(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t^2)v''(\tilde{\sigma}^2)\right)}_{=v\mu_t^{v}}\mathrm{d}t - \underbrace{\sigma^{\tilde{\sigma}}\tilde{\sigma}_t v'(\tilde{\sigma}^2)}_{=v\sigma_t^{v}}\mathrm{d}Z_t.$$

The PDE for $v(\tilde{\sigma}^2)$ is then:

$$\rho v(\tilde{\sigma}^2) = \partial_t v(\tilde{\sigma}^2) + \overbrace{\left(\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\cancel{\sigma^2} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2) + (1-\gamma)\cancel{\sigma\sigma_t^{v}}\right) v}^{\boldsymbol{u}\boldsymbol{v}} \\ \underbrace{-\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)v'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2} v''(\tilde{\sigma}^2)}_{\mathbf{M}\boldsymbol{v}}$$

Finally, the system of two PDEs — one for $\vartheta(\tilde{\sigma}^2)$, and one for $v(\tilde{\sigma}^2)$ — is solved numerically by iterating over both equations simultaneously in an outer loop. No inner loop is needed, since within-sector parameters like κ and χ do not generate macro-level feedback effects.

Uniqueness of Stationary Monetary Equilibria From this point onward, we return to the log utility case. Our analysis of the model solution yields the following proposition.

Proposition 10.2. *In any equilibrium, ϑ must satisfy the equation*

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \check{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2\right)\vartheta_t dt. \tag{10.12}$$

Conversely, any $[0,1]$-valued solution ϑ to this equation is associated with a competitive equilibrium.

In our model, equilibria as defined in Definition 10.1 may not be unique because equation (10.12) for ϑ is a fixed-point equation that can have multiple solutions. However, we establish here that if we make suitable Markov assumptions on exogenous processes, then there is always at most one solution to equation (10.12) that is both *non-degenerate*, i.e. different from $\vartheta \equiv 0$, and *stationary*. This solution corresponds to a unique stationary "monetary" equilibrium in which government bonds have always a positive value. This equilibrium has a particularly appealing property: it is the only equilibrium consistent with any belief that, with positive probability, bonds retain a positive value bounded away from zero in the arbitrarily distant future. In this chapter, we always focus on this equilibrium with the exception of Section 10.5.1, where we briefly discuss how alternative equilibria may be associated with a loss of safe asset status.[12]

We next formulate our uniqueness result. To do so, we first need to provide a precise definition of the notion of stationarity we are requiring.

Definition 10.2. *The exogenous processes $\tilde{\sigma}_t$, a_t, and $\check{\mu}_t^{\mathcal{B}}$ are* stationary *if there exists an ergodic Markov state process X_t on a compact domain $\mathbb{X} \subset \mathbb{R}^n$ and continuous functions $\tilde{\sigma}, a, \check{\mu}^{\mathcal{B}} : \mathbb{X} \to \mathbb{R}$ such that*

$$\tilde{\sigma}_t = \tilde{\sigma}(X_t), \qquad a_t = a(X_t), \qquad \check{\mu}_t^{\mathcal{B}} = \check{\mu}^{\mathcal{B}}(X_t)$$

for all $t \geq 0$.[13]

Given stationary exogenous process, we say that a solution ϑ_t to BSDE (10.12) is stationary *if there is a continuous function $\vartheta : \mathbb{X} \to [0,1]$ such that $\vartheta_t = \vartheta(X_t)$ for all t.*

Proposition 10.3 (Uniqueness of stationary non-degenerate solutions)**.** *Suppose the exogenous processes are stationary and $\rho + \check{\mu}^{\mathcal{B}}(X) > 0$ for all $X \in \mathbb{X}$. Then, equation (10.12) has at most one stationary nondegenerate (i.e. not identically 0) solution.*

The proof for Proposition 10.3 is in the Appendix B.1 of Brunnermeier et al. (2024). The key idea behind the proof is to investigate the finite-horizon version of BSDE (10.12) and show two key properties. First, the mapping from terminal conditions $\vartheta_T > 0$ to the (always unique) finite-horizon solution over $[0, T]$ represents a contraction in a suitable sense. Second, conditional on a fixed state X_t, the finite-horizon solutions are monotonic in time t. We establish these properties with the help of the comparison theorem for BSDEs.

In fact, these properties do not only allow us to establish uniqueness of the non-degenerate stationary solution but also yield an additional limit result: for any given terminal condition $\vartheta_T > 0$, the solution to the finite-horizon equation converges to the unique non-degenerate stationary solution as $T \to \infty$, provided the latter exists.

[12] We make two further remarks with regard to this equilibrium selection. First, the selection of the unique stationary monetary equilibrium is in complete analogy to the standard choice in models with monetary frictions. Second, under the fiscal policy arrangements discussed in Brunnermeier et al. (2021), this equilibrium would emerge as the unique equilibrium in the sense of Definition 10.1.

[13] Informally, an ergodic Markov process can travel from any state to any other state.

Conceptually, this additional result is important because it implies that all nonstationary solutions must converge to 0 in the distant future. Practically, the additional result matters because it ensures that the solution procedure that we employ for the numerical illustration in Section 10.4.3 converges to the desired solution.

10.4 Properties and Asset Pricing of Safe Assets

10.4.1 Safe Asset Definition

Individuals hold a safe asset for precautionary reasons, which they can "liquify" at an above average return when they face an idiosyncratic and/or aggregate shock and they attach a high marginal value to extra resources. This marginal value for individual i is measured by that individual's stochastic discount factor (SDF) process, which we denote by ξ_t^i. This process satisfies $\xi_0^i = 1$ and $d\xi_t^i/\xi_t^i = -r_t^f dt - \varsigma_t dZ_t - \tilde{\varsigma}_t^i d\tilde{Z}_t^i$, with a negative drift term equal to the risk-free rate and aggregate and idiosyncratic prices of risk, ς_t, $\tilde{\varsigma}_t^i$ respectively.[14] The return of citizen i's net worth, $r^{n,i}$ is given by equation (10.4).

The following definition makes the "Good Friend Analogy" of a safe asset precise.

Definition 10.3. *An asset j is in equilibrium a* safe asset *for individual i at time t if the conditional covariance between her SDF and return of the asset in excess to her net worth return, $dr_t^j - dr_t^{n,i}$, is positive, i.e.* $\text{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^j - dr_t^{n,i}] > 0$.

We make several remarks. First, the safe asset concept is an equilibrium concept. The same asset with the same cash flows can be a safe asset in one equilibrium and not a safe asset in another equilibrium. The returns r_t^j and $r_t^{n,i}$ as well as the SDF ξ_t^i depend on the equilibrium the economy is in.

[14] In integral form the individual SDF is

$$\xi_t^i = \underbrace{\exp\left(-\int_0^t r_\tau^f d\tau\right)}_{\text{time discounting}} \cdot \underbrace{\exp\left(-\int_0^t \varsigma_t dZ_\tau - \frac{1}{2}\int_0^t \varsigma_\tau^2 d\tau\right)}_{\text{aggregate risk}} \cdot \underbrace{\exp\left(-\int_0^t \tilde{\varsigma}_\tau d\tilde{Z}_\tau^i - \frac{1}{2}\int_0^t \tilde{\varsigma}_\tau^2 d\tau\right)}_{\text{idiosyncratic risk}},$$

where the second and third factors are martingales.

Second, an asset is safe for individual i relative to her own net worth n^i. A safe asset provides (on average) better payoffs in high marginal utility states than the total net worth portfolio.[15]

Third, the definition is contingent on the economy's state. An asset can be safe at a specific point in time t but may lose its "safe-asset status" at a different time.

Fourth, the definition focuses exclusively on the risk properties of the asset. This is sufficient because, in our model, we have assumed that all assets enjoy perfect market liquidity (all the time). However, as we explain in the next section, safety derives from retrading. Therefore, liquidity is also an important feature of a safe asset. Beyond our model environment, a definition of a safe asset should also include that the asset can be traded easily without any trading frictions, e.g. due to asymmetric information.

The following proposition shows that government bonds are indeed a safe asset in the steady-state solution derived in Section **??**.

Proposition 10.4. *If $\tilde{\sigma}_t > 0$, a_t, and $\check{\mu}_t^{\mathcal{B}}$ are constant over time, the government bond is a safe asset for all agents at all times.*

In equilibrium each citizen's net worth return is positively correlated with her consumption growth rate and hence negatively correlated with her SDF, as the incomplete markets frictions prevents her from hedging her idiosyncratic risk. Since the government bond return is risk-free, $\mathrm{Cov}_t[d\xi_t^i/\xi_t^i, r^f - dr_t^{n,i}] > 0$ and consequently the government bond is a safe asset. Capital is not a safe asset as $\mathrm{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^K] \leq \mathrm{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^{n,i}]$.

More generally, with stochastic idiosyncratic risk $\tilde{\sigma}_t$, the real return of the government bond is not risk-free. However, as we will see in Section 10.4.3 in the context of our calibrated model, it depreciates in volatile times less than citizen's net worth. Indeed it even appreciates, and hence more than satisfies the safe asset criterion.

[15] Alternatively, one could define an asset as (absolutely) safe by $\mathrm{Cov}_t[d\xi_t^i/\xi_t^i, dr_t^j] \geq 0$.

10.4.2 Two Perspectives on Asset Valuation Equations

The value of government debt has to satisfy a debt valuation equation that relates the real value of debt to the present value of future primary surpluses. In this section, we contrast the standard asset pricing approach of deriving such an equation with an alternative approach that emphasizes and makes explicit the benefits from retrading of the asset. Both approaches imply an identical valuation formula with complete markets, but lead to two distinct equations when markets are incomplete. These equations provide two different perspectives for asset pricing in incomplete market environments. Here, we apply these perspectives to government debt valuation.[16]

The standard approach to asset valuation is based on a buy-and-hold fiction. An asset is priced as if it was held forever, so that the value of the asset must equal the present value of all future cash flows derived from the asset's payouts.[17] We call this the *"buy and hold perspective"* of asset pricing. Applied to the total government debt stock, the cash flows in the present value formula are precisely the primary surpluses.[18]

We propose an alternative approach that recognized that, in equilibrium, individual agents may not intend to buy and hold an asset, but plan to retrade it whenever they face a shock. They raise cash flow by selling the asset and face a cash outflow when buying more of the asset. The aggregate stock of the asset can also be priced by first valuing the cash flows from agents' optimal dynamic trading strategy in equilibrium and then aggregating across all agents. This approach leads to a *"dynamic trading perspective"* of asset pricing. Importantly, the aggregated present value of individual trades may be different from zero, even though trades among private agents wash out in the aggregate. Hence, the dynamic trading perspective incorporates an additional term that makes explicit the aggregate value of equilibrium trades. Applied to gov-

[16]For concreteness, we present the equations only for government debt in the context of our model. But we remark that our alternative valuation approach, the "dynamic trading perspective", is general and can be applied in any incomplete markets setting to any asset.

[17]To be fully precise, the buy-and-hold fiction does assume an eventual liquidation of the asset which results in a single terminal resale cash flow. The liquidation time is then sent to infinity, so that the present value of this cash flow only matters if there is a bubble.

[18]Despite the label, this may require the agent to trade, but only directly with the issuer, the government, in order to absorb new debt issuance, not with other agents.

ernment bonds in our model, this term is positive because bonds allow agents to self-insure against idiosyncratic shocks. We refer to this term as the "service flow" term from retrading.

The distinction between the two perspectives is particularly illuminating when there can be rational bubbles. Dynamic programming implies that a transversality condition has to hold only from the dynamic trading perspective, for each individual agent. Optimality does not imply a transversality condition from the buy and hold perspective. For that reason, a gap may appear between the value of debt and the present value of surpluses from the latter perspective. This gap is closed by an additional bubble term.

Unfortunately, it can even happen that both the bubble term and the present value of primary surpluses are infinite with opposite sign, yet their sum still converges as the time horizon approaches infinity. In contrast, the terms in the dynamic trading perspective are always well-defined and finite.

Buy and Hold Perspective. We denote by $s_t := \tau_t a_t - \mathcal{G}$ the primary surplus per unit of aggregate capital. Recall that ξ_t^i is agent i's SDF process. From the buy and hold perspective, individual uninsurable risk does not enter the valuation equation directly, so that only the aggregate component $\bar{\xi}_t$ of the processes ξ_t^i matters, i.e. $d\bar{\xi}_t/\bar{\xi}_t = -r_t^f dt - \varsigma_t dZ_t$.[19] Absent aggregate shocks (including inflation shocks), the government bond is a risk-free asset and the relevant discount factor is simply $\bar{\xi}_t = \exp(-\int_0^t r_\tau^f d\tau)$.

Proposition 10.5 (Buy and Hold Perspective)**.** *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \bar{\xi}_t s_t K_t dt \right] + \mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right). \tag{10.13}$$

The derivation of equation (10.13) is standard. We start by using $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ to

[19]The aggregate discount factor is the projection of any individual citizen's SDF onto a common filtration generated by the aggregate Brownian Z. Put differently, $\bar{\xi}_t := \mathbb{E}\left[\xi_t^i \mid Z_\tau : \tau \le t\right]$, takes conditional expectations with respect to the history of aggregate shocks dZ_τ up to time t but without any knowledge of idiosyncratic shocks. Equivalently, $\bar{\xi}_t = \int \xi_t^i di$ is the unweighted average of individual SDFs.

rewrite the government flow budget constraint (10.2) as

$$-\left(d\mathcal{B}_t - i_t\mathcal{B}_t dt\right) = \mathcal{P}_t \underbrace{\left(\tau a_t - \mathcal{G}\right)}_{=s_t} K_t dt.$$

Multiplying both sides by the nominal SDF $\xi_t^i/\mathcal{P}_t$ of agent i and using Itô's product rule to replace $\xi_t^i/\mathcal{P}_t d\mathcal{B}_t$ with $d\left(\xi_t^i/\mathcal{P}_t\mathcal{B}_t\right) - \mathcal{B}_t d(\xi_t^i/\mathcal{P}_t)$[20] yields

$$-d\left(\xi_t^i\mathcal{B}_t/\mathcal{P}_t\right) + \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right) + i_t\xi_t^i/\mathcal{P}_t dt\right) = \xi_t^i s_t K_t dt.$$

Integrating this equation from $t = 0$ to $t = T$, taking expectations, and solving for $\xi_0^i\mathcal{B}_0/\mathcal{P}_0$ implies

$$\xi_0^i\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] - \mathbb{E}\left[\int_0^T \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right) + i_t\xi_t^i/\mathcal{P}_t dt\right)\right] + \mathbb{E}\left[\xi_T^i\frac{\mathcal{B}_T}{\mathcal{P}_T}\right]. \quad (10.14)$$

Equation (10.14) is simply an accounting identity, an integrated version of the government flow budget constraint (10.2). We now note that the individual SDF ξ_t^i must price the bond because agent i is marginal in the bond market. This implies that the associated nominal SDF $\xi_t^i/\mathcal{P}_t$ must decay on average at the nominal market interest rate, so that the second term in equation (10.14) vanishes. In addition, we can replace the individual SDF ξ_t^i with the average SDF $\bar{\xi}_t$ because equation (10.14) holds for all individuals i and $s_t K_t$ and $\mathcal{B}_T/\mathcal{P}_T$ are free of idiosyncratic risk. When taking the limit $T \to \infty$, we obtain equation (10.13).

This equation consists of two terms: a discounted stream of primary surpluses plus (the limit of) a discounted terminal value. The latter can be positive even in the limit, giving rise to a possible bubble on government debt.[21] The reason is that individual transversality conditions do not necessarily imply $\mathbb{E}\left[\bar{\xi}_T\frac{\mathcal{B}_T}{\mathcal{P}_T}\right] \to 0$ because agents do not buy and hold a fixed fraction of the government debt stock but constantly trade bonds. If the terminal condition does converge to zero, then we obtain the traditional debt valuation equation that says that the value of debt must equal the present value

[20] There is no quadratic covariation term because $d\mathcal{B}_t$ is absolutely continuous.
[21] The bubble term on government debt is discussed in detail in Brunnermeier et al. (2021).

of primary surpluses.

Dynamic Trading Perspective. Let $\eta_t^i := n_t^i / N_t$ be agent i's net worth share and denote again i's SDF process by ξ_t^i. In our model, η_t^i also represents the share of total bonds held by i because all agents hold identical portfolios (up to scale). Pricing individual bond portfolios and aggregating over agents i yields our main valuation equation from the dynamic trading perspective,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \int \left(\mathbb{E}\left[\int_0^\infty \xi_t^i \cdot \eta_t^i s_t K_t dt \right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \cdot \eta_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right] \right) di. \tag{10.15}$$

The real value of all outstanding public debt $\mathcal{B}_0/\mathcal{P}_0$ is the integral of the valuations of individual debt holdings. Each of these valuations consists of two terms, the discounted value of the share of future primary surpluses, $\eta_t^i s_t K_t := \eta_t^i (\tau_t a - \mathcal{G}) K_t$, paid out to agent i plus the discounted value of future service flows, $\eta_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}$, that agent i derives from trading bonds. The safe asset service flow is due to partial insurance. It increases in the value of public debt and the amount of idiosyncratic risk the citizen is exposed to. The latter depends on her portfolio share on physical capital $(1 - \vartheta_t)$ and undiversified risk $\bar{\chi}\tilde{\sigma}_t$. Government bonds provide a positive service flow because the agent sells bonds precisely when she experiences a negative idiosyncratic shock, so that the bond portfolio generates a positive payout in times of high marginal utility ξ_t^i.

Equation (10.15) emphasizes that the total value is obtained by aggregating individual portfolio valuations. Mathematically, it is more convenient to interchange the order of integration (Fubini's Theorem). This yield the following proposition.

Proposition 10.6 (Dynamic Trading Perspective)**.** *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di \right)}_{=:\xi_t^{**}} s_t K_t dt \right] + \mathbb{E}\left[\int_0^\infty \underbrace{\left(\int \xi_t^i \eta_t^i di \right)}_{=:\xi_t^{**}} \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right]. \tag{10.16}$$

This equation discounts aggregate cash flows (surpluses and service flows) free of idiosyncratic risk like equation (10.13) obtained from the buy and hold perspective. But importantly, the "stochastic discount factor" ξ_t^{**} in this equation is a net-worth-weighted average of individual stochastic discount factors. Since a single agent's individual net worth weight η_t^i co-moves negatively with her SDF ξ_t^i, the discount factor is lower (discount rate is higher) than the usual unweighted average discount factor (used in the buy and hold perspective). It turns out this weighted average SDF is not a mere mathematical artifact from swapping integrals but has an economic interpretation. It is the correct marginal rate of intertemporal substitution of aggregate cash flows for a pseudo-representative agent who is forced to distribute aggregate consumption to individuals according to the equilibrium consumption shares c_t^i/C_t in our model. We discuss this interpretation in more detail below.

To derive valuation equations (10.15) and (10.16), we start by valuing agent i's bond portfolio at time $t = 0$.

We can write the evolution of the bond portfolio as

$$\frac{db_t^i}{b_t^i} = dr_t^{\mathcal{B}} + d\Delta_t^{b,i}. \tag{10.17}$$

Absent trading and payouts, the bond portfolio grows at the (stochastic) bond return $dr_t^{\mathcal{B}}$, but the actual portfolio value has to be adjusted for cash inflows $b_t^i d\Delta_t^{b,i}$ due to trading and payouts. Writing

$$dr_t^{\mathcal{B}} = \mu_t^{r^{\mathcal{B}}} dt + \sigma_t^{r^{\mathcal{B}}} dZ_t, \qquad d\Delta_t^{b,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$$

and using Ito's product rule, we obtain for the discounted bond wealth

$$\begin{aligned}\frac{d(\xi_t^i b_t^i)}{\xi_t^i b_t^i} = &\Big(\underbrace{\mu_t^{r^{\mathcal{B}}} - r_t^f - \varsigma_t \sigma_t^{r^{\mathcal{B}}}}_{=0} + \mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i}\Big) dt \\ &+ \left(\sigma_t^{r^{\mathcal{B}}} + \sigma_t^{\Delta,i} - \varsigma_t\right) dZ_t + \left(\tilde{\sigma}_t^{\Delta,i} - \tilde{\varsigma}_t\right) d\tilde{Z}_t^i. \end{aligned} \tag{10.18}$$

Here, the first part of the drift is zero by standard asset pricing logic because agent i is marginal in the market for government bonds. Integrating over $t \in [0, T]$, taking expectations, and rearranging yields

$$\xi_0^i b_0^i = -\mathbb{E}_0\left[\int_0^T \xi_t^i b_t^i\left(\mu_t^{\Delta,i} - \varsigma_t\sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i\tilde{\sigma}_t^{\Delta,i}\right)dt\right] + \mathbb{E}_0\left[\xi_T^i b_T^i\right].$$

Optimal behavior implies a transversality condition $\lim_{T\to\infty}\mathbb{E}\left[\xi_T^i n_T^i\right] = 0$ on total wealth n_T^i of agent i as a necessary choice condition. Because total wealth consists of bond wealth and capital wealth and the latter cannot become negative, a transversality condition for bond wealth b_T^i immediately follows. Consequently, the second term converges to zero as $T \to \infty$ and we obtain equation (10.19) in the limit. Under the optimal trading policy, the initial bond wealth b_0^i must equal the discounted value of future payouts (=outflows) from the bond portfolio,

$$b_0^i = -\mathbb{E}\left[\int_0^\infty \xi_t^i b_t^i\left(\mu_t^{\Delta,i} - \varsigma_t\sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i\tilde{\sigma}_t^{\Delta,i}\right)dt\right]. \tag{10.19}$$

As all agents hold the same constant fraction of their net worth in bonds ($\theta_t^i = \vartheta_t$), the value of the individual bond portfolio is simply the product of the agent's net worth share and aggregate bond wealth, $b_t^i = \eta_t^i q_t^B K_t$.

To characterize the trading process $d\Delta_t^{b,i}$, start from (10.17):

$$d\Delta_t^{b,i} = \frac{db_t^i}{b_t^i} - dr_t^{\mathcal{B}}. \tag{10.20}$$

Because all agents hold the same fraction $\theta_t^i = \vartheta_t$ of their net worth in bonds, we have $b_t^i = \eta_t^i q_t^B K_t$. As η_t^i loads only on the idiosyncratic Brownian and $q_t^B K_t$ only on the aggregate Brownian, their quadratic covariation vanishes and thus Ito's product rule simply implies

$$\frac{db_t^i}{b_t^i} = \frac{d(q_t^B K_t)}{q_t^B K_t} + \frac{d\eta_t^i}{\eta_t^i}.$$

Furthermore, the return on bonds can be written as (compare equation (10.6))

$$dr_t^{\mathcal{B}} = \frac{d(q_t^B K_t)}{q_t^B K_t} - \breve{\mu}_t^{\mathcal{B}} dt.$$

Substituting the previous two equations into (10.20) yields

$$d\Delta_t^{b,i} = \check{\mu}_t^{\mathcal{B}} dt + \frac{d\eta_t^i}{\eta_t^i} = \check{\mu}_t^{\mathcal{B}} dt + \sigma_t^{\eta,i} d\tilde{Z}_t^i,$$

which implies

$$\mu_t^{\Delta,i} = \check{\mu}_t^{\mathcal{B}}, \qquad \sigma_t^{\Delta,i} = 0, \qquad \tilde{\sigma}_t^{\Delta,i} = \sigma_t^{\eta,i}.$$

From the government budget constraint (10.2) and the definition of s_t, we get

$$\check{\mu}_t^{\mathcal{B}} = -s_t/q_t^B. \tag{10.21}$$

Note that individual net worth n_t^i and total net worth $N_t := (q_t^K + q_t^B)K_t$ have identical drifts and volatility loadings on the aggregate Brownian dZ_t, so that simply

$$\frac{d\eta_t^i}{\eta_t^i} = \frac{d(n_t^i/N_t)}{n_t^i/N_t} = \tilde{\sigma}_t^{n,i} d\tilde{Z}_t^i$$

because n_t^i loads on the idiosyncratic Brownian $d\tilde{Z}_t^i$, but N_t does not. Combining the net worth evolution (10.4) with the equilibrium portfolio weights, we obtain

$$\tilde{\sigma}_t^{n,i} = (1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t. \tag{10.22}$$

Thus, we show that the bond trading process satisfies

$$\mu_t^{\Delta,i} = -s_t/q_t^B, \qquad \sigma_t^{\Delta,i} = 0, \qquad \tilde{\sigma}_t^{\Delta,i} = (1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t. \tag{10.23}$$

The first equation says that the proportional reduction in the value of all agents' bond portfolios due to trading with the government equals the surplus-debt ratio s_t/q_t^B. This term captures cash flows from payouts, not from retrading among private agents. The second and third term capture such retrading in response to aggregate and idiosyncratic shocks, respectively. Here, agents do not trade in response to aggregate shocks as they are all exposed symmetrically, but agents do trade in response to idiosyncratic shocks: they sell capital and buy bonds when they receive a positive shock and vice versa.

Combining the previous equations and using $q_t^B K_t = \mathcal{B}_t / \mathcal{P}_t$ leads to the individual valuation equation

$$\eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]. \tag{10.24}$$

Finally, integrating over individuals i yields equation (10.15).

Comparison of the SDFs $\bar{\xi}_t$ and ξ_t^{**} The SDFs used in equations (10.13) and (10.16) are both free of idiosyncratic risk and imply the same aggregate risk premium, but they differ with respect to their average rate of decay, the "risk-free rate" they imply. The average SDF $\bar{\xi}$ decays at the equilibrium risk-free rate r_t^f. It is thus a proper SDF in this model that prices all assets free of idiosyncratic risk. The same is not true for the weighted average SDF ξ_t^{**}. The latter decays at a rate $r_t^f + \tilde{\varsigma}_t \tilde{\sigma}_t^n$, where $\tilde{\sigma}_t^n$ is the idiosyncratic net worth volatility of agents (which is identical for all agents in equilibrium). The weighted average SDF ξ_t^{**} therefore discounts safe cash flows at a higher rate than the risk-free rate that contains a risk premium for idiosyncratic wealth risk. The reason for this is apparent from equation (10.15) which inverts the order of integration: while aggregate cash flows from bonds are free of idiosyncratic risk, each agent holds a stochastic share η_t^i of the aggregate bond portfolio so that individual bond portfolios do contain priced idiosyncratic risk.

These considerations imply that only equation (10.13) is a standard asset pricing formula, a discounted present value formula using a SDF that prices all assets (at least those free of idiosyncratic risk). But equation (10.13) can have a bubble and infinities with opposite sign. It can be more informative to work with equation (10.16) instead, as this equation makes the source of trading gains transparent. However, we need to keep in mind that this equation uses a SDF that does not (in general) price the assets in the economy correctly without additional service flow terms.

Relating the Dynamic Trading Perspective to a Representative Agent. The weighted-average SDF may not be a proper SDF that prices assets in the competitive equilibrium of our incomplete markets economy. Yet, it turns out to be the correct SDF of a hypothetical representative agent who is forced to distribute aggregate consumption according to the consumption shares that arise in our incomplete markets economy. For details on the representative agent formulation, we refer readers to Appendix B.3 of Brunnermeier et al. (2024).

10.4.3 Counter-cyclical Safe Asset and Negative Beta

Setup for Numerical Illustration In this section, we illustrate the dynamics of our model with aggregate risk in the context of a numerical example. To ensure that this example captures a quantitatively plausible situation, in particular with regard to the implications for government debt valuation, the example is based on a calibration. However, because the most important takeaways from this section are qualitative, not quantitative, we defer a description and justification of our calibration choices to Section 10.6.

We introduce aggregate risk as shocks to idiosyncratic risk $\tilde{\sigma}_t$. We assume that the idiosyncratic variance $\tilde{\sigma}_t^2$ follows a Cox–Ingersoll–Ross process,

$$d\tilde{\sigma}_t^2 = -\psi \left(\tilde{\sigma}_t^2 - \left(\tilde{\sigma}^0 \right)^2 \right) dt - \sigma^{\tilde{\sigma}} \tilde{\sigma}_t dZ_t \tag{10.25}$$

with parameters $\psi, \sigma^{\tilde{\sigma}}, \tilde{\sigma}^0 > 0$.

We interpret periods of high idiosyncratic risk as recessions and want them to be associated with lower consumption and higher marginal utility. Rather than microfounding this relationship explicitly, we simply impose an exogenous relationship $a_t = a(\tilde{\sigma}_t)$ with $a'(\cdot) < 0$ that generates the desired correlation structure.

For government policy, summarized by debt growth net of interest payments, $\check{\mu}_t^{\mathcal{B}}$,[22] we similarly impose a functional relationship $\check{\mu}_t^{\mathcal{B}} = \check{\mu}^{\mathcal{B}}(\tilde{\sigma}_t)$ with $(\check{\mu}^{\mathcal{B}})'(\cdot) > 0$. For sufficiently large $(\check{\mu}^{\mathcal{B}})'$, this ensures that primary surpluses $s_t = -\check{\mu}_t^{\mathcal{B}} q_t^B$ are positive for low idiosyncratic risk (in expansions) and negative for high idiosyncratic risk (in recessions). Primary surpluses therefore correlate negatively with marginal utility and any

[22] To be precise, the government also chooses the nominal interest rate i_t. However, in our flexible price model, this policy choice merely affects the equilibrium inflation rate but not real allocations or asset prices.

agent in the economy would require a positive risk premium for holding a (hypothetical) claim to primary surpluses, a feature that is empirically plausible.

Imposing tight functional relationships between $\tilde{\sigma}_t$ and the other exogenous variables a_t and $\check{\mu}_t^{\mathcal{B}}$ is somewhat stylized but allows us to keep the state space of our model one-dimensional. This is helpful to illustrate global dynamics.

We also remark here that, for our numerical example, we replace logarithmic preferences of households with stochastic differential utility with relative risk aversion $\gamma > 1$, but we continue to assume a unit elasticity of intertemporal substitution (EIS). We elaborate more on the details in Section 10.6. This modification does not matter for the qualitative behavior of the model, but it allows us to generate quantitatively realistic aggregate risk premia. This is important to capture the full extent to which pro-cyclical primary surpluses reduce the value of the cash flow component.

Equilibrium Dynamics of Bond and Capital Values Figure 10.1 illustrates the equilibrium dynamics of the value of the government bond stock q^B (blue line) and the value of the capital stock q^K (red line) per unit of capital in the economy by plotting these valuations as a function of the state variable $\tilde{\sigma}$. The gray shaded area depicts the stationary distribution of $\tilde{\sigma}$. q^B is strictly increasing in idiosyncratic risk whereas q^K is strictly decreasing. We can interpret this observation as flight to safety from capital to bonds in times of elevated idiosyncratic risk. We discuss implications for the pricing of (diversified) equity from flight to safety in Section 10.4.3 below.

Because output comoves negatively with $\tilde{\sigma}$ by construction, the monotonicity patterns of $q^B(\tilde{\sigma})$ and $q^K(\tilde{\sigma})$ imply that bond valuations are counter-cyclical whereas capital valuations are pro-cyclical. It is this counter-cyclical valuation that makes government bonds a good safe asset in the presence of aggregate risk. We analyze the source of the counter-cyclicality in the following subsection.

Analyzing the Two Bond Asset Pricing Terms Separately We now consider the two terms in the government debt valuation equation derived from the dynamic trading

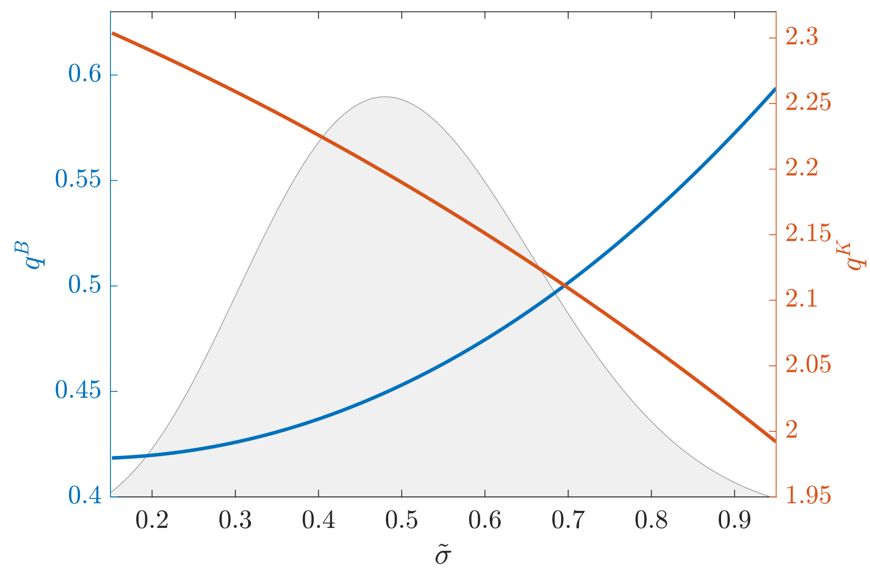

Figure 10.1: Equilibrium asset valuations q^B (blue line, left scale) and q^K (red line, right scale) as a function of idiosyncratic risk $\tilde{\sigma}$. The gray shaded area in the background depicts the (rescaled) ergodic density of the state variable $\tilde{\sigma}$.

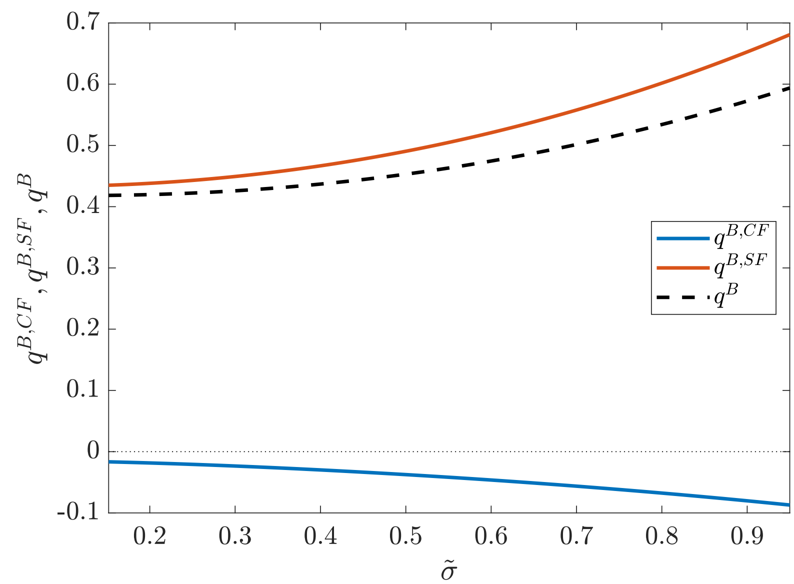

Figure 10.2: Decomposition of the value of government debt as a function of idiosyncratic risk $\tilde{\sigma}$. The blue solid line shows the present value of primary surpluses ($q^{B,CF}$), the red solid line the present value of service flows ($q^{B,SF}$) and the black dashed line the total value of government debt (q^B), all normalized by the capital stock.

perspective, equation (10.16). Figure 10.2 plots the two present values[23]

$$q^{B,CF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) s_t K_t dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0,$$
$$q^{B,SF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) (1-\vartheta_t)^2 \gamma \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0.$$

The blue solid line shows the present value of future primary surpluses (cash flows) $q^{B,CF}$ as a function of the single state variable $\tilde{\sigma}$. This value is strictly decreasing in idiosyncratic risk and has a low – in fact negative – value. Comparing the present value of surpluses $q^{B,CF}K$ in our model to the market value of government debt $q^B K$, which is represented by the black dashed line in Figure 10.2, reveals a large gap $(q^B - q^{B,CF})K$, a "debt valuation puzzle". In addition, when compared with the present value of surpluses $q^{B,CF}K$, the total value of government debt $q^B K$ has also the opposite correlation with the aggregate state. Yet, there is no puzzle from the perspective of our model: government debt is a safe asset valued for its service flow from re-trading which is represented by the component $q^{B,SF}(\tilde{\sigma})$. As the red solid line in Figure 10.2 shows, this value is positive, large and positively correlated with $\tilde{\sigma}_t$. This additional component dominates the overall dynamics of the value of government debt and is the reason that q^B appreciates in bad times despite the simultaneous drop in $q^{B,CF}$. That $q^{B,SF}$ must be positively correlated with $\tilde{\sigma}$ can also be seen from the present value equation: for our policy specification, residual net worth risk $(1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t$ is increasing in $\tilde{\sigma}_t$, so that an increase in idiosyncratic risk increases the value of insurance service flows from re-trading.[24]

The correlation structure apparent in Figure 10.2 implies that, if the two claims $q^{B,CF}$ and $q^{B,SF}$ could be traded separately, the cash flow claim would be a high-β asset, while the service flow claim would be a negative-β asset. The presence of this second,

[23]Relative to equation (10.16), here an additional factor γ appears because we no longer assume logarithmic preferences.

[24]This is not an entirely rigorous argument as it ignores changes in the discount rate. The effective discount rate in the weighted-average SDF $\int \xi_t^i \eta_t^i di$ can both increase or decrease with the aggregate state $\tilde{\sigma}_t$ depending on whether the *aggregate* risk premium increases or decreases. Note however, that the level of idiosyncratic risk does not directly matter for the effective discount rate because the risk premium on idiosyncratic risk exactly offsets the lower risk-free rate due to a precautionary motive.

negative-β component makes government debt as a whole a negative β asset. Government debt emerges as a “good friend” also with respect to aggregate shocks.

We remark that, while the exact numbers in Figure 10.2 depend on our calibration, the broad qualitative pattern described in this section is fairly robust, so long as the calibration is consistent with the following two stylized facts about US primary surpluses: (1) the average primary surplus is close to zero or even slightly negative and (2) primary surpluses are pro-cyclical. Fact (1) implies that even a risk-free claim to the cash flows would have a non-positive value and, together with fact (2), the most likely outcome is a negative cash flow component that has a positive β. The model can then only generate a large positive total value of government debt if the service flow component dominates.

The Possibility of Insuring Bond Holders and Tax Payers at the Same Time In our simple setting, households are both capital owners and bond holders. In this section, we conceptually separate each household into two sub-units, a capital owner and a government debt holder. Surprisingly, it is possible to follow a government policy that provides insurance against negative aggregate shocks for both tax payers and bond holders at the same time. By cutting taxes (or even granting subsidizes) for capital owners in recessions, their tax burden is positively correlated with their income providing insurance to tax payers. At the same time, the safe asset service flow rises in recessions, which provides insurance to government bond holders. This finding in our incomplete market setting with a safe asset is in sharp contrast to traditional asset pricing in which either tax payers or government bond holders can be insured, but not both.

We remark that the government nevertheless faces a trade-off also in our setting. By making debt issuance more counter-cyclical, tax payers become better insured whereas insurance to bond holders is reduced. This is the case because bond holders are also exposed to the cash flow component, which captures the conventional logic. However, the trade-off is considerably more favorable because the cash flow component represents only a small fraction of the total value of debt.

Volatile, Flight-to-Safety Prone Equity Markets The presence of idiosyncratic risk and government debt as a safe asset has also implications for equity markets. We explain here why the diversified equity portfolio does not emerge as a safe asset and how flight to safety can generate quantitatively large additional equity return volatility.

First, why are stocks not safe assets? In our model, agents can hold a diversified stock portfolio. Like government bonds, this stock portfolio is free of idiosyncratic risk and thus allows agents to self-insure against idiosyncratic consumption fluctuations. However, unlike government bonds, stocks are poor aggregate risk hedges as they are ultimately claims to capital, which looses in value in recessions. This implies that stocks are positive-β assets in our model.

To understand why stock prices fall in times of high idiosyncratic risk, even though idiosyncratic equity risk can be diversified away, note that the marginal holder of capital in our model is always an insider who has to bear the increased idiosyncratic risk. As a consequence, when idiosyncratic risk goes up, so does the insider premium earned by the managing households, which is achieved by a reduction in the dividend that is paid to outside equity holders. This makes stock dividends more procyclical than production cash flows, so that stocks lose value precisely when idiosyncratic risk goes up.

When evaluating the diversified stock portfolio with regard to the key characteristic of safe assets, the Good Friend Analogy, stocks fail to qualify as safe assets in the same way as government debt does. Stocks have the good friend property only partially: stocks are valuable when an agent experiences a negative idiosyncratic shock, but due to their positive β, they are not in bad aggregate times.

Second, how does flight to safety generate volatility? While the focus of this chapter is on government bonds, our model can also match the empirical mean and volatility of the excess return on the stock market in excess of government bonds as we discuss in Section 10.6 below. The realistic Sharpe ratio is clearly a feature of recursive preferences with a high risk aversion, but the ability of our simple model to generate large return volatility in the presence of realistic levels of output variation is noteworthy[25]

[25]This is so because we work with preferences that feature a unit EIS. It is well-known from the long-run risk literature that recursive preferences can also generate large return volatility, but *only* if the EIS is sufficiently larger than 1. In contrast, the mechanism we describe here works even for EIS ≤ 1.

and directly related to the existence of safe government bonds.

To gain intuition, let's abstract from the distinction between capital and outside equity[26] and consider the following equation, which aggregates the intertemporal budget constraints of all households:

$$q_t^K K_t + q_t^B K_t = \mathbb{E}_t \left[\int_t^\infty \frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di} C_s ds \right]. \tag{10.26}$$

In many macro asset pricing models, government debt does not represent positive net wealth, $q_t^B = 0$, and thus equation (10.26) implies for such models that the value of the capital stock equals the present value of future consumption. In other words, in these models, pricing the aggregate equity claim is equivalent to pricing the aggregate consumption claim.[27] In the presence of realistic consumption volatility, large volatility in capital valuations $q_t^K K_t$ is then hard to generate (and requires substantial time variation in the SDF $\int \xi_s^i \eta_s^i di$).

Our model with $q^B \neq 0$ suggests an additional explanation for the high observed stock market volatility. When idiosyncratic risk $\tilde{\sigma}_t$ rises, there is a flight to safety that increases the value of bonds (q_t^B) and lowers the value of capital (q_t^K). Even in the absence of changes in the present value on the right-hand side of equation (10.26), this portfolio reallocation generates *flight-to-safety volatility* in capital valuations and thus in the stock market.

To understand how much flight-to-safety volatility matters quantitatively, we compare the excess stock return volatility in our model to the one generated by a version of the model without government debt. In that alternative version, $q_t^B = 0$ at all times and thus flight-to-safety volatility disappears.[28] We find that the average (annualized) excess return volatility in the alternative model would be 2.4% as opposed to 11.7% in

[26] As discussed previously, a state-dependent insider premium will ensure that equity values and capital values move in lockstep despite the fact that idiosyncratic equity risk can be diversified away.

[27] Because the equation results from aggregating individual intertemporal budget constraints, the SDF used in this pricing equation is again the weighted-average SDF as in the dynamic trading perspective.

[28] Formally, we selection the "non-monetary" equilibrium in our model (degenerate solution $\vartheta \equiv 0$ to equation (10.12)). We keep all parameters as in our baseline model.

our baseline model.[29] We can therefore conclude that flight-to-safety volatility accounts for more than three quarters of the overall excess return volatility in our framework.

10.5 Safe Assets, Bubbles, and Convenience Yield

10.5.1 Safe Asset Tautology and Loss of Safe Asset Status

In this section, we clarify the relationship between safe assets and bubbles as well as the fragility of the safe asset status. We offer two key takeaways: First, while safe assets and bubbles are two distinct concepts, they complement each other. If an asset is associated with a bubble, it is more likely to be a safe asset. Second, safe-asset status is fragile and may be lost when the bubble pops. The same asset with the same payoffs might be a safe asset in one equilibrium, but not a different equilibrium. In this sense, a safe asset is safe because it is perceived to be safe, a tautology. In contrast to our standard equilibrium selection (compare Section 10.3), in this section – and in this section only – we do not restrict attention to stationary equilibria in which government bonds have a positive value. Then Proposition 10.3 does not apply and multiple equilibria may possibly arise.

Bubbles and Safe Assets. While bubbles and safe assets are distinct concepts, there is a complementarity between bubbles and the negative β property of safe assets:

First, low-β assets can sustain bubbles more easily that high-β assets. A bubble on an asset is possible if, in the buy and hold perspective, the discounted terminal value does not necessarily vanish in the limit. For example, in the case of government debt, the terminal value may not vanish if $\mathbb{E}\left[\bar{\xi}_T \mathcal{B}_T / \mathcal{P}_T\right]$ does not necessarily converge to zero as $T \to \infty$ (compare equation (10.13)). In the long-run, the value of government debt $\mathcal{B}_T / \mathcal{P}_T$ grows on average at the same rate as the aggregate economy, so that the discounted terminal value grows on average at the rate

$$g - r^f - \text{risk premium on gov. debt}$$

[29] As the benchmark asset in the alternative model, we choose a zero net supply risk-free bond, the most common choice in the literature.

where g is the average growth rate of the economy and r^f is the average risk-free rate. The (average) risk premium on government debt scales linearly with its (average) β, at least approximately. Hence, the smaller is the β, the larger is the growth rate of the discounted terminal value and the easier it is to sustain a bubble on government debt. The same argument applies of course also to other assets than government debt.

Second, a bubble component can lower an asset's β and thereby make it safer. To understand this point, take again government debt as an example but now consider the dynamic trading perspective, equation (10.16). Suppose the situation is as depicted in Figure 10.2 with a cash flow component that has a positive β and a service flow component that has a negative β.[30] The relative contribution of the two components to the overall value of government debt determines the size and sign of the asset's β. The service flow component is proportional to the market value of government debt whereas the cash flow component does not directly depend on it. A bubble component raises the market value of the debt and thus increases the relative contribution of the service flow component, which lowers the asset's β.

The previous discussion implies that the safe asset status can be bubbly. An asset whose cash flow component has a sufficiently large β may be safe in some equilibria but not safe in others. Specifically, in an equilibrium in which the asset has a (sufficiently large) bubble component, the service flow component dominates, it has a negative total β and its required risk-adjusted rate of return is low, such that the asset can easily sustain the bubble. In a different equilibrium without a bubble component, the cash flow component dominates, the total β is positive and, as a result, the risk-adjusted rate of return so large that the asset does not appear to be able to sustain bubbles.

Bubbly safe assets give rise to the *Safe Asset Tautology*: the asset is safe in a given equilibrium because it is perceived to be safe, but there are alternative equilibria in which the asset is not safe. In these alternative equilibria, the asset has a positive β and is therefore not a good friend after negative aggregate shocks. When government debt is merely a bubbly safe asset, the safe asset status can be fragile. If markets coordinate on a different equilibrium, the bubble bursts and the government loses the safe asset

[30] As we have explained previously, this is the relevant case to make sense of the empirical facts about primary surpluses and debt from the perspective of our theory.

status together with the fiscal space it implies.

Bubbly versus Fundamentally Safe Government Debt. In our model environment, a precautionary savings motive arising from uninsured idiosyncratic risk can depress discount rates sufficiently to make bubbles possible. But importantly, it depends on government policy how strong this precautionary savings motive is. If the government makes bonds more attractive by raising higher primary surpluses (equivalently, by lowering $\breve{\mu}_t^{\mathcal{B}}$), the bond wealth share ϑ_t increases, idiosyncratic risk sharing improves and the precautionary motive is dampened.

A government that makes its debt very attractive can therefore raise discount rates sufficiently to eliminate any space for rational bubbles. Under such a policy, government debt can still be a safe asset, however. In this case, government debt is a fundamentally safe asset whose safe asset status does not require the continued belief of market participants in its safety, unlike for bubbly safe assets.

The simplest way to retain a safe asset status in the absence of a bubble is for the government to give up insurance of tax payers in recessions and make the β of the cash-flow component negative. Alternatively, there may still be a sufficient amount of residual idiosyncratic risk for the counter-cyclical service flow component to be substantial even though the precautionary savings motive is not large enough to sustain bubbles. In this case, also a mildly pro-cyclical surplus process may be consistent with safe government debt in the absence of bubbles.

Selecting the Public Debt Bubble. When the safe asset status is bubbly, a government with access to a richer set of policy tools than considered in this chapter may still be able to select a unique equilibrium. Brunnermeier et al. (2021) analyze how fiscal policy and asset regulation can affect the set of possible equilibria in environments with public debt bubbles. Their results also apply to our model:

The government can impose a number of specific policy measures that target bubbles on alternative assets or give an advantage to government bonds: it can eliminate private Ponzi schemes by imposing no Ponzi conditions on private agents through insolvency laws, it can tax competing bubbly assets, and it can use financial repression tools such as reserve and liquidity requirements to generate additional demand for its own liabilities. But most importantly, if a government can credibly commit to create a fundamentally safe asset off equilibrium by raising surpluses, it can eliminate all other equilibria. Hence, a government with sufficient backup fiscal capacity does not need to fear a loss of the safe asset status of its debt.

10.5.2 Privately Issued Safe Assets and Convenience Yields

So far, we have emphasized government debt as a safe asset. In this section, we discuss safe asset issuance by private agents. We also elaborate on the difference between service flows derived from retrading and convenience yields.[31]

Privately Issued Safe Assets. Assume that each agent can issue nominally risk-free bonds, just like the government. The safe asset definition, based on the Good Friend Analogy, applies equally also to such debt instruments issued by private citizens. For any individual asset holder, government bonds and privately issued safe bonds are perfect substitutes. As a consequence, the equilibrium interest rate i_t^p that private agents have to pay on their bonds equals the government's, $i_t^p = i_t$.

In equilibrium, agents are then indifferent as to how many bonds to issue and how many privately issued bonds of other agents to hold. Privately issued bonds are equally suitable as safe assets for their holders. However, private bond issuance also comes with a short position in the bond for the issuing agent. In the same spirit as before, we can value the short position by determining the present value of net payouts that an issuer makes to all bond holders. That valuation exercise reveals that the short position generates a negative service flow for the issuing agent. This negative service flow results from the fact that the agent repays outstanding debt after negative idiosyncratic shocks and issues additional debt after positive idiosyncratic shocks. While the cash flows generated from this contingent bond issuance are zero on average, they are systematically correlated with marginal utility and thus tend to increase the overall riskiness of the agent's portfolio.

Once we aggregate all long and short positions of all privately issued safe bonds, the

[31]For the equations presented in this section, we revert back to the logarithmic preference specification.

service flows "earned" by bond holders and the service flows "paid" by bond issuers exactly cancel out. Therefore, unlike government debt, private safe asset creation does not generate net service flows for the economy as a whole.

Convenience Yields. A conclusion from the previous part is that $\Delta i_t := i_t^p - i_t = 0$, the yield spread between privately issued and government debt is zero. Government debt is not special. In the presence of idiosyncratic risk, a precautionary motive depresses all asset returns symmetrically. Equivalently, a service flow from re-trading can be derived from all assets that are both free of idiosyncratic risk and tradeable on liquid markets.

Such a service flow is conceptually different from a convenience yield. A convenience yield on government debt captures the special role that government bonds play in certain transactions. It can be measured by a positive yield spread $\Delta i_t > 0$ between government debt and safe corporate debt of equal maturity. In contrast, the service flow from retrading we emphasize in this chapter affects also safe corporate debt. It is therefore unrelated to the spread Δi_t.

We can once again price government debt according to our two valuation perspectives:

Proposition 10.7. *In the model with convenience yields, the value of government debt at $t = 0$ satisfies from the buy and hold perspective:*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty}\left(\mathbb{E}\left[\int_0^T \bar{\xi}_t s_t K_t dt\right] + \mathbb{E}\left[\int_0^T \bar{\xi}_t \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right] + \mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]\right),$$

and from the dynamic trading perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \xi_t^{**} s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^{**}\left(\Delta i_t + (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2\right)\frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right].$$

From the latter, dynamic trading perspective, the service flows from bonds in the utility function (captured by Δi_t) and from self-insurance through retrading (captured by $(1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2$) appear symmetrically. However, the buy and hold perspective reveals an asymmetry. The convenience yield still enters the valuation explicitly as a service flow term. In contrast, the service flow from retrading is absent in this perspective. Instead, it is implicitly contained in the stochastic discount factor $\bar{\xi}_t$ and results in a lower discount rate due to precautionary savings as well as – potentially – a bubble term.

The terms arising from the buy and hold perspective are the ones that are typically measured in empirical asset pricing. The best an empirical researcher can do when estimating a SDF based on aggregate asset price data is to identify $\bar{\xi}_t$. When looking at yield differences between safe corporate and government bonds, the empirical researcher identifies an estimate of Δi_t. The importance of self-insurance service flows can only be determined indirectly, e.g. by finding a bubble component.[32]

10.6 Calibration and Quantifying the Bubble Mining Laffer Curve

In this section we describe the calibration underlying the numerical illustration presented in Section 10.4.3 and argue that it leads to predictions for asset prices and macro aggregates that are broadly realistic.

Details on the Model Setup Recall that $\tilde{\sigma}_t$ is assumed to follow the Cox–Ingersoll–Ross process, as specified in Section 10.4.3, equation (10.25). For the functional relationships for a_t and $\check{\mu}_t^{\mathcal{B}}$, we choose parsimonious linear specifications

$$a(\tilde{\sigma}_t) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0), \tag{10.27}$$

$$\check{\mu}_t^{\mathcal{B}} = \check{\mu}^{\mathcal{B},0} + \alpha^{\mathcal{B}}(\tilde{\sigma}_t - \tilde{\sigma}^0) \tag{10.28}$$

with parameters a^0, $\check{\mu}^{\mathcal{B},0}$, α^a, and $\alpha^{\mathcal{B}}$. Sufficiently large coefficients $\alpha^a, \alpha^{\mathcal{B}} > 0$ ensure that output, consumption, and primary surpluses all fall when idiosyncratic risk rises.

In order to match certain aspects of the data better, we also make two small mod-

[32] The presence of a bubble component in the buy and hold perspective means that even at the low discount rates implied by $\bar{\xi}_t$, cash flows $s_t K_t$ and convenience yield service flows $\Delta i_t \mathcal{B}_t / \mathcal{P}_t$ are insufficient to explain the total value of government debt. The same always remains true if we discount at the higher rates implied by ξ_t^{**}, so that the self-insurance service flow must explain the gap.

ifications to the model itself. First, for our model to generate a quantitatively realistic price of aggregate risk, we replace logarithmic preferences with stochastic differential utility with unit EIS and arbitrary relative risk aversion $\gamma > 0$: household i maximizes V_0^i, where V_t^i is recursively defined by

$$V_t^i = \mathbb{E}_t \left[\int_t^\infty (1-\gamma)\rho V_s^i \left(\log(c_s^i) - \frac{1}{1-\gamma} \log\left((1-\gamma) V_s^i \right) \right) ds \right].$$

In the special case $\gamma = 1$, this specification collapses to our baseline specification with logarithmic utility. Second, to separate the level of investment from the adjustment cost parameter ϕ, which governs fluctuations in investment and capital prices, we consider the slightly more general capital adjustment cost function

$$\hat{\Phi}(\iota) = \iota^0 + \Phi(\iota - \iota^0)$$

with the additional parameter ι^0. All solution formulas for the baseline model remain valid for this more general specification if we replace a_t with $a_t - \iota^0$ and ι_t with $\iota_t - \iota^0$.

Following the same steps as in the baseline model, we obtain the slightly modified equation

$$\mathbb{E}_t \left[d\vartheta_t \right] = \left(\rho + \check{\mu}_t^{\mathcal{B}} - \left(\sigma_t^v - (\gamma - 1)\, \sigma_t^{\bar{q}} \right) \sigma_t^\vartheta - \gamma \left(1 - \vartheta_t\right)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \right) \vartheta_t dt,$$

where $\sigma_t^{\bar{q}}$ is the volatility of $\bar{q}_t := q_t^B + q_t^K$.

we now also have to characterize the process v_t as it affects the BSDE for ϑ_t through the term σ_t^v.[33] To characterize v_t, we start from the costate equation (a necessary optimality condition by the stochastic maximum principle), which is here given by

$$\begin{aligned} \mathbb{E}_t[d\xi_t^i] &= -\left(\partial_V f(c_t^i, V_t^i)\xi_t^i + \frac{\partial H_t^i}{\partial n_t^i} \right) dt \\ &= -\left((1-\gamma)\rho \log(c_t^i/n_t^i) - \rho \log v_t - \rho + \mu_t^{n,i} + \frac{c_t^i}{n_t^i} - \varsigma_t^i \sigma_t^{n,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{n,i} \right) \xi_t^i dt \end{aligned}$$

[33] There is no need to solve for v_t in the baseline model because there it enters the value function additively and thus only impacts total utility but not optimal choices.

$$= -\left((1-\gamma)\rho\log\rho - \rho\log v_t + \mu_t^{n,i} - \left(\gamma\sigma_t^{n,i} - \sigma_t^v\right)\sigma_t^{n,i} - \gamma\left(\tilde{\sigma}_t^{n,i}\right)^2\right)\xi_t^i dt, \tag{10.29}$$

where the last line uses $c_t^i/n_t^i = \rho$ and the price of risk. We also know $\tilde{\xi}_t^i = v_t(n_t^i)^{-\gamma}$ and applying Ito's lemma to this equation yields for the drift term

$$\mathbb{E}_t[d\xi_t^i] = \left(\mu_t^v - \gamma\mu_t^{n,i} + \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) - \gamma\sigma_t^v\sigma_t^{n,i}\right)\xi_t^i dt \tag{10.30}$$

Combining equations (10.29) and (10.30) and solving for μ_t^v yields

$$\begin{aligned}
\mu_t^v &= \gamma\mu_t^{n,i} - \frac{\gamma(\gamma+1)}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \gamma\sigma_t^v\sigma_t^{n,i} \\
&\quad - \left((1-\gamma)\rho\log\rho - \rho\log v_t^i + \mu_t^{n,i} - \left(\gamma\sigma_t^{n,i} - \sigma_t^v\right)\sigma_t^{n,i} - \gamma\left(\tilde{\sigma}_t^{n,i}\right)^2\right) \\
&= \rho\log v_t + (\gamma-1)\left(\rho\log\rho + \mu_t^{n,i} - \frac{\gamma}{2}\left(\left(\sigma_t^{n,i}\right)^2 + \left(\tilde{\sigma}_t^{n,i}\right)^2\right) + \sigma_t^v\sigma_t^{n,i}\right) \\
&= \rho\log v_t + (\gamma-1)\left(\rho\log\rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta - \frac{\gamma}{2}\left(\left(\sigma_t^{\bar{q}}\right)^2 + (1-\vartheta)^2\tilde{\chi}^2\tilde{\sigma}_t^2\right) + \sigma_t^v\sigma_t^{\bar{q}}\right),
\end{aligned}$$

where in the last line we use that individual net worth has the same drift and aggregate volatility as aggregate net worth $\bar{q}_t K_t$, while its idiosyncratic volatility is $\tilde{\sigma}_t^{n,i}$, as determined previously. The previous equation for μ_t^v leads to a second BSDE

$$\mathbb{E}_t[dv_t] = \mu_t^v v_t dt$$

that has to be solved numerically jointly with the BSDE for ϑ_t stated previously.

Numerical Model Solution. As before, we solve the model numerically using a finite difference method. For our numerical solution, we impose the functional relationships $\vartheta_t = \vartheta(t, \tilde{\sigma}_t)$, $v_t = v(t, \tilde{\sigma}_t)$ and use the known forward equation for the state variable $\tilde{\sigma}_t$ to transform the two BSDEs into partial differential equations in time t and the state $\tilde{\sigma}_t$. We choose suitable terminal guesses for the functions ϑ and v[34] at a finite terminal time T and solve the two PDEs backward in time using a finite difference method. We

[34] Specifically, we use the functions implied by the steady state equilibrium with $\tilde{\sigma}_t = \tilde{\sigma}^0$ forever.

Table 10.1: Parameter Choices

parameter	description	value	parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.54	$\mathcal{G}$	gov. expenditures	0.138
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67	$\check{\mu}^{\mathcal{B},0}$	$\check{\mu}_t^{\mathcal{B}}$ stoch. steady state	0.0026
$\sigma^{\tilde{\sigma}}$	$\tilde{\sigma}_t^2$ volatility	0.4	α^a	a_t slope	0.072
$\bar{\chi}$	undiversifiable risk	0.3	$\alpha^{\mathcal{B}}$	$\check{\mu}_t^{\mathcal{B}}$ slope	0.12
γ	risk aversion	6	ϕ	capital adj. cost	8.1
ρ	time preference	0.138	ι^0	capital adj. intercept	-0.022
a^0	a_t stoch. steady state	0.625	δ	depreciation rate	0.055

choose T sufficiently large such that an increase in T no longer changes the solutions at $t = 0$, $\vartheta(0, \cdot)$ and $v(0, \cdot)$, noticeably. These solution functions $\vartheta(0, \cdot)$ and $v(0, \cdot)$ represent our numerical approximation to the stationary (Markov) equilibrium functions $\tilde{\sigma} \mapsto \vartheta(\tilde{\sigma}), v(\tilde{\sigma})$.

Calibration Strategy We calibrate our model such that, when we feed in a quantitatively realistic process for idiosyncratic risk, the model generates variation in macro aggregates and aggregate risk premia that are broadly consistent with US data. We briefly outline our calibration strategy here. The resulting parameter choices are summarized in Table 10.1. Additional details as well as a description of the underlying data sources can be found in Appendix A.10 in Brunnermeier et al. (2024).

We normalize the time period in our model to one year. Because ours is a continuous-time model, this is merely a choice of units that does not affect any results. With regard to the parameters $\tilde{\sigma}^0$, ψ, $\sigma^{\tilde{\sigma}}$ of the exogenous risk process $\tilde{\sigma}_t$, we tie our hands by estimating them externally. Specifically, we choose these parameters such that $\tilde{\sigma}_t^2$ closely matches, in a maximum likelihood sense, the common idiosyncratic volatility (CIV) factor proposed by Herskovic et al. (2016). For the share of undiversifiable idiosyncratic risk, $\bar{\chi}$, we choose an intermediate value of $\bar{\chi} = 0.3$ based on previous broad literature.

We choose the nine parameters γ, ρ, a^0, $\mathcal{G}$, $\check{\mu}^{\mathcal{B},0}$, α^a, $\alpha^{\mathcal{B}}$, ϕ, ι^0 such that the model generates values for a number of moments that are broadly in line with the empirical evidence.[35] These moments are the average ratios of consumption, government expen-

[35] We do not employ a formal simulated method of moments estimation but merely adjust parameters manually to achieve a good visual fit.

ditures, primary surpluses, capital, and debt to output, the average investment rate, the volatilities of output, consumption, investment, and the surplus-output ratio, and the equity premium and equity sharpe ratio.[36] Our empirical moments are based on a sample from 1970 to 2019 just prior to the start of the covid pandemic with two exceptions. The first is the debt-output ratio. Over the largest part of our sample period, this ratio has exhibited a clear upward trend. For this reason, we target the average over the last decade in the sample (0.71) instead of the average over the full sample (0.37). The second exception is the average investment rate, $\mathbb{E}[I/K]$, which we do not compute ourselves but take directly from Cooper and Haltiwanger (2006), who report a value estimated from micro data.

Matching the average ratios is directly informative for the average value of the endogenous variable ϑ_t and the parameters ρ, a^0, $\mathcal{G}$, $\check{\mu}^{\mathcal{B},0}$, and ι^0. Requiring the model to match the macro volatilities is standard in the business cycle literature and also ensures that the model generates a broadly realistic amount of aggregate macro risk.[37] While including the surplus volatility is less standard, this moment is important to discipline the (cyclical) variation in primary surpluses, a key ingredient into the valuation of government debt.[38] Finally, requiring the model to match the equity premium and equity sharpe ratio ensures that this aggregate macro volatility is realistically priced in capital markets.

The remaining parameter δ does not affect anything of interest for the purpose of this chapter.[39] We set it to 0.055, a value slightly smaller than but broadly in line with typical calibrations. With this choice, the average growth rate of our model economy is 2.0%, close to the empirical counterpart of 2.1% in our sample.

Model Fit. Table 10.2 summarizes the quantitative model fit. In addition to our target moments, we report in Table 10.2 also a number of untargeted moments: the correla-

[36]Volatilities of macro aggregates are at the quarterly frequency. Equity moments refer to annualized quantities but are measured at a monthly frequency.

[37]These moments also discipline the model parameter α^a and ϕ.

[38]That moment also disciplines the parameter $\alpha^{\mathcal{B}}$.

[39]This is due to the combination of the AK structure of our economy with a unit EIS. The former implies that δ merely affects the growth rate of the economy and the latter that income and substitution effects from permanent variations in growth rates cancel out.

Table 10.2: Quantitative Model Fit

moment symbol	moment description	model	data
	(1) targeted moments		
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.61	0.64
$\sigma(I)/\sigma(Y)$	relative investment volatility	3.35	3.38
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.58	0.56
$\mathbb{E}[G/Y]$	average government expenditures-output ratio	0.22	0.22
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0005	-0.0005
$\mathbb{E}[I/K]$	average investment rate	0.12	0.12
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[d\tilde{r}^E - dr^{\mathcal{B}}]$	average (unlevered) equity premium	3.59%	3.40%
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma(dr^E - dr^{\mathcal{B}})}$	equity sharpe ratio	0.31	0.31
	(2) untargeted moments		
$\rho(Y, C)$	correlation of output and consumption	0.98	0.92
$\rho(Y, I)$	correlation of output and investment	0.99	0.94
$\rho(Y, S/Y)$	correlation of output and surpluses	0.98	0.60
$\sigma(q^B K/Y)$	volatility of debt-output ratio	4.8%	2.0%
$\mathbb{E}[r^f]$	average risk-free rate	5.18%	0.64%
$\sigma(r^f)$	volatility of risk-free rate	5.47%	2.25%

Notes: For $x \notin \{dr^E - dr^{\mathcal{B}}, r^f\}$, $\sigma(x)$ denotes the standard deviation of x and $\rho(x, y)$ denotes the correlation of x and y, both at a quarterly frequency. Inputs x and y are HP-filtered with smoothing parameter 1600. For $x, y \in \{Y, C, I, G\}$, we take logarithms before filtering. $\mathbb{E}[x]$ denotes expectations over the ergodic model distribution, inputs x are *not* HP-filtered. $x \in \{dr^E - dr^{\mathcal{B}}, r^f\}$ refer to annualized returns measured at monthly frequency and are also *not* HP-filtered. Y: (aggregate) output, C: consumption, I: investment, G: government expenditures, S: primary surplus, r^f: risk-free rate; K, q^K, q^B, dr^B, $d\tilde{r}^E$ are defined as in Section 10.2.

tions of consumption, investment, and primary surpluses with output, the standard deviation of the debt-output ratio, and the average and standard deviation of the risk-free rate.

Section (1) of Table 10.2 reveals that our model achieves overall a very good fit to the targeted moments. As we have varied only nine of our parameters to match twelve moments, this is by no means a trivial observation. Most importantly, our model is consistent with the observed large equity premium and price of risk (Sharpe ratio) while at the same time matching the volatility and comovement of macro aggregates. This verifies that our model is capable of generating realistic aggregate risk premia without requiring excessive real volatility. Notably, our model achieves quantitatively plausible aggregate risk pricing with a moderate risk aversion parameter (γ) of just 6.

Quantifying the Bubble Mining Laffer Curve When idiosyncratic risk is large, safe asset demand may be sufficient to sustain a public debt bubble. This is indeed the case for our calibration. As Brunnermeier et al. (2021) point out, such public debt bubbles represent a fiscal resource that can be "mined" for revenue as a substitute for taxation. However, the ability to mine a bubble does not imply that a government can expand spending without limits. Bubble mining affects the sustainability of bubbles and thereby creates a "debt Laffer curve".

Here, we briefly revisit the Laffer curve logic and then use our calibrated model to quantify the Laffer curve for the US. The main takeaway is that the negative β property of government debt matters considerably. The (average) maximum permanent deficit is above 2% of GDP in our dynamic model but merely 0.1% if we hold idiosyncratic risk constant over time.

The Laffer curve logic follows from the following simple formula for primary deficits per unit of capital[40]

$$-s_t = \check{\mu}_t^{\mathcal{B}} q_t^B.$$

The first factor, $\check{\mu}_t^{\mathcal{B}}$, measures revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond

[40]This equation, in turn, follows immediately from the government budget constraint.

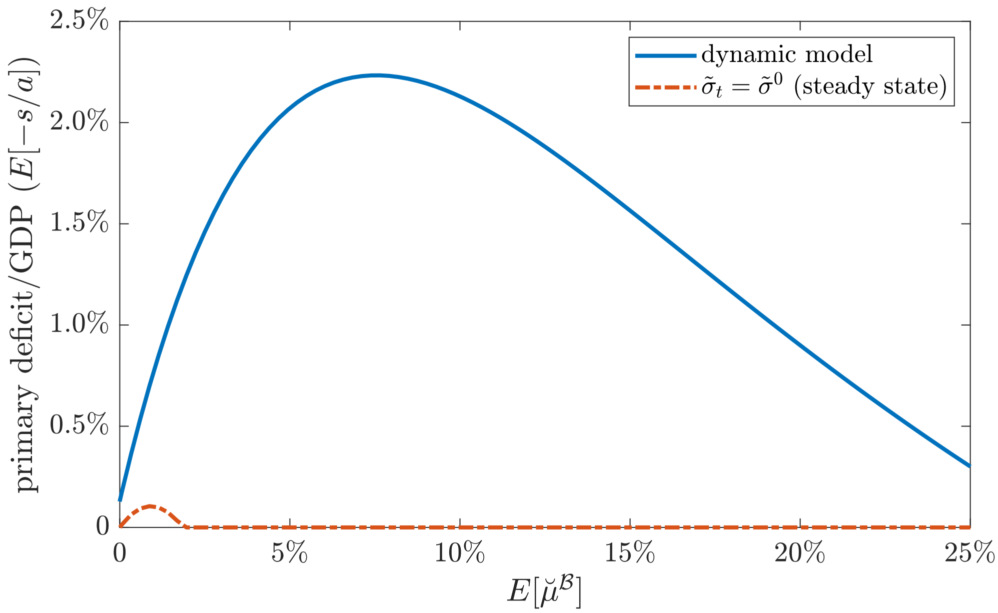

Figure 10.3: Debt Laffer curve for dynamic model and in steady state (constant $\tilde{\sigma}_t$) when there is a bubble on government debt. $\mathbb{E}[\breve{\mu}^{\mathcal{B}}]$ is varied by varying parameter $\breve{\mu}^{\mathcal{B},0}$ while keeping all other parameters as in Table 10.1.

holders is diluted by the issuance of new bonds, i.e., a higher $\breve{\mu}_t^{\mathcal{B}}$ represents a tax on existing bond holders. The second factor, q_t^B, is the tax base, the real value of existing debt (per unit of capital). Permanent deficits are possible if this tax base remains positive even for (permanently) negative primary surpluses. That this is a possibility can be seen from both perspectives to debt valuation discussed in Section 10.4.2: the value of debt remains positive despite negative surpluses if, in the buy and hold perspective (equation (10.16)), a positive bubble term offsets the negative surplus term, or, equivalently, in the dynamic trading perspective (equation (10.16)), the service flow term is sufficiently large. In this case, the tax base is positive, but it nevertheless reacts negatively to an increase in the rate of bubble mining $\breve{\mu}_t^{\mathcal{B}}$.[41] This negative reaction creates a Laffer curve.

The blue line in Figure 10.3 depicts the debt Laffer curve for our calibrated model.

[41]We can see analytically that higher $\breve{\mu}^{\mathcal{B}}$ lowers the equilibrium value of q^B in steady state, compare the formulas in Section **??**. Outside of the steady state, equation (10.9) tells us that there is a negative relationship if an upward shift in $\breve{\mu}_t^{\mathcal{B}}$ at all dates decreases ϑ_t. Equation (10.12) suggests that this is indeed the case, but additional technical considerations are required to make this a fully rigorous argument.

Specifically, the figure plots the average deficit-GDP ratio that can be sustained for different debt growth policies of the form (10.28) with identical $\alpha^{\mathcal{B}}$ (identical cyclicality of debt growth and surpluses) but varying $\check{\mu}^{\mathcal{B},0}$, i.e. the average level of (interest-adjusted) debt growth varies across different policies on the x-axis. The implicit assumption in Figure 10.3 is that $\mathcal{G}$ remains unchanged, so that larger deficits imply smaller output taxes.[42]

In Figure 10.3, if the bubble is mined too aggressively so that the average $\check{\mu}^{\mathcal{B}}$ exceeds 7.3%, the government fails to raise additional real revenues. In particular, there is a limit to bubble mining and the government still faces a constraint on real spending. Our calibrated model suggests that the average primary deficit that can be sustained by bubble mining is bounded above by 2.25% of GDP.

It turns out that the negative β property is very important for the qualitative and quantitative shape of the Laffer curve depicted in Figure 10.3. If we abstract from counter-cyclical idiosyncratic risk and consider a constant level of $\tilde{\sigma}_t = \tilde{\sigma}^0$ instead, the resulting Laffer curve is as depicted by the red dashed line in Figure 10.3. This steady-state Laffer curve reveals two differences compared to the dynamic model. First, the Laffer curve is quantitatively tiny. The maximum (average) permanent deficit is merely 0.1% (and it is reached at a much lower average value of $\check{\mu}^{\mathcal{B}}$). Second, the steady-state Laffer curve quickly decays to zero, so that the tax base is more quickly eroded as the government dilutes the claims of existing bond holders at a faster rate. Instead, in the dynamic model, agents hold on to some bonds even at very large levels of average (interest-adjusted) debt growth rates of more than 10% despite the high inflation rates that they imply. The reason is that the insurance against adverse aggregate events makes bonds attractive for agents even if they pay negative rates of return on average.

[42] If instead the increased revenues from bubble mining are used to fund additional government expenditures (higher $\mathcal{G}$), the slope of the Laffer curve is uniformly smaller.

10.7 Exercises

Consider the model of Lecture 10 with log utility and without government policy ($\mu^{\mathcal{B}} = i = \mathcal{G} = \tau = 0$).[43] In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk $\tilde{\sigma}_t$ evolves according to the exogenous stochastic process

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t,$$

where $\tilde{\sigma}^{ss}$, b and ν are positive constants.

1. Use goods market clearing and optimal investment to express q^K, $q^{\mathcal{B}}$ and ι in terms of $\vartheta := \frac{q^{\mathcal{B}}}{q^K + q^{\mathcal{B}}}$
2. Derive the "money valuation equation" using martingale method:
   (a) First, write down:
      - Returns on capital ($dr_t^{K,\tilde{i}}$) and bonds ($dr_t^{\mathcal{B}}$), and the law of motion for individual wealth $n_t^{\tilde{i}}$. Keep in mind that q_t^K and $q_t^{\mathcal{B}}$ are now stochastic processes.
      - Prices of idiosyncratic and aggregate risks, $\tilde{\varsigma}_t$ and ς_t respectively. You may take as given that prices of risk are volatility loadings of net worth.

   (b) Use the martingale pricing condition:

   $$\frac{\mathbb{E}[dr_t^{K,\tilde{i}}]}{dt} - \frac{\mathbb{E}[dr_t^{\mathcal{B}}]}{dt} = \varsigma_t(\sigma_t^{r^{K,\tilde{i}}} - \sigma_t^{r^{\mathcal{B}}}) + \tilde{\varsigma}_t(\tilde{\sigma}_t^{r^{K,\tilde{i}}} - \tilde{\sigma}_t^{r^{\mathcal{B}}})$$

   and market clearing conditions to derive an expression of the form $\mu_t^{\vartheta} = f(\vartheta_t, \tilde{\sigma}_t)$, where function f only depends on model parameters (the "money valuation equation").[44]
3. Suppose that $\nu = 0$ and the economy is at the steady state with $\tilde{\sigma}_t = \tilde{\sigma}^{ss}$.

[43] There can still be a (constant) supply of bonds $\mathcal{B}_t \neq 0$.

[44] You should derive an expression for μ_t^{ϑ} using Ito's Lemma and the definition of $\vartheta_t := \frac{q_t^{\mathcal{B}}}{q_t^K + q_t^{\mathcal{B}}}$ to substitute $\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}$ in the martingale condition.

- Derive expressions for $q^{\mathcal{B}}$, q^K and ϑ in the monetary and non-monetary equilibria.
- What is the smallest value of $\tilde{\sigma}^{ss}$ that allows for a monetary equilibrium? Denote this value by $\tilde{\sigma}^{ss}_{min}$.
- Suppose that $\tilde{\sigma}^{ss} > \tilde{\sigma}^{ss}_{min}$, what happens to $q^{\mathcal{B}}$, q^K and ϑ as $\tilde{\sigma}^{ss}$ rises?
- Suppose that $0 < \tilde{\sigma}^{ss} < \tilde{\sigma}^{ss}_{min}$, what happens to $q^{\mathcal{B}}$, q^K and ϑ as $\tilde{\sigma}^{ss}$ falls?

4. Suppose that $\nu > 0$ and solve the model numerically:

   (a) Set $a = 0.2$, $\phi = 1$, $\delta = 0.05$, $\rho = 0.01$, $\tilde{\sigma}^{ss} = 0.2$, $b = 0.05$, $\nu = 0.02$.

   (b) Since $\tilde{\sigma}_t$ follows a Cox–Ingersoll–Ross process, it is distributed according to Gamma distribution with parameters $\alpha = 2b\tilde{\sigma}^{ss}/\nu^2$ and $\beta = 2b/\nu^2$:

   $$f(\tilde{\sigma}) = \frac{\beta^\alpha}{\Gamma(\alpha)}\tilde{\sigma}^{\alpha-1}e^{-\beta\tilde{\sigma}}$$

   Based on this, suggest a grid for $\tilde{\sigma}$ and construct the M matrix using `build_M.m`. *Hint:* The function shares the same logic as in Chapter 6.

   (c) Apply Ito's lemma to $\vartheta_t = \vartheta(\tilde{\sigma}_t)$, and equate the drift term with $\vartheta_t\mu_t^\vartheta$, using the expression for μ_t^ϑ from question 2. This gives you an HJB-looking equation for $\vartheta(\tilde{\sigma})$.

   (d) Solve the model using value function iteration:

      i. Rewrite the money valuation equation such that in the discretized form you get:
      $$\rho\vartheta = \mathbf{u}(\vartheta) + \mathbf{M}\vartheta$$

      ii. Write a loop that updates $\vartheta(\tilde{\sigma})$ with the implicit method:
      $$\vartheta_{t-\Delta t} = \Big((1+\rho\Delta t)\mathbf{I} - \Delta t\mathbf{M}\Big)^{-1}\Big(\Delta t\mathbf{u}(\vartheta_t) + \vartheta_t\Big)$$

      iii. Iterate over $\vartheta(\tilde{\sigma})$ until convergence.
      Hint: Following is main code for VFI under the assumption of log-utility.

```
%% Solution VFI
v = ones(N, 1)/2;
v0 = v;

for i = 1:maxit
    u = v0.*(1-v0).^2.*sigt.^2;
    v = update_v(v0, sigt, rho, u, MU, S, dt);

    d = max(abs(v-v0)/dt);
    if d < tol
        break
    end
    v0 = v;
end
```

(e) Plot $\vartheta, q^{\mathcal{B}}, q^K, r^f, \varsigma, \tilde{\varsigma}$ as functions of $\tilde{\sigma}$.[45] Explain the dependence of the variables on $\tilde{\sigma}$.

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[45] To compute r^f you would be using Ito's formula and the martingale pricing formula for $dr^{K,\tilde{i}}$ or $dr^{\mathcal{B}}$.

Chapter 11

Money and Banking: I Theory of Money

11.1 Introduction / Overview

In this chapter, we extend the simple model introduced in Chapter 8 to a two-sector setting with financial frictions, allowing us to investigate the role of financial intermediaries in the transmission of monetary policy. This chapter builds on the framework developed in Brunnermeier and Sannikov (2016).

We use several tools to analyze the model. First, we adopt a total wealth numeraire, which simplifies many of the derivations by normalizing prices relative to the economy's total wealth. Second, we apply the "benchmark asset" approach to portfolio choice, which enables us to isolate deviations in asset returns relative to a reference asset and simplifies derivations. Lastly, we rely on several techniques developed in Chapter 4, including methods for handling heterogeneous agents, dynamic optimization, and risk-sharing constraints.

The extended model yields several important economic insights. Intermediaries emerge as key actors due to their expertise in risk management, which gives them an advantage in allocating capital under uncertainty. The presence of money, especially when markets are incomplete, has nontrivial implications for how risks are shared across the economy. In particular, inflation risk can actually help complete markets and improve overall risk sharing. Furthermore, monetary policy that responds endogenously to the nature of economic shocks can enhance welfare by improving the allocation of risk. Finally, we highlight the "Paradox of Prudence": efforts by individual agents or intermediaries to act prudently in isolation can, in aggregate, lead to worse outcomes for the system as a whole, due to feedback loops and systemic risk amplification.

11.2 Economy without versus with Outside Money/Gov. Bonds

11.2.1 A Model with an Intermediary Sector

Households. There is a continuum of households of measure one. Denote the household sector by h. Each household $(h, \tilde{i}), \tilde{i} \in [0,1]$ operates a firm that has a CRS production technology

$$y_t^{h,\tilde{i}} = a^h k_t^{h,\tilde{i}}.$$

The capital stock of each household evolves according to

$$\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left[\Phi(\iota_t^{h,\tilde{i}}) - \delta\right]\mathrm{d}t + \sigma \mathrm{d}Z_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

The households face both aggregate risk ($\mathrm{d}Z_t$) and idiosyncratic risk ($\mathrm{d}\tilde{Z}_t^{\tilde{i}}$) in the capital accumulation process. Incomplete markets make it impossible for the households to insure each other against the idiosyncratic risk. Nevertheless, they can reduce their exposure to idiosyncratic risk through the intermediary sector.

Finally, the households have logarithmic utility

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log c_t^{h,\tilde{i}}\mathrm{d}t\right].$$

Intermediaries. There is a continuum of identical intermediaries. Denote the intermediary sector by I. The intermediaries are not involved in production. Instead, they have competitive advantage in risk management. Each intermediary holds shares of *all* individual firms and diversifies the idiosyncratic risk of each firm to $\varphi\tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ where $0 < \varphi < 1$.[1] The following figure shows the balance sheets of intermediaries and households in this economy.

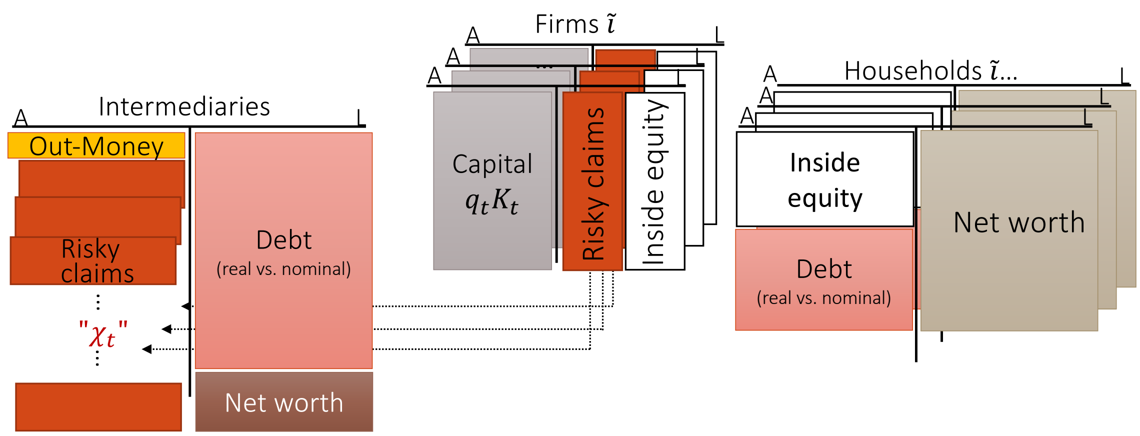

Figure 11.1: Balance sheets

The intermediaries take deposits from the households (real or nominal) and invest in shares of household-operated firms (risky claims). The intermediaries also hold outside money issued by the central bank. The intermediaries have limited capacity of risky claim issuance. The fraction of risk held by intermediaries χ_t^I cannot exceed a certain threshold $\bar{\chi} \leq 1$.

The intermediaries also have logarithmic utility

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t^{I,\tilde{i}} \mathrm{d}t \right].$$

[1] Although intermediaries hold shares of all firms, they are still subject to idiosyncratic risk. One way to generate this is through unequal exposure to different firms. Suppose that intermediary $\tilde{i}$ allocates fraction $(1-\varphi)$ of their portfolio equally across all firms and puts the remaining mass φ into one single firm $\tilde{i}$. Because the $(1-\varphi)$ portion of risk diversifies away, the intermediary is exposed to $\varphi\tilde{\sigma}$ of idiosyncratic risk.

Monetary Policy. The intermediaries also hold government-issued outside money (narrow money and government debts). The quantity of outside money follows

$$\frac{d\mathcal{MB}_t}{\mathcal{MB}_t} = \mu_t^{\mathcal{MB}} dt + \sigma_t^{\mathcal{MB}} dZ_t.$$

In this chapter, we continue to use the notion "outside money" as a reminder that intermediaries can create "inside money" (privately issued debts).

Frictions. There are two frictions in this economy

- The intermediaries can only issue a limited amount of risky claims ($\chi_t \leq \bar{\chi}$).
- The intermediary can only issue debts to the households (incomplete markets). Specifically, we will study and contrast two situations
    1. Nominal debt issuance (monetary economy)
    2. Real debt issuance only (non-monetary economy)

An equivalent expert-household setting. Note that the model with intermediaries is equivalent to the following expert-household model that resembles the ones in Chapter 3 and 4. Consider a Basak-Cuoco economy with experts and households where both groups have the same productivity ($a^e = a^h$), but experts are better at managing idiosyncratic risk ($\tilde{\sigma}^e < \tilde{\sigma}^h$).

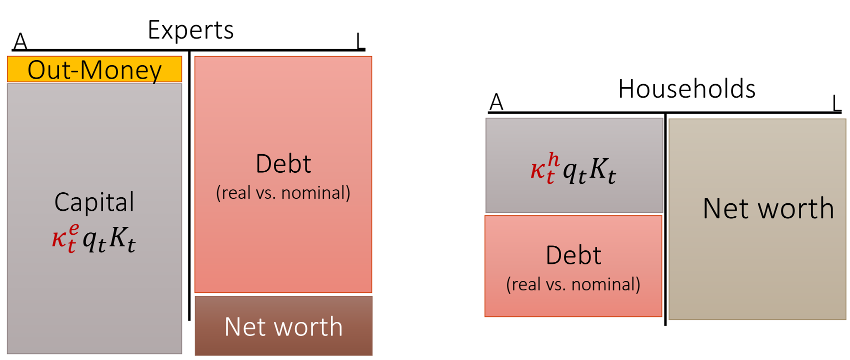

Figure 11.2: An equivalent model

The experts hold κ_t^e fraction of capital and the households hold $\kappa_t^h = 1 - \kappa_t^e$ fraction of capital. Since $a^e = a^h$, allocation of capital has no effect on aggregate productivity, which is the same as in the model with intermediaries where all capital is owned by households. It can also be shown that any risk allocation induced by a (κ_t^e, κ_t^h) pair can be replicated in the model with intermediaries by the correct choice of χ_t^I. The skin-in-the-game constraint on the experts corresponds to the constraint on intermediaries $\chi_t^I \leq \bar{\chi}$.

11.2.2 Solution Method

As in Chapter 8, let q_t^K be the real price of capital and $q_t^{\mathcal{MB}} = \mathcal{MB}_t / \mathcal{P}_t K_t$ be the normalized price of outside money. Then the real value of total capital stock is $q_t^K K_t$ and the real value of total outside money is $q_t^{\mathcal{MB}} K_t$. Define the total wealth of the economy as $N_t = (q_t^K + q_t^{\mathcal{MB}}) K_t$ and the fraction of nominal weath as $\vartheta_t = q_t^{\mathcal{MB}} / (q_t^K + q_t^{\mathcal{MB}})$.

To study the portfolio problems of intermediaries and households, we first state the utility maximization problem of agent $(i, \tilde{i})$ as

$$\begin{aligned}
\max_{\{\iota_t^{i,\tilde{i}}, \theta_t^{i,\tilde{i}}, c_t^{i,\tilde{i}}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \log c_t^{i,\tilde{i}} \mathrm{d}t \right] \\
\text{s.t.} \quad & \frac{\mathrm{d}n_t^{i,\tilde{i}}}{n_t^{i,\tilde{i}}} = -\frac{c_t^{i,\tilde{i}}}{n_t^{i,\tilde{i}}} \mathrm{d}t + (1 - \theta_t^{i,\tilde{i}}) \mathrm{d}r_t^K(\iota_t^{i,\tilde{i}}) + \theta_t^{i,\tilde{i}} \mathrm{d}r_t^{bm} \\
& n_0^{i,\tilde{i}} \text{ given.}
\end{aligned}$$

Here r_t^{bm} is the return on a benchmark asset, which corresponds to either nominal or real debt in this model. Similar to Chapter 8, the return on capital is[2]

$$\mathrm{d}r_t^K(\iota_t^{i,\tilde{i}}) = \left\{ \frac{a - \iota_t^{i,\tilde{i}}}{q_t^K} + \Phi(\iota_t^{i,\tilde{i}}) - \delta + \mu_t^{q^K} + \sigma \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K} \left[\mu_t^{\mathcal{MB}} + (\sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}}) \sigma_t^{\mathcal{MB}} \right] \right\} \mathrm{d}t$$

[2]Here we omit the index on the idiosyncratic Brownian motion $\mathrm{d}\tilde{Z}_t$. This is because the intermediaries hold claims of all individual firms, and technically, their exposure should be $\int_{\tilde{i}} \tilde{\sigma} \mathrm{d}\tilde{Z}_t^{\tilde{i}}$. We have assumed that the intermediaries can only diversify away a part of the idiosyncratic risk (e.g., $\tilde{Z}_t^{\tilde{i}}$ are correlated with each other), so $\int_{\tilde{i}} \tilde{\sigma} \mathrm{d}\tilde{Z}_t^{\tilde{i}} = \varphi \tilde{\sigma} \mathrm{d}\tilde{Z}_t$ where $\tilde{Z}_t$ is a single Brownian motion.

$$+\left(\sigma+\sigma_t^{q^K}+\frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}}\right)\mathrm{d}Z_t+\left[\mathbf{1}_{\{i=I\}}\varphi+\mathbf{1}_{\{i=h\}}\right]\tilde{\sigma}\mathrm{d}\tilde{Z}_t. \tag{11.1}$$

The intermediaries are also subject to the following "skin-in-the-game" constraint:

$$\chi_t^I=\frac{(1-\theta_t^I)\eta_t^I}{1-\vartheta_t}\le\bar{\chi},$$

where $\eta_t^I=N_t^I/N_t$ is the wealth share of the intermediary sector. To understand this equation, note that $(1-\theta_t^I)\eta_t^I=(1-\vartheta_t)\chi_t^I$ are two expressions of the same variable — the intermediaries' risky claims as a fraction of the total wealth in the economy.

Equations (8.6) and (8.7) still hold. That is, given ϑ_t

$$q_t^K=(1-\vartheta_t)\frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho},\qquad q_t^{\mathcal{MB}}=\vartheta_t\frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho},\qquad \iota_t=\frac{(1-\vartheta_t)\check{a}-\rho}{1-\vartheta_t+\phi\rho}. \tag{11.2}$$

Throughout the rest of this chapter, we will proceed in the N_t-numeraire as we once did in Section 4.2.2. We postulate that

$$\frac{\mathrm{d}\vartheta_t}{\vartheta_t}=\mu_t^{\vartheta}\mathrm{d}t+\sigma_t^{\vartheta}\mathrm{d}Z_t, \qquad \text{[in the } N_t\text{-numeraire]}$$
$$\frac{\mathrm{d}\xi_t^{h,\tilde{i}}}{\xi_t^{h,\tilde{i}}}=-r_t^{h,\tilde{i}}\mathrm{d}t-\varsigma_t^{h,\tilde{i}}\mathrm{d}Z_t-\tilde{\varsigma}_t^{h,\tilde{i}}\mathrm{d}\tilde{Z}_t^{\tilde{i}}. \qquad \text{[in the } N_t\text{-numeraire]}$$

The drift of individual SDFs no longer equal to the risk-free rate. This is because by Itô's quotient rule,

$$\mu_t^{\xi^{h,\tilde{i}}/N}=-r_t-\mu_t^N+\sigma_t^N(\sigma_t^N+\varsigma_t^{h,\tilde{i}})\neq-r_t. \qquad \text{[in consumption numeraire]}$$

Ideally, we would use different symbols for variables denoted in the N_t-numeraire. However, that would make the notations overly cumbersome, and, as a result, making this chapter unreadable. Hence, we will continue to use the same notations but **keep in mind that all variables are denominated in the N_t-numeraire hereafter**.

In equilibrium, all households have the same SDF, so

$$r_t^{h,\tilde{i}}=r_t^h,\qquad \varsigma_t^{h,\tilde{i}}=\varsigma_t^h,\qquad \tilde{\varsigma}_t^{h,\tilde{i}}=\tilde{\varsigma}_t^h.$$

Note ϑ_t is unchanged under the new numeraire. To solve for risk allocation between households and intermediaries and ϑ_t, we once again utilize the price-taking social planner's problem from Section 4.2.1. In this model, it corresponds to

$$\max_{\chi_t^I \le \bar{\chi}} \quad \mathbb{E}_t\left[\mathrm{d}r_t^N\right]/\mathrm{d}t - \left[\varsigma_t^I \chi_t^I + \varsigma_t^h \chi_t^h\right]\sigma_t^{xK} - \left[\tilde{\varsigma}_t^I \varphi \chi_t^I + \tilde{\varsigma}_t^h \chi_t^h\right]\tilde{\sigma}, \quad \text{s.t.} \quad \chi_t^I + \chi_t^h = 1,$$

where $\mathrm{d}r_t^N$ is the return of the total wealth in the economy and σ_t^{xK} is the excess risk of capital relative to the benchmark asset. The definition of benchmark asset varys depending on the specific model (e.g., nominal debt vs. real debt). In the N_t-numeraire, the total wealth is constant, so $\mathrm{d}r_t^N = 0$. The first-order condition is then

$$\varsigma_t^I \sigma_t^{xK} + \tilde{\varsigma}_t^I \varphi \tilde{\sigma} \le \varsigma_t^h \sigma_t^{xK} + \tilde{\varsigma}_t^h \tilde{\sigma}. \tag{11.3}$$

The FOC holds with equality if intermediaries' credit constraint is loose ($\chi_t^I < \bar{\chi}$). Analogous to Section 4.2.2, the evolution of sectoral wealth shares can be obtained from martingale asset pricing equations on agents' portfolio return. In the N_t-numeraire, the net worth of agent $(i,\tilde{i}), i \in \{I,h\}, \tilde{i} \in [0,1]$, $n_t^{i,\tilde{i}}$, becomes

$$\frac{n_t^{i,\tilde{i}}}{N_t} = \frac{n_t^{i,\tilde{i}}}{N_t^i} \times \frac{N_t^i}{N_t} := \underbrace{\tilde{\eta}_t^{i,\tilde{i}}}_{\text{within-sector wealth share}} \times \underbrace{\eta_t^i}_{\text{sector } i\text{'s wealth share}},$$

where $N_t^i = \int_0^1 n_t^{i,\tilde{i}} \mathrm{d}\tilde{i}$ is the total net worth of sector i. Again, consider two assets

- Asset A: agent $(i,\tilde{i})$'s portfolio return in terms of total wealth, that is $n_t^{i,\tilde{i}}/N_t = \tilde{\eta}_t^{i,\tilde{i}} \eta_t^i$.[3] The return to this asset is

$$\frac{\mathrm{d}(\tilde{\eta}_t^{i,\tilde{i}} \eta_t^i) + (c_t^{i,\tilde{i}}/N_t)\mathrm{d}t}{\tilde{\eta}_t^{i,\tilde{i}} \eta_t^i} = \frac{\mathrm{d}\eta_t^i}{\eta^i} + \frac{\mathrm{d}\tilde{\eta}_t^{i,\tilde{i}}}{\tilde{\eta}_t^{i,\tilde{i}}} + \rho \mathrm{d}t.$$

  Remember that this model is scale-invariant, so all agents within the same sector make the same consumption-portfolio decisions. Hence, $\tilde{\eta}_t^{i,\tilde{i}}$ only moves because of individual agents' exposure to idiosyncratic risk, which implies $\mathbb{E}_t\left[\mathrm{d}\tilde{\eta}_t^{i,\tilde{i}}\right] = 0$.

[3] Note that variables that are ratios of two other quantities (e.g., $\tilde{\eta}_t^{i,\tilde{i}}, \eta_t^i$) are numeraire invariant.

As before, we use the following notation[4]

$$\frac{\mathrm{d}\eta_t^i}{\eta^i} = \mu_t^{\eta^i}\mathrm{d}t + \sigma_t^{\eta^i}\mathrm{d}Z_t, \qquad \frac{\mathrm{d}\tilde{\eta}_t^{i,\tilde{i}}}{\tilde{\eta}_t^{i,\tilde{i}}} = \tilde{\sigma}_t^{\tilde{\eta}^i}\mathrm{d}\tilde{Z}_t^{\tilde{i}}. \tag{11.4}$$

- Asset B: a benchmark asset (bm) that everyone can hold. In this chapter, asset bm is either real debt or money in terms of total wealth. Note that both these assets are free of idiosyncratic risk. Denote the return on bm by $\mathrm{d}r_t^{bm} = \mu_t^{bm}\mathrm{d}t + \sigma_t^{bm}\mathrm{d}Z_t r$.

The martingale asset pricing equation is then

$$\mu_t^{\eta^i} + \rho - \mu_t^{bm} = \varsigma_t^i\left(\sigma_t^{\eta^i} - \sigma_t^{bm}\right) + \tilde{\varsigma}_t^i\tilde{\sigma}_t^{\tilde{\eta}^i}. \tag{11.5}$$

Take η_t^i -weighted sum across the two sectors ($i \in \{I, h\}$)

$$\sum_i \eta^i\mu_t^{\eta^i} + \rho - \mu_t^{bm} = \sum_i \eta^i\varsigma_t^i\left(\sigma_t^{\eta^i} - \sigma_t^{bm}\right) + \sum_i \eta^i\tilde{\varsigma}_t^i\tilde{\sigma}_t^{\tilde{\eta}^i}. \tag{11.6}$$

Since $\sum_i \eta_t^i = 1$, $\sum_i \eta^i\mu_t^{\eta^i} = \sum_i \eta^i\sigma_t^{\eta^i} = 0$. Also, since $\eta_t^h = 1 - \eta_t^I$,

$$\sigma_t^{\eta^h} = -\frac{\eta_t^I}{1-\eta_t^I}\sigma_t^{\eta^I}.$$

Log utility implies

$$\varsigma_t^I = \sigma_t^{\eta^I}, \qquad \varsigma_t^h = \sigma_t^{\eta^h} = -\frac{\eta_t^I}{1-\eta_t^I}\sigma_t^{\eta^I}, \qquad \tilde{\varsigma}_t^I = \tilde{\sigma}_t^{\tilde{\eta}^I}, \qquad \tilde{\varsigma}_t^h = \tilde{\sigma}_t^{\tilde{\eta}^h}. \tag{11.7}$$

The asset pricing equations can thus be simplified to the following "benchmark asset

[4]In N_t-numeraore, individual net worth $n_t^{i,\tilde{i}}$ evloves according to

$$\frac{\mathrm{d}(\tilde{\eta}_t^{i,\tilde{i}}\eta_t^i)}{\tilde{\eta}_t^{i,\tilde{i}}\eta_t^i} = \mu_t^{\eta^i}\mathrm{d}t + \sigma_t^{\eta^i}\mathrm{d}Z_t + \tilde{\sigma}_t^{\tilde{\eta}^i}\mathrm{d}\tilde{Z}_t^{\tilde{i}}.$$

evaluation equation"

$$\underbrace{\rho - \mu_t^{bm}}_{\text{excess return on } N_t} = \underbrace{\sum_i \eta^i \left(\sigma_t^{\eta^i}\right)^2 + \sum_i \eta^i \left(\tilde{\sigma}_t^{\tilde{\eta}^i}\right)^2}_{\text{net worth weighted risk premium}}. \tag{11.8}$$

Subtracting (11.6) from (11.5), we get the drift of η_t^i as

$$\mu_t^{\eta^i} = (1-\eta_t^i)\left[\varsigma_t^i(\sigma_t^{\eta^i} - \sigma_t^{bm}) - \varsigma_t^{-i}(\sigma_t^{\eta^{-i}} - \sigma_t^{bm}) + \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\tilde{\eta}^i} - \tilde{\varsigma}_t^{-i}\tilde{\sigma}_t^{\tilde{\eta}^{-i}}\right]. \tag{11.9}$$

where $-i = \{I, h\}\backslash i$ denotes the other sector in the economy. Since each agent's portfolio contains a linear combination between the benchmark asset and capital, her net worth (in the N_t-numeraire) $\tilde{\eta}_t^{i,\tilde{i}}\eta_t^i$ has a loading on the aggregate risk of

$$\sigma_t^{\eta^i} = \theta_t^i \sigma_t^{bm} + (1-\theta_t^i)\sigma_t^{r^K} = \sigma_t^{bm} + (1-\theta_t^i)\sigma_t^{xK}, \tag{11.10}$$

where $1-\theta_t^i = \dfrac{\chi_t^i}{\eta_t^i}(1-\vartheta_t)$ is the portfolio weight on capital of agents in sector i. Similarly, since the benchmark asset is idiosyncratic-risk-free, $\tilde{\eta}_t^{i,\tilde{i}}\eta_t^i$'s exposure to idiosyncratic risk depends on the portfolio weight on capital,

$$\tilde{\sigma}_t^{\tilde{\eta}^i} = \begin{cases} (1-\theta_t^i)\varphi\tilde{\sigma} & \text{if } i = I \\ (1-\theta_t^i)\tilde{\sigma} & \text{if } i = h \end{cases}. \tag{11.11}$$

11.2.3 The Non-monetary Economy

In this subsection, we consider a non-monetary economy where outside money has no value and intermediaries can only issue real debts. This is exactly the Basak-Cuoco economy in Chapter 3 with an additional idiosyncratic risk component. Equation (11.2) tells us that in consumption numeriare

$$q_t^{\mathcal{MB}} = \vartheta_t = 0, \qquad q_t^K = \frac{1+\phi\check{a}}{1+\phi\rho}, \qquad \iota_t = \frac{\check{a}-\rho}{1+\phi\rho}.$$

In this economy, the benchmark asset bm is the risk-free real debts issued by intermediaries. In the N_t-numeraire, $\sigma_t^{bm} = -\sigma_t^N = -\sigma$. Therefore, from (11.1) we know that

$$\sigma_t^{xK} = \sigma_t^{r^K} - \sigma_t^{bm} = \left(\sigma + \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}} - \sigma_t^N\right) - (-\sigma_t^N) = \sigma.$$

Further, notice that since $\sigma_t^{q^K} = 0$,

$$\sigma_t^{r^K} = \sigma + \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}} - \sigma_t^N = 0.$$

Equations (11.9) and (11.10) then imply

$$\mu_t^{\eta^I} = \frac{\chi_t^I - \eta_t^I}{\eta_t^I}\frac{(\chi_t^I - \eta_t^I)^2}{\eta_t^I(1-\eta_t^I)}\sigma^2 + (1-\eta_t^I)\left[\left(\frac{\chi_t^I}{\eta_t^I}\right)^2\varphi^2 - \left(\frac{1-\chi_t^I}{1-\eta_t^I}\right)^2\right]\tilde{\sigma}^2, \tag{11.12}$$

$$\sigma_t^{\eta^I} = \theta_t^I\sigma_t^{bm} = \frac{\chi_t^I - \eta_t^I}{\eta_t^I}\sigma, \tag{11.13}$$

so the aggregate risk is not perfectly shared as long as $\chi_t^i \neq \eta_t^i$. However, investment rate ι_t and capital price q_t^K are constant (in consumption numeraire). This is because the changes in η_t^I induced by aggregate shocks are fully absorbed by movements in the risk-free rate, which can be computed from (11.8).

To conclude the model, we can plug (11.11) and (11.13) into the FOC to the planner's problem (11.3) to solve for χ_t^i as a function of the state η_t^I:

$$\chi_t^I = \min\left\{\frac{\eta^I(\sigma_t^2 + \tilde{\sigma}^2)}{\sigma^2 + [(1-\eta_t^I)\varphi^2 + \eta_t^I]\tilde{\sigma}^2}, \bar{\chi}\right\}.$$

The evolution of η_t^I is governed by the SDE consisted of (11.12) and (11.13).

11.2.4 The Nominal Economy

In this subsection, we consider a monetary equilibrium where outside money is positively valued and the intermediaries issue nominal debts (inside money) to the households. In this setting, the benchmark asset bm is the nominal money (inside and outside). In the N_t-numeraire, the risk of capital and money is

$$\sigma_t^{r^K} = \sigma + \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}} - \sigma_t^N,$$
$$\sigma_t^{bm} = \sigma + \sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}} - \sigma_t^N$$

The second equation is derived from the fact that in consumption numeraire $\mathrm{d}r_t^{\mathcal{MB}} = \dfrac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = \dfrac{\mathrm{d}(q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t)}{q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t}$. Hence,

$$\sigma_t^{xK} = \sigma_t^{q^K} - \sigma_t^{q^{\mathcal{MB}}} + \left(1 + \frac{q_t^{\mathcal{MB}}}{q_t^K}\right)\sigma_t^{\mathcal{MB}} = \frac{\sigma_t^{\mathcal{MB}} - \sigma_t^{\vartheta}}{1 - \vartheta_t}.$$

Because $N_t = (q_t^{\mathcal{MB}} + q_t^K)K_t$, Itô's lemma implies $\sigma_t^N = \frac{q_t^{\mathcal{MB}}}{q_t^{\mathcal{MB}}+q_t^K}(\sigma + \sigma_t^{q^{\mathcal{MB}}}) + \frac{q_t^K}{q_t^{\mathcal{MB}}+q_t^K}(\sigma + \sigma_t^{q^K})$[5]. We can rewrite the risk of money as

$$\begin{aligned}\sigma_t^{bm} &= \sigma + \sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}} - \left[\frac{q_t^{\mathcal{MB}}}{q_t^{\mathcal{MB}} + q_t^K}(\sigma + \sigma_t^{q^{\mathcal{MB}}}) + \frac{q_t^K}{q_t^{\mathcal{MB}} + q_t^K}(\sigma + \sigma_t^{q^K})\right] \\ &= \sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}}.\end{aligned}$$

Equation (11.10) then implies

$$\sigma_t^{\eta^I} = \sigma_t^{bm} + (1 - \theta_t^I)\sigma_t^{xK} = \left(\frac{\chi_t^I}{\eta_t^I} - 1\right)(\sigma_t^{\mathcal{MB}} - \sigma_t^{\vartheta}).$$

Postulate that $\vartheta_t = \vartheta(\eta_t^I)$. By Itô's lemma, $\sigma_t^{\vartheta} = \dfrac{\vartheta'(\eta_t^I)}{\vartheta(\eta_t^I)}\eta_t^I\sigma_t^{\eta^I}$. Solving for $\eta_t^I\sigma_t^{\eta^I}$, we get

$$\eta_t^I\sigma_t^{\eta^I} = \frac{\chi_t^I - \eta_t^I}{1 + \dfrac{\chi_t^I - \eta_t^I}{\eta_t^I}\dfrac{\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}}\sigma_t^{\mathcal{MB}}.$$

Both sectors' balance sheets are perfectly hedged against aggregate risk if $\sigma_t^{\mathcal{MB}} = 0$. This result is achieved via inflation risk ($\sigma_t^{q^{\mathcal{MB}}}$), which effectively completes the market

[5]The risk of aggregate capital stock is simply $\sigma_t^K = \sigma$.

for aggregate risk. As such, a strict inflation targeting might not always be desirable.

Again, we can plug (11.11) and $\sigma_t^{\eta^I} = 0$ into the FOC to the planner's problem (11.3) to solve for χ_t^i as a function of the state η_t^I:

$$\chi_t^I = \min\left\{ \frac{\eta_t^I}{(1-\eta_t^I)\varphi^2 + \eta_t^I}, \bar{\chi} \right\}. \tag{11.14}$$

In the N_t-numeraire, the return to money is $dr_t^{bm} = \frac{d(q_t^{\mathcal{MB}} K_t / \mathcal{MB}_t N_t)}{q_t^{\mathcal{MB}} K_t / \mathcal{MB}_t N_t} = \frac{d(\vartheta_t / \mathcal{MB}_t)}{\vartheta_t / \mathcal{MB}_t}$. So $\mu_t^{bm} = \mu_t^{\vartheta} - \mu_t^{\mathcal{MB}}$ and (11.8) becomes the following money evaluation equation ($\sigma_t^{\mathcal{MB}} = 0$)

$$\rho - \mu_t^{\vartheta} + \mu_t^{\mathcal{MB}} = (1-\vartheta_t)^2 \left[\varphi^2 \eta_t^I \left(\frac{\chi_t^I}{\eta_t^I} \right)^2 + (1-\eta_t^I) \left(\frac{1-\chi_t^I}{1-\eta_t^I} \right)^2 \right] \tilde{\sigma}^2 \tag{11.15}$$

The drift of η_t^I is given by (11.9)

$$\mu_t^{\eta^I} = (1-\eta_t^I)(1-\vartheta_t)^2 \left[\left(\frac{\chi_t^I}{\eta_t^I} \right)^2 \varphi^2 - \left(\frac{1-\chi_t^I}{1-\eta_t^I} \right)^2 \right] \tilde{\sigma}^2. \tag{11.16}$$

To solve the model, postulate that $\vartheta_t = \vartheta(\eta_t^I, t)$. By Itô's lemma,

$$\vartheta_t \mu_t^{\vartheta} = \partial_t \vartheta + (\partial_\eta \vartheta) \eta_t^I \mu_t^{\eta^I}.$$

Plugging in (11.15) and (11.16), we get the following PDE

$$\vartheta_t \left\{ \rho + \mu_t^{\mathcal{MB}} - (1-\vartheta_t)^2 \left[\varphi^2 \eta_t^I \left(\frac{\chi_t^I}{\eta_t^I} \right)^2 - (1-\eta_t^I) \left(\frac{1-\chi_t^I}{1-\eta_t^I} \right)^2 \right] \tilde{\sigma}^2 \right\} =$$
$$\partial_t \vartheta_t + (\partial_\eta \vartheta_t) \eta_t^I (1-\eta_t^I)(1-\vartheta_t)^2 \left[\left(\frac{\chi_t^I}{\eta_t^I} \right)^2 \varphi^2 - \left(\frac{1-\chi_t^I}{1-\eta_t^I} \right)^2 \right] \tilde{\sigma}^2,$$

where χ_t^I is given by (11.14). We can start with a terminal condition $\vartheta(\eta^I, T)$ and solve this PDE backwards in time as in Section 6.1.1.

11.2.5 Numerical Results and Discussions

Numerical Code. Following is main code executing the algorithm. We use the PDE solver `payoff_policy_growth.m` in Section 6.1.1

```
% solves chashless vs. monetary model in lecture 6
% required script: payoff_policy_growth.m
close all; clear; clc

%% parameters
a = 0.15;            % productivity
rho = 0.03;          % decay rate
sigma = 0.2;         % aggregate volatility
sigmaIdio = 0.3;     % indio volatility
phi = 2;             % investment function parameter
chibar = 0.5;        % risk claims upperbound of intermediaries
varphi = 2/3;        % diversification ability of intermediaries for idio risk
delta = 0.03;        % decay rate
tol = 1e-3;          % tolerance
muB = 0.01;          % monetary policy: constant
sigmaB = 0;          % monetary policy: constant
lambda = 0.5;        % lambda = dt/(1+dt), dt is the time step

%% Grid
etaLength = 300;
eta = (linspace(0.001,chibar,etaLength))';  % wealth share of intermediaries
S = zeros(etaLength,1);
G = zeros(etaLength,1);

%% ------------ Cashless Economy -- closed form solution ------------ %%
% cashless -- vartheta = 0
varthetaCashless = 0;
% real price of capital
qKCashless = (1+phi*a)/(1+phi*rho)*ones(etaLength,1);
% normalized price of outside money
qBCashless = zeros(etaLength,1);
% Investment
iotaCashless = (a-rho)/(1+phi*rho)*ones(etaLength,1);
% risk share of intermediaries
chiCashless = min(eta*(sigma^2+sigmaIdio^2)...
    ./(sigma^2 + ((1-eta)*varphi^2 + eta)*sigmaIdio^2), chibar);
% drift of etaI
muEtaCashless = (chiCashless-eta).^3*sigma^2./eta.^2./(1-eta) ...
    + (1-eta)*sigmaIdio^2 ...
    .*((chiCashless./eta).^2*varphi^2 - ((1-chiCashless)./(1-eta)).^2);
% volatility of etaI
sigmaEtaCashless = (chiCashless - eta)./eta * sigma;
% Risk free rate
PhiCashless = log(qKCashless)/phi;
riskFreeRateCashless = rho + PhiCashless - delta - chiCashless./eta*(sigma^2+sigmaIdio
    ^2);
```

```

%% ------------ Monetary Economy -- iterative method ------------ %%
% risk share of intermediaries
chiMonetary = min(eta./((1-eta)*varphi^2 + eta), chibar);
% initial vartheta: start from steady state
varthetaSteadyStateMonetary = 1 - sqrt(rho)/sigmaIdio/varphi;
varthetaInitMonetary = varthetaSteadyStateMonetary*ones(length(eta),1);
varthetaMonetary = varthetaInitMonetary;

for i=1:1500
  % 1. compute updated coefficients
  muEtaMonetary = (1-eta) .* (1-varthetaMonetary).^2 * sigmaIdio^2 ...
    .*((chiMonetary./eta).^2*varphi^2 - ((1-chiMonetary)./(1-eta)).^2);
  muVarthetaMonetary = rho + muB - (1-varthetaMonetary).^2 * sigmaIdio^2 ...
    .*(varphi^2*chiMonetary.^2./eta + (1-chiMonetary).^2./(1-eta));     % money valuation equation

  % 2. PDE time step, call payoff_policy_growth.m function
  MU = muEtaMonetary.*eta;
  newVarthetaMonetary = payoff_policy_growth(eta, muVarthetaMonetary, MU, S, G, varthetaMonetary, lambda);

  % 3. check convergence
  absChangeVartheta = abs(newVarthetaMonetary-varthetaMonetary)/lambda*(1-lambda);
  relChangeVartheta = absChangeVartheta./(abs(newVarthetaMonetary)+abs(varthetaMonetary))*2;
  maxRelChange = max(relChangeVartheta);
  if maxRelChange < tol
      break;
  end

  % 4. update vartheta
  varthetaMonetary = newVarthetaMonetary;
end
% real price of capital
qKMonetary = (1-varthetaMonetary).*(1 + phi*a)./(1 - varthetaMonetary + phi*rho);
% normalized price of outside money
qBMonetary = varthetaMonetary.*(1 + phi*a)./(1 - varthetaMonetary + phi*rho);
% Investment
iotaMonetary = ((1-varthetaMonetary)*a-rho)./(1-varthetaMonetary+phi*rho);
% drift of etaI
muEtaMonetary = (1-eta) .* (1-varthetaMonetary).^2 * sigmaIdio^2 ...
    .*((chiMonetary./eta).^2*varphi^2 - ((1-chiMonetary)./(1-eta)).^2);
% volatility of etaI
sigmaEtaMonetary = zeros(etaLength);
% Find risk free rate
PhiMonetary = log(qKMonetary)/phi;
muVarthetaMonetary = rho + muB - (1-varthetaMonetary).^2 * sigmaIdio^2 ...
    .*(varphi^2*chiMonetary.^2./eta + (1-chiMonetary).^2./(1-eta));
muP = (1./varthetaMonetary + 1./(1 - varthetaMonetary + phi*rho)).*muVarthetaMonetary.*varthetaMonetary;
riskFreeRateMonetary = PhiMonetary - delta + muP - sigma^2;
```

Numerical Results The equilibrium for the non-monetary economy and monetary is as follows

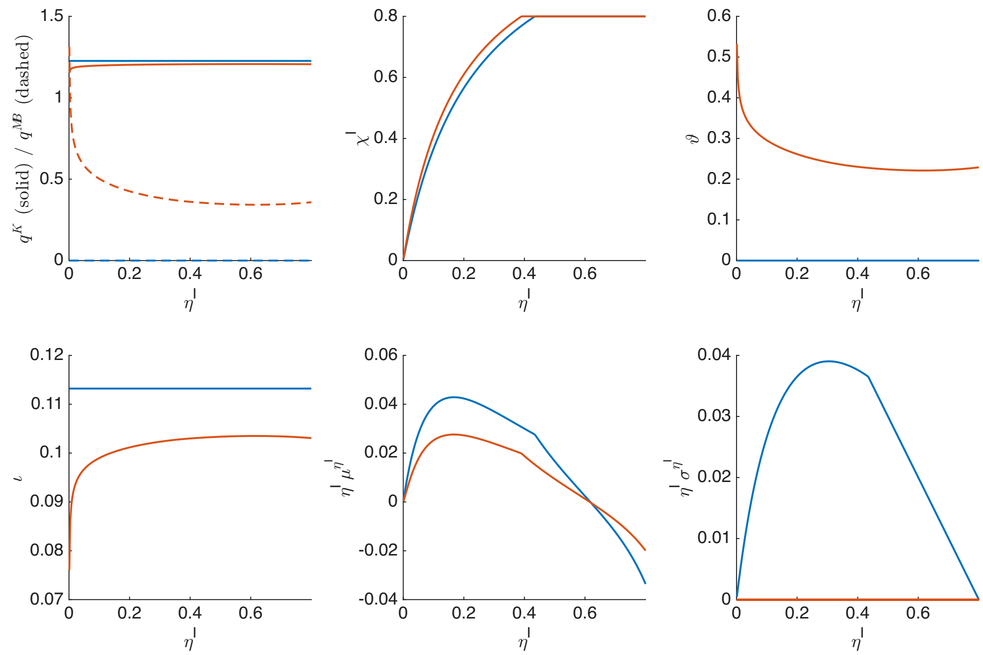

Figure 11.3: Non-monetary Economy (blue) vs. Monetary Economy (red)

Parameter	Value	Parameter	Value
ρ	0.05	σ	0.2
a	0.15	$\tilde{\sigma}$	0.5
ϕ	2	φ	0.4
δ	0.03	ϕ	2
$\sigma^{\mathcal{MB}}$	0	$\mu^{\mathcal{MB}}$	0.01 (monetary economy)

Table 11.1: Parameters for Baseline Model.

An Economy with Both Real and Nominal Debts. Including real debts in the monetary economy does not alter the equilibrium at all[6]. This is because the markets are already complete with respect to aggregate risk if there exist nominal debts. Adding real debts will not change aggregate risk sharing and can not affect idiosyncratic risk sharing.

[6]This result relies on the absence of price stickiness.

The Role of Financial Frictions. Note that if $\bar{\chi} = 1$ (i.e, the intermediaries are not constrained), then both models (non-monetary vs. monetary) converge to the one-sector model in Chapter 8 in the long run. This is because the intermediaries will manage all capital stock and the wealth distribution between sectors have no impact on risk sharing.

Discontinuity at the Non-monetary Limit. In this model, the absolute quantity of money $\mathcal{MB}_t$ does not affect the equilibrium (although $\mu_t^{\mathcal{MB}}$ does). However, as we move from $\mathcal{MB}_t > 0$ (monetary economy) to $\mathcal{MB}_t = 0$ (non-monetary economy), equilibrium dynamics jump discontinuously (as illustrated in previous sections). This result is seemingly contradictory to the standard DSGE literature (e.g., Woodford, 2003). The reason behind this discrepancy is that traditionary monetary models consider money as a medium of exchange, whereas in our model money is a store of value.

11.3 The I Theory of Money

11.3.1 Model Setup

In Section 11.2.4, the intermediary sector is perfectly hedged against aggregate risk and the idiosyncratic risk that households bear ($\tilde{\sigma}_t^{\tilde{\eta}^h}$) is constant over time. These two results might seem unrealistic in practice, and in order to break them, we need the intermediaries' exposure to aggregate risk to differ from the aggregate risk of the economy. One way of modeling differentiated aggregate risk exposure is to introduce two production technologies that have different levels of aggregate risk.

Specifically, assume each individual firm has two technologies a and b. The two technologies are Leontieff in the sense that their relative capital share is fixed. In other words, each firm allocates $(1 - \bar{\psi})$ fraction of its capital to technology a and $\bar{\psi}$ fraction to technology b. The capital accumulation processes for both technologies are

$$a: \qquad \frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left[\Phi(\iota_t^{h,\tilde{i}}) - \delta\right] \mathrm{d}t + \sigma^a \mathrm{d}Z_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}},$$

$$b: \qquad \frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left[\Phi(\iota_t^{h,\tilde{i}}) - \delta\right]\mathrm{d}t + \sigma^b \mathrm{d}Z_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}.$$

where $\sigma := \sigma^b - \sigma^a > 0$. Total output is equal to:

$$Y_t = AK_t = A_t^a(1-\bar{\psi})K_t + A_t^b\bar{\psi}K_t,$$

with A_t^a and A_t^b being per-capital output for technologies a and b, determined endogenously. Aggregate capital evolves as follows:

$$\frac{\mathrm{d}K_t}{K_t} = (\Phi(\iota_t) - \delta)\mathrm{d}t + \sigma^K \mathrm{d}Z_t,$$

with $\sigma^K = (1-\bar{\psi})\sigma^a + \bar{\psi}\sigma^b$, and returns on the two technologies are given by:

$$\begin{aligned} \mathrm{d}r_t^x(\iota_t) = &\left\{\frac{A_t^x - \iota_t}{q_t^K} + \Phi(\iota_t) - \delta + \mu_t^{q^K} + \sigma^x\sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\left[\mu_t^{\mathcal{MB}} + (\sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}})\sigma_t^{\mathcal{MB}}\right]\right\}\mathrm{d}t \\ &+ \left(\sigma^x + \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}}\right)\mathrm{d}Z_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t, \end{aligned}$$

for $x \in \{a, b\}$. Further, assume that intermediaries can only hold risky claims of technology b. Their return on outside equity is given by:

$$\mathrm{d}r_t^{OE,I} = r_t^{OE}\mathrm{d}t + \left(\sigma^{\mathcal{MB}} + \sigma_t^{q^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\sigma_t^{\mathcal{MB}}\right)\mathrm{d}Z_t + \varphi\tilde{\sigma}\mathrm{d}\tilde{Z}_t,$$

whereas households' return is $\mathrm{d}r_t^{OE,h} = \mathrm{d}r_t^{OE,I} + (1-\varphi)\tilde{\sigma}\mathrm{d}\tilde{Z}_t$. The balance sheets of intermediaries and households are as follows

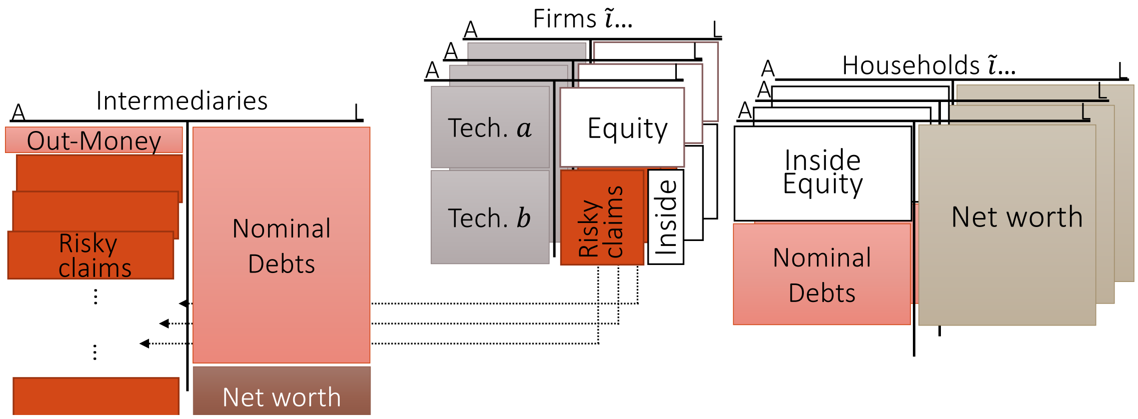

Figure 11.4: Balance sheets

To be clear, capital ownership in the economy is

Tech. a	Tech. b held by intermediaries	Tech. b held by households	Total
$1-\bar{\psi}$	χ_t^I	$\bar{\psi}-\chi_t^I$	1

Solution. The price-taking planner's problem in N_t-numeraire is

$$\min_{\chi_t \leq \bar{\chi}} \quad (1-\bar{\psi})\varsigma_t^h \sigma_t^{xK^a} + \left[\varsigma_t^I \chi_t^I + \varsigma_t^h(\bar{\psi}-\chi_t^I)\right]\sigma_t^{xK^b} + \left[\tilde{\varsigma}_t^I \varphi \chi_t^I + \tilde{\varsigma}_t^h(1-\chi_t^I)\right]\tilde{\sigma}$$

where

$$\sigma_t^{xK^a} = -\bar{\psi}\sigma - \frac{\sigma_t^{\vartheta} - \sigma^{\mathcal{MB}}}{1-\vartheta_t}, \qquad \sigma_t^{xK^b} = (1-\bar{\psi})\sigma - \frac{\sigma_t^{\vartheta} - \sigma^{\mathcal{MB}}}{1-\vartheta_t}.$$

The FOC is (which holds with equality if $\chi_t^I < \bar{\chi}$)

$$\varsigma_t^I \sigma_t^{xK^b} + \tilde{\varsigma}_t^I \varphi \tilde{\sigma} \leq \varsigma_t^h \sigma_t^{xK^b} + \tilde{\varsigma}_t^h \tilde{\sigma},$$

i.e.,

$$\sigma_t^{\eta^I}\left[(1-\bar{\psi})\sigma - \frac{\sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}}}{1-\vartheta_t}\right] + \left[(1-\vartheta_t)\frac{\chi_t^I}{\eta_t^I}\varphi\tilde{\sigma}\right]\varphi\tilde{\sigma} \leq \frac{-\eta_t^I \sigma_t^{\eta^I}}{1-\eta_t^I}\left[(1-\bar{\psi})\sigma - \frac{\sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}}}{1-\vartheta_t}\right]$$
$$+\left[(1-\vartheta_t)\frac{1-\chi_t^I}{1-\eta_t^I}\tilde{\sigma}\right]\tilde{\sigma} \qquad (11.17)$$

The rest of the model is the same as before. Since the return on money is

$$\mathrm{d}r_t^{bm} = \frac{\mathrm{d}(q_t^{\mathcal{MB}} K_t/\mathcal{MB}_t N_t)}{q_t^{\mathcal{MB}} K_t/\mathcal{MB}_t N_t} = \frac{\mathrm{d}(\vartheta_t/\mathcal{MB}_t)}{\vartheta_t/\mathcal{MB}_t} = \underbrace{\left[(\mu_t^{\vartheta} - \mu_t^{\mathcal{MB}}) + \sigma_t^{\mathcal{MB}}(\sigma_t^{\mathcal{MB}} - \sigma_t^{\vartheta})\right]}_{\mu_t^{bm}} \mathrm{d}t + \underbrace{(\sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}})}_{\sigma_t^{bm}} \mathrm{d}Z_t$$

Equation (11.8) implies

$$\rho - \left[(\mu_t^{\vartheta} - \mu_t^{\mathcal{MB}}) + \sigma_t^{\mathcal{MB}}(\sigma_t^{\mathcal{MB}} - \sigma_t^{\vartheta})\right] = \sum_i \eta^i \left(\sigma_t^{\eta^i}\right)^2 + \sum_i \eta^i \left(\tilde{\sigma}_t^{\tilde{\eta}^i}\right)^2 \tag{11.18}$$

The drift of η_t^I is still given by (11.9), (11.7) and (11.11),

$$\begin{aligned} \mu_t^{\eta^I} &= (1 - \eta_t^I)\left[\varsigma_t^I(\sigma_t^{\eta^I} - \sigma_t^{bm}) - \varsigma_t^h(\sigma_t^{\eta^h} - \sigma_t^{bm}) + \tilde{\varsigma}_t^I \tilde{\sigma}_t^{\tilde{\eta}^I} - \tilde{\varsigma}_t^h \tilde{\sigma}_t^{\tilde{\eta}^h}\right], \\ &= (1 - \eta_t^I)\left[\left(\sigma_t^{\eta^I}\right)^2 + \left(\tilde{\sigma}_t^{\tilde{\eta}^I}\right)^2 - \left(\frac{\eta_t^I \sigma_t^{\eta^I}}{1 - \eta_t^I}\right)^2 - \left(\tilde{\sigma}_t^{\tilde{\eta}^h}\right)^2\right] - \sigma_t^{\eta^I}(\sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}}). \end{aligned} \tag{11.19}$$

The volatility of η_t^I is given by (11.10)

$$\begin{aligned} \sigma_t^{\eta^I} &= \sigma_t^{bm} + (1 - \theta_t^I)\sigma_t^{xK^b} \\ &= \sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}} + \frac{\chi_t^I}{\eta_t^I}(1 - \vartheta_t)\left[(1 - \bar{\psi})\sigma - \frac{\sigma_t^{\vartheta} - \sigma_t^{\mathcal{MB}}}{1 - \vartheta_t}\right]. \end{aligned} \tag{11.20}$$

Postulate that $\vartheta_t = \vartheta(\eta_t^I)$. By Itô's lemma, $\sigma_t^{\vartheta} = \frac{\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}\sigma_t^{\eta^I}$, so

$$\eta_t^I \sigma_t^{\eta^I} = \frac{(1 - \vartheta_t)\,\chi_t^I(1 - \bar{\psi})\sigma + (\chi_t^I - \eta_t^I)\sigma_t^{\mathcal{MB}}}{1 - \dfrac{\chi_t^I - \eta_t^I}{\eta_t^I}\left(\dfrac{-\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}\right)}. \tag{11.21}$$

So the intermediaries are not perfectly hedged under $\sigma_t^{\mathcal{MB}} = 0$. Amplification results from the changes in the price of money relative to capital, $\vartheta(\eta_t^I)$. As long as the intermediaries' portfolio share of households' equity $(1 - \theta_t^I)$ is greater than $(1 - \vartheta_t)$, the overall capital share (so that $\chi_t^I > \eta_t^I$), and as long as $\vartheta'(\eta^I) < 0$, amplification exists.

Note that since $\sigma_t^{\vartheta} = (1-\vartheta_t)(\sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{q^K})$., we can further decompose $\frac{-\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}$ as

$$\frac{-\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)} = (1-\vartheta_t)\left(\frac{(q_t^K)'(\eta_t^I)\eta_t^I}{q_t^K(\eta_t^I)} + \frac{-(q_t^{\mathcal{MB}})'(\eta_t^I)\eta_t^I}{q_t^{\mathcal{MB}}(\eta_t^I)}\right).$$

Amplification arises from two spirals: changes in the price of capital q_t^K, i.e. the *liquidity* spiral, and changes in the value of money $q_t^{\mathcal{MB}}$, the *disinflationary* spiral. In the region where intermediaries are undercapitalized (i.e. η_t^I is low), negative shocks are amplified both on the asset sides of intermediary balance sheets, as the price of physical capital $q^K(\eta_t^I)$ drops following a negative shock, and on the liability sides, through the Fisher disinflationary spiral, as the value of money $q^{\mathcal{MB}}(\eta_t^I)$ rises. Both effects impair the intermediaries' net worth. Intermediaries' response to these losses is to shrink their balance sheets, leading to fire-sales (lowering the price q_t^K) and reduction in inside money (increasing the value of liabilities $q_t^{\mathcal{MB}}$). In other words, intermediaries take fewer deposits, create less inside money, and the money multiplier collapses.[7] This again reduces their net worth, and so on. The "Paradox of Prudence" emerges. Each individual intermediary micro-prudent behavior to scale back his risk is macro-imprudent, as it raises endogenous risk.

Specifically, this feedback effects lead to a geometric series, which has been summed up in Equation (11.21). Amplification becomes greater as $\vartheta'(\eta_t^I)$ becomes more negative, and as intermediary leverage $(1-\theta_t^I)$ rises.

Numerical Algorithm. Postulate that $\vartheta_t = \vartheta(t, \eta_t^I)$. By Itô's lemma,

$$d\vartheta_t = \underbrace{\left[\partial_t\vartheta + (\partial_\eta\vartheta)\mu_t^{\eta^I}\eta_t^I + \frac{1}{2}(\partial_{\eta\eta}\vartheta)\left(\sigma_t^{\eta^I}\eta_t^I\right)^2\right]}_{\mu_t^{\vartheta}\vartheta_t} dt + \underbrace{(\partial_\eta\vartheta)\sigma_t^{\eta^I}\eta_t^I}_{\sigma_t^{\vartheta}\vartheta_t} dZ_t$$

[7]In reality, rather than turning savers away, financial intermediaries might still issue demand deposits and simply park the proceeds with the central bank as excess reserves.

Plugging in (11.18), we have

$$\rho - \left[\frac{\partial_t \vartheta}{\vartheta_t} + \frac{\partial_\eta \vartheta}{\vartheta_t}\mu_t^{\eta^I}\eta_t^I + \frac{1}{2}\frac{\partial_{\eta\eta}\vartheta}{\vartheta_t}\left(\sigma_t^{\eta^I}\eta_t^I\right)^2 - \mu_t^{\mathcal{MB}} + \sigma_t^{\mathcal{MB}}\left(\sigma_t^{\mathcal{MB}} - \frac{\partial_\eta \vartheta}{\vartheta_t}\sigma_t^{\eta^I}\eta_t^I\right)\right]$$
$$= \sum_i \eta^i \left(\sigma_t^{\eta^i}\right)^2 + \sum_i \eta^i \left(\tilde{\sigma}_t^{\tilde{\eta}^i}\right)^2$$

In a "PDE" form:

$$\vartheta\left\{\rho + \mu_t^{\mathcal{MB}} - (\sigma_t^{\mathcal{MB}})^2 - \left[\sum_i \eta^i \left(\sigma_t^{\eta^i}\right)^2 + \sum_i \eta^i \left(\tilde{\sigma}_t^{\tilde{\eta}^i}\right)^2\right]\right\}$$
$$= \partial_t \vartheta + (\partial_\eta \vartheta)\eta_t^I\left(\mu_t^{\eta^I} - \sigma_t^{\eta^I}\sigma_t^{\mathcal{MB}}\right) + \frac{1}{2}(\partial_{\eta\eta}\vartheta)\left(\sigma_t^{\eta^I}\eta_t^I\right)^2$$

where $\mu_t^{\eta^I}, \sigma_t^{\eta^I}, \sigma_t^{\eta^h}, \tilde{\sigma}_t^{\tilde{\eta}^I}, \tilde{\sigma}_t^{\tilde{\eta}^h}$ are given by (11.19), (11.21), (11.7), (11.11). Further, χ_t can be obtained as a function of η_t^I by solving the planner's FOC (11.17).

Numerical Analysis. Consider parameter values $\rho = 0.05$, $a = 0.5$, $\sigma = 0.1$, $\tilde{\sigma} = 0.4$, $\phi = 2$, $\varphi = 0.2$, $\bar{\psi} = 0.5$, $\bar{\chi} = 0.45$ and $\mu_t^{\mathcal{MB}} = \sigma_t^{\mathcal{MB}} = 0$.

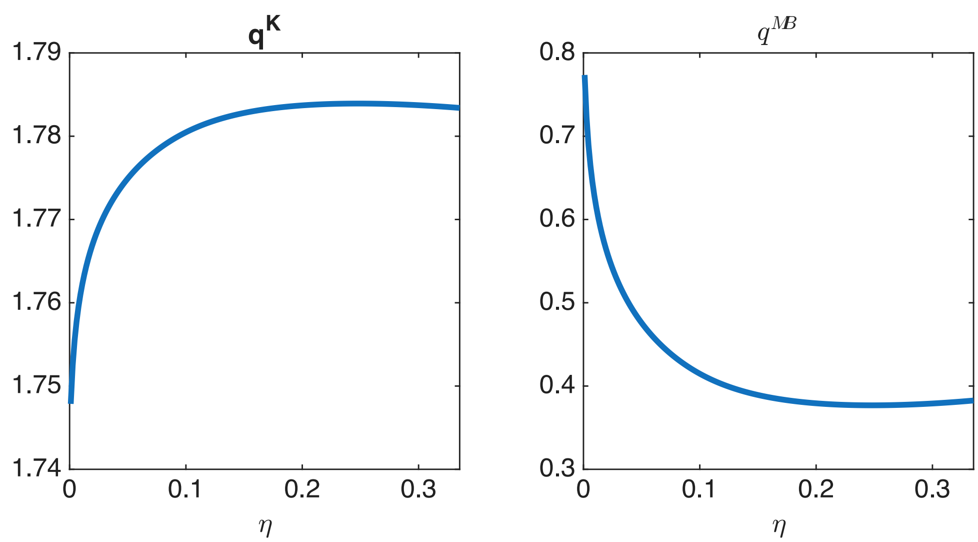

Figure 11.5: Equilibrium prices of capital q_t^K and money $q_t^{\mathcal{MB}}$.

Figure above shows the prices $q^K(\eta)$ and $q^{\mathcal{MB}}(\eta)$ of capital and money in equilibrium. At $\eta_t^I = 0$, the values of q_t^K and $q_t^{\mathcal{MB}}$ converge to those under the benchmark without intermediaries (Chapter 8). As η_t^I rises, the price of capital rises and the price of money drops. Money becomes less valuable as η_t^I rises mainly because intermediaries create money, the inside money on the liabilities sides of the intermediaries' balance sheets is a perfect substitute to outside money, and intermediaries' diversify idiosyncratic risk away, which lowers the demand for money.

Figure below illustrates the equilibrium dynamics through the drift and volatility of the state variable η_t^I.

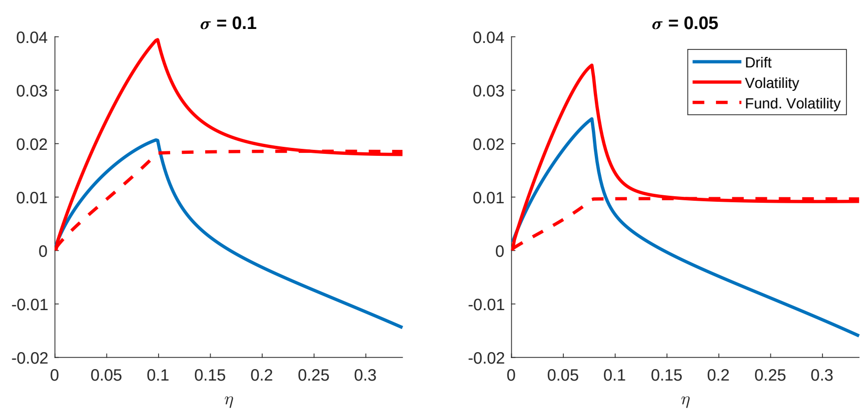

Figure 11.6: Equilibrium dynamics.

The drift of η_t^I captures the relative risk premia that intermediaries and households earn on their portfolios relative to money. As intermediaries become undercapitalized, the price of and return from holding claims on b-firms rises, leading intermediaries to take on more risk. The opposite happens when intermediaries are overcapitalized - risk premia decline and the households' rate of earnings rises. The stochastic steady state of η_t^I is the point where the drift of η_t^I equals zero - at that point the earnings rates of intermediaries and households balance each other out.

The left panel also shows the volatility terms: The dashed red curve shows the fundamental portion of the volatility of η_t^I. The solid curve depicts the total volatility that includes the effects of amplification due to the liquidity and the disinflationary spiral. Amplification becomes prominent when intermediaries are undercapitalized. While the left panel illustrates dynamics for our baseline parameters, the right panel reduces fundamental risk parameter to $\sigma = 0.05$. The right panel illustrates the volatility paradox: endogenous risk persists due to amplification even as fundamental risk declines. We see that the maximal volatility of η_t^I below the steady state stays roughly constant as fundamental risk declines, i.e. amplification in this model can be very large.

11.3.2 The Transmission of Shocks

To understand how shocks propagate in this economy, we decompose the transmission of a negative shock to capital stock in the following steps.

Step 1 Shock Onset

Suppose a nagetive shock $\mathrm{d}Z_t < 0$ destroys some capital $\tilde{\sigma}_t K_t \mathrm{d}Z_t$ in the economy. Recall the evolution of net worth for each agent is

$$\frac{\mathrm{d}n_t^{i,\tilde{i}}}{n_t^{i,\tilde{i}}} = -\rho \mathrm{d}t + (1 - \theta_t^{i,\tilde{i}})\mathrm{d}r_t^K(\iota_t^{i,\tilde{i}}) + \theta_t^{i,\tilde{i}}\mathrm{d}r_t^{bm}.$$

Since the intermediaries are levered $((1 - \theta_t^I) > 1)$, the shock reduces their net worth by a larger percentage than it does to the value of assets owned by intermediaries. As a result, the leverage of intermediaries rises after a negative shock.

In the meantime, since the households are nominally insured $((1 - \theta_t^h) < 1)$, the negative shock hits them less than the intermediaries in the sense that the households' net worth shrinks by a smaller fraction.

Step 2 Deleveraging

As the intermediaries' net worth declines, they have to reduce the size of their balance sheets by fire-selling their risky claims. If prices of capital and money are fixed, pure deleveraging has no real impact.

Step 3 The Liquidity Spiral

As the net worth of both groups falls after the shock, the total demand for capital contracts. Hence, deleveraging results in lower price of capital and lower investment rate. Drops in the capital price also hurt intermediaries' balance sheets (as $\mathrm{d}r_t^K$ depends on $\mu_t^{q^K}$), leading to further deleveraging — which we call the "liquidity spiral".

Step 4 The Disinflation Spiral

The liability side of the intermediaries' balance sheets is also affected by shocks. On the one hand, shrinked intermediary balance sheets mean that less inside money is being created and total money supply decreases, and therefore, the value of outside money appreciates ($q_t^{\mathcal{MB}} \uparrow$, or disinflation).

On the other hand, intermediaries' ability to diversify idiosyncratic risk is impaired after deleveraging, so the households have to bear more risk. The households in turn demand more safe assets, and increased money supply leads to further disinflation.

11.4 Policy

11.4.1 Fiscal Policy

The government can impose a flow of transfers/taxes on individual agents $\mathrm{d}\tau_t^{i,\tilde{i}}$, subject to a budget constraint

$$\sum_i \int_{\tilde{i}} \mathrm{d}\tau_t^{i,\tilde{i}} = \mathrm{d}T_t,$$

where T_t is the government's seigniorage income. In the models of Chapter 8 and 9, we have assumed that the transfers are made to agents proportional to their capital holdings. That is, $\mathrm{d}\tau_t^{i,\tilde{i}} = (k_t^{i,\tilde{i}}/K_t)\mathrm{d}T_t$. The government could also make the transfers proportional to bond holdings ($\mathrm{d}\tau_t^{i,\tilde{i}} = (n_t^{i,\tilde{i}}(1-\theta_t^{i,\tilde{i}})/\mathcal{MB}_t)\mathrm{d}T_t$) or to net worth ($\mathrm{d}\tau_t^{i,\tilde{i}} = (n_t^{i,\tilde{i}}/N_t)\mathrm{d}T_t$). It can be shown that if transfers are made to bond holders, then policy based on $\mu_t^{\mathcal{MB}}, \sigma_t^{\mathcal{MB}}$ has no real impact. If transfers are based on net worth, the effects of fiscal policy are between the two other cases.

Intra-temporal Transfer Policy. If the government can instead choose $\tau_t^{i,\tilde{i}}$ in an unconstrained way (while respecting its budget constraint), it can essentially complete the markets for idiosyncratic risk and achieve first-best outcomes.

If the government is constrained to make only sector-specific transfers ($\tau_t^{i,\tilde{i}} = \tau_t^i, \forall \tilde{i}$), it can effectively control η_t^i by shifting resources between sectors. This constraint on the government can be micro-founded by agents' hidden savings (e.g., Di Tella and Sannikov, 2016)

Inter-temporal Transfer Policy. Now we focus on bond supply policies ($\mu_t^{\mathcal{MB}}, \sigma_t^{\mathcal{MB}}$). We focus on the case where seigniorage is rebated to capital holders.

- A policy that uses $\mu_t^{\mathcal{MB}}$ and transfers ($\sigma_t^{\mathcal{MB}} = 0$) affect only the drift of ϑ_t (see eq. 11.18)
- A policy that uses $\sigma_t^{\mathcal{MB}}$ and transfers ($\mu_t^{\mathcal{MB}} = 0$) affects the risk of money and nominal bonds and therefore affects agents' portfolio choice.

Through Chapter 8 and 9, bonds are a bubble and backed by agents' demand for nominal insurance. Hence, all transfers/seigniorage come from bubble mining (i.e., through bond supply policies). However, if the government also imposes taxes, a part of the bond price ($q_t^{\mathcal{MB}}$) and its fluctuations will be induced by future taxes (see FTPL-Equations). Thus, the government can generate transfers through changes of future taxes, which we refer to as "inter-temporal" fiscal policy.

11.4.2 Monetary Policy.

Interest Rate Policy. To study monetary policy, we need to first introduce interests on bond/money. Previously, bonds pay no interest rates and the fluctuations of its value solely from inflation ($1/\mathcal{P}_t$). If we instead assume that bonds pay a nominal rate of i_t, equation (8.5) becomes

$$\begin{aligned} dr_t^{\mathcal{MB}} &= i_t dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} = i_t dt + \frac{d(q_t^{\mathcal{MB}} K_t/\mathcal{MB}_t)}{q_t^{\mathcal{MB}} K_t/\mathcal{MB}_t} \\ &= \left\{ i_t + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{MB}}} - \left[\mu_t^{\mathcal{MB}} + (\sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}})\sigma_t^{\mathcal{MB}} \right] \right\} dt + (\sigma_t^{q^{\mathcal{MB}}} - \sigma_t^{\mathcal{MB}}) dZ_t^{\tilde{\sigma}}. \end{aligned}$$

To study monetary policy *without* fiscal implications, we let $\sigma_t^{\mathcal{MB}} = 0$, so

$$dr_t^{\mathcal{MB}} = \left\{ i_t - \mu_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{MB}}} \right\} dt + \sigma_t^{q^{\mathcal{MB}}} dZ_t^{\tilde{\sigma}}.$$

For now, we shut off transfers to capital holders by setting $\tau_t^{i,\tilde{i}} = 0$. If interest payments on bonds are entirely funded by drift in $\mathcal{MB}_t$, the government budget constraint implies $i_t = \mu_t^{\mathcal{MB}}$, so changes in interest rate only affect inflation and have on real impact. Note that this is a special case of transferring seigniorage to money holders, so variations in $\mu_t^{\mathcal{MB}}$ (i_t) are neutral.

Long-term Government Bonds. Now we introduce a consol bond, which has face value F_t^L and nominal price P_t^L. The consol bond never matures but pays an interest rate i_t^L. In this subsection, we denote outside money by $\mathcal{M}_t$ (whose nominal price is 1) and $\mathcal{MB}_t = \mathcal{M}_t + P_t^L F_t^L$. With two bonds, the government budget constraint is

$$d\mathcal{M}_t + P_t^L dF_t^L = i_t \mathcal{M}_t dt + i_t^L F_t^L dt.$$

Define the fraction of bond values that are not in short-term reserves as

$$\vartheta_t^L = \frac{P_t^L F_t^L}{\mathcal{MB}_t}.$$

Postulate that the price of a single long-term consol bond follows

$$\frac{dP_t^L}{P_t^L} = \mu_t^{P^L} dt + \sigma_t^{P^L} dZ_t.$$

Assume that only intermediaries find it worthwhile to hold consol bonds. The martingale pricing condition is that

$$\mathbb{E}_t\left[\mathrm{d}r_t^L - \mathrm{d}r_t^{\mathcal{M}}\right]/\mathrm{d}t = \sigma_t^{P^L}\sigma_t^{\eta}.$$

Hence,

$$\mathrm{d}r_t^L = \mathrm{d}r_t^{\mathcal{M}} + \sigma_t^{P^L}\sigma_t^{\eta}\mathrm{d}t + \sigma_t^{P^L}\mathrm{d}Z_t.$$

In the total net worth numeraire, the return on total bond portfolio ($\mathcal{MB}_t/N_t = \vartheta_t$) is

$$\mathrm{d}r_t^B = \mu_t^{\vartheta}\mathrm{d}t + \sigma_t^{\vartheta}\mathrm{d}Z_t.$$

On the other hand, by definition

$$\begin{aligned}\mathrm{d}r_t^B &= (1-\vartheta_t^L)\mathrm{d}r_t^{\mathcal{M}} + \vartheta_t^L\mathrm{d}r_t^L \\ &= \mathrm{d}r_t^{\mathcal{M}} + \vartheta_t^L(\sigma_t^{P^L}\sigma_t^{\eta}\mathrm{d}t + \sigma_t^{P^L}\mathrm{d}Z_t).\end{aligned}$$

Therefore, the return on reserves

$$\mathrm{d}\, r_t^{\mathcal{M}} = (\mu_t^{\vartheta} - \vartheta_t^L\sigma_t^{P^L}\sigma_t^{\eta})\mathrm{d}t + (\sigma_t^{\vartheta} - \vartheta_t^L\sigma_t^{P^L})\mathrm{d}Z_t$$

Note that reserves is the benchmark asset in this economy. Similar to equation (11.20), the volatility of η_t^I is

$$\begin{aligned}\sigma_t^{\eta^I} &= \sigma_t^{r^{\mathcal{M}}} + (1-\theta_t^{\mathcal{M},I} - \theta_t^{L,I})\sigma_t^{xK^b} + \theta_t^{L,I}(\sigma_t^{r^L} - \sigma_t^{r^{\mathcal{M}}}) \\ &= \sigma_t^{r^{\mathcal{M}}} + \frac{\chi_t^I}{\eta_t^I}(1-\vartheta_t)\,\sigma_t^{xK^b} + \frac{\vartheta_t^L\vartheta_t}{\eta_t^I}(\sigma_t^{r^L} - \sigma_t^{r^{\mathcal{M}}}) \\ &= \sigma_t^{\vartheta} + \frac{\chi_t^I(1-\vartheta_t)}{\eta_t^I}\left((1-\bar{\psi})\sigma - \frac{\sigma_t^{\vartheta}}{1-\vartheta_t}\right) + \frac{\chi_t^I(1-\vartheta_t) + \vartheta_t - \eta_t^I}{\eta_t^I}\vartheta_t^L\sigma_t^{P^L}.\end{aligned}$$

Again, by Itô's lemma, $\sigma_t^{\vartheta} = \frac{\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}\sigma_t^{\eta^I}, \sigma_t^{P^L} = \frac{(P^L)'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}\sigma_t^{\eta^I}$, so

$$\eta_t^I \sigma_t^{\eta^I} = \frac{(1-\vartheta_t)\,\chi_t^I(1-\bar{\psi})\sigma}{1 - \frac{\chi_t^I - \eta_t^I}{\eta_t^I}\left(\frac{-\vartheta'(\eta_t^I)\eta_t^I}{\vartheta(\eta_t^I)}\right) + \vartheta_t^L\left(\frac{(P^L)'(\eta_t)\eta_t^I}{P^L(\eta_t^I)}\right)\frac{\eta_t^I - \chi_t^I\,(1-\vartheta_t) - \vartheta_t}{\eta_t^I}} \tag{11.22}$$

The new term in red can either mitigate or amplify the volatility. Suppose that the interest rate policies (i_t and i_t^L) are such that the long-term bond price appreciates as η goes down, e.g. i_t^L is constant and i_t is increasing in η. In that case $(P^L)' < 0$ and the term in red is positive for low η, as $\eta_t^I - \chi_t^I\,(1-\vartheta_t) - \vartheta_t < 0$ if η is small. In this case long-term bonds act as a hedge for intermediaries and mitigate volatility because of interest rate policy.

11.4.3 Macroprudential Policy

Macroprudential policy usually takes the form of restrictions on the intermediaries' leverage. In this model, we simply interpret it as the government directly controlling the portfolio decisions of intermediaries (and households).

11.4.4 Three Policy Benchmarks

Inflation Targeting.

Removing Endogenous Risk. This benchmark of monetary policy aims at eliminating endogenous risk. That is, $\mu_t^{\mathcal{MB}} = 0, \sigma_t^{\mathcal{MB}} = \sigma_t^{\vartheta}$. Now the FOC gives χ_t in closed form

$$\chi_t^I = \min\left(\frac{\eta_t^I}{\eta_t^I + (1-\eta_t^I)\varphi^2 + (1-\bar{\psi})^2\sigma^2/\tilde{\sigma}^2}, \bar{\chi}\right) \tag{11.23}$$

and

$$\sigma_t^{\eta^I} = \frac{\chi_t^I}{\eta_t^I}(1-\vartheta_t)(1-\bar{\psi})\sigma.$$

$$\mu_t^{\eta^I} = (1-\eta_t^I)(1-\vartheta_t)^2\left(\frac{1-2\eta_t^I}{(1-\eta_t^I)^2}\left(\frac{\chi_t^I}{\eta_t^I}\right)^2(1-\bar{\psi})^2\sigma^2 + \left(\frac{\chi_t^I}{\eta_t^I}\right)^2\varphi^2\tilde{\sigma}^2 - \left(\frac{1-\chi_t^I}{1-\eta_t^I}\right)^2\tilde{\sigma}^2\right)$$

The “money valuation equation" becomes

$$\vartheta\left\{\rho - \left[\sum_i \eta^i\left(\sigma_t^{\eta^i}\right)^2 + \sum_i \eta^i\left(\tilde{\sigma}_t^{\tilde{\eta}^i}\right)^2\right]\right\} = \partial_t\vartheta + (\partial_\eta\vartheta)\left(\eta_t^I\mu_t^{\eta^I}\right) + \frac{1}{2}(\partial_{\eta\eta}\vartheta)\left(\sigma_t^{\eta^I}\eta_t^I\right)^2$$

We compare the equilibrium with and without ($\mu_t^{\mathcal{MB}} = \sigma_t^{\mathcal{MB}} = 0$) monetary policy.

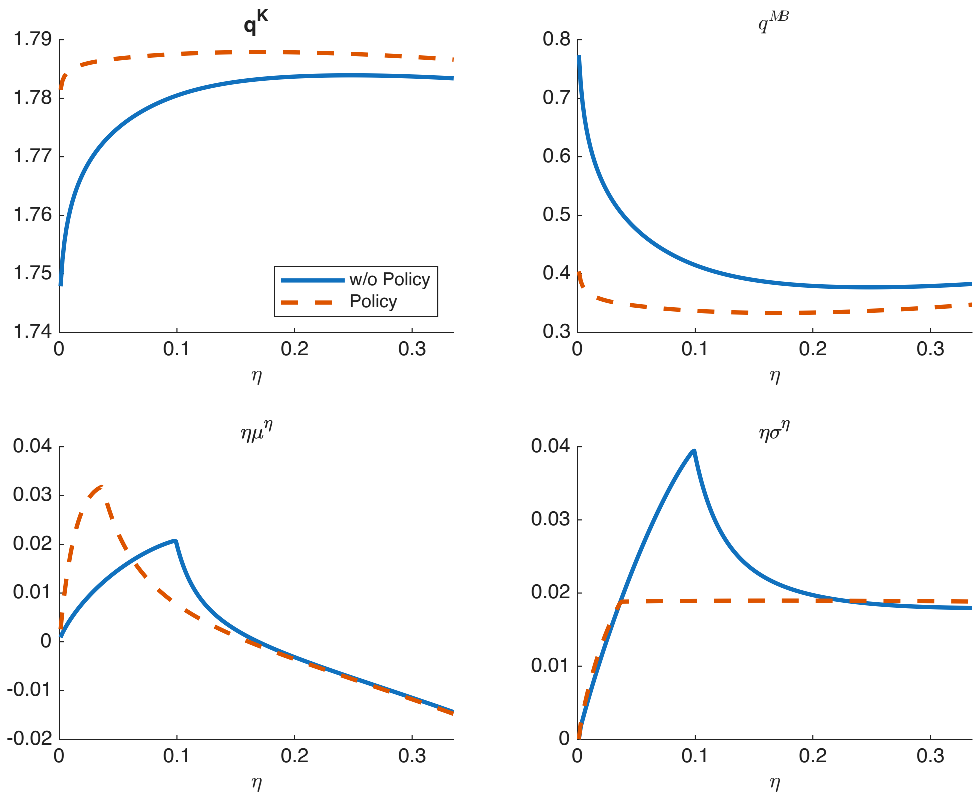

Figure 11.7: Prices, drift and volatility of η_t^I without policy (solid) and with (dashed).

The price of money falls since the intermediary sector creates more inside money: it does not need to absorb as much aggregate risk to do that. As a consequence, the price of capital rises - there is more demand for capital from both sectors.

Ultimately, monetary policy affects the degree of market incompleteness with respect to sharing of aggregate risk, but it cannot disentangle risk and risk-taking. The allocation of capital, the value of money relative to capital, and earnings rates of sectors a and b as well as intermediaries are endogenously determined by the risk profiles of available assets.

Perfect Aggregate Risk Sharing. Finally, we can also set the intermediaries' aggregate risk exposure to zero. Consider the model with long-term bonds and a policy that ensures that the red term in (11.22) converges to infinity such that $\sigma_t^{\eta^I} \to 0$.

We obtain an economy with perfect sharing of aggregate risk. In this case the aggregate risk exposures of all households and intermediaries is proportional to σ^K, and η_t, q_t^K and $q_t^{\mathcal{MB}}$ have no volatility.

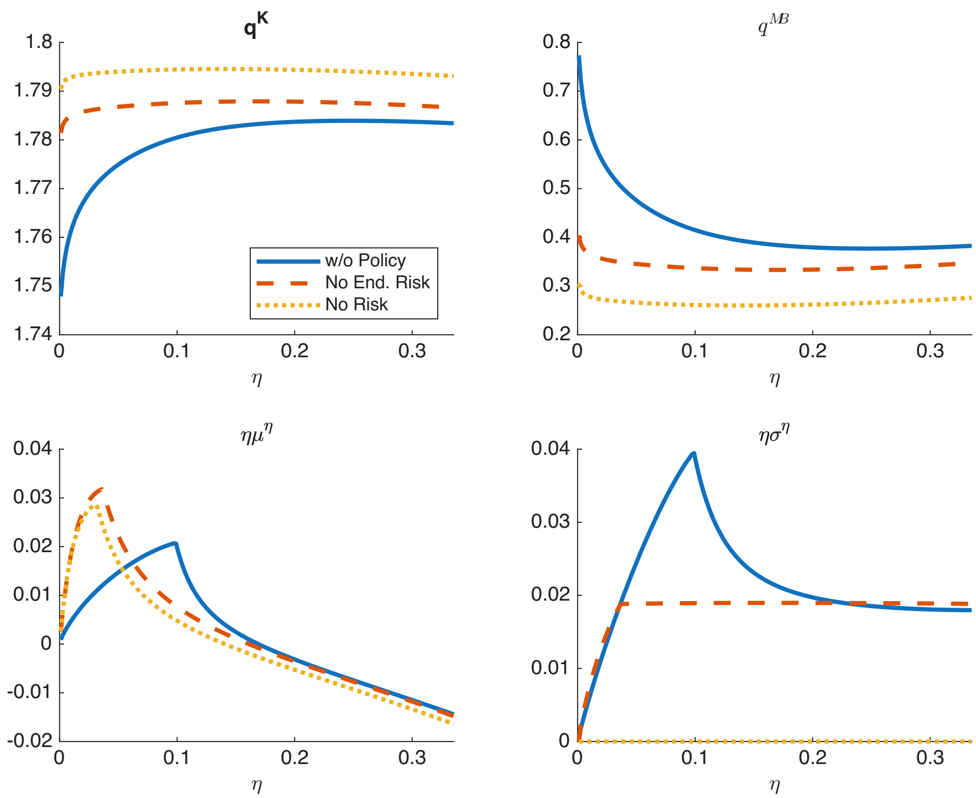

Figure 11.8: Comparison on the degree of aggregate risk sharing.

Figure 11.8 compares prices, allocations and dynamics in the baseline model, under policy that eliminates endogenous risk, and with perfect risk sharing. Equilibrium moves further in the direction that it took with the application of policy that removes endogenous risk. Specifically, the value of money falls, the steady state of η_t^I moves closer to zero, and the volatility of η_t^I becomes zero.

Qualitatively, what makes perfect aggregate risk sharing different is the fact that the boundary condition without intermediaries no longer plays a role at $\eta_t^I = 0$. The absence of crisis dynamics contributes to the significant drop in the relative value of money $\vartheta(\eta_t^I)$.[8] Also, leverage of intermediaries rises without bound approaching $\eta = 0$ - in normal circumstances this would be impossible due to the rise of endogenous risk, since endogenous risk is generated by the increase in leverage even in environment when exogenous shocks are small (but not zero).

It is important to highlight one more time the observation that monetary policy cannot provide insurance and control risk-taking at the same time. Leverage rises endogenously the more risk sharing becomes possible. Asset allocation, together with asset prices and risk premia, are also endogenous and dependent on the insurance that monetary policy provides. Hence, the value of money ϑ_t falls with perfect risk sharing, which may be detrimental to welfare as we observed in the model without intermediaries.

These links, which cannot be broken without macroprudential policy, have implications beyond the stylized elements of our model. In particular, loose monetary policy can lead to excessive leverage in some sectors, reduced risk premia and, consequently, bubbles in some asset classes. These can pose significant threat to financial stability. Also, with incomplete markets, improving risk sharing along some dimensions does not necessarily lead to higher welfare.

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[8] In fact, we might have to raise the idiosyncratic volatility to make money value in the equilibrium with perfect aggregate risk sharing.

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