Eco529: Modern Macro, Money, and International Finance
Lecture 15: Welfare & Optimal Policy

Markus Brunnermeier

Princeton University

Summer, 2026

Course Overview

1. Intro

Real Macrofinance Models with Heterogeneous Agents

Immersion Chapters

Money Models

10. Single Sector: Money Model with Store of Value and Medium of Change
11. Safe Asset with Time-varying Idiosyncratic Risk
12. Multi-Sector: Money Model with Redistributive Monetary Policy
13. Price Stickiness (New Keynesian)
14. **Welfare and Optimal Policies**

International Macrofinance Models

The Big Roadmap: Towards the I Theory of Money

- One sector model with idio risk - "The I Theory without I" **Lecture 10-12**
  (steady state focus)
  - Store of Value
    Insurance Role of Money within a Sector
  - Time-varying Idiosyncratic Risk and Safe Assest
  - Fiscal Theory of the Price Level
  - Medium of Exchange Role
- 2 Sector/Type Model with Money and Idiosyncratic Risk **Lecture 13**
  - Equivalence btw Experts Producers and Intermediaries
  - Real Debt vs. Nominal Debt/Money
    Implicit insurance role of money *across sectors*
  - Banking, I Theory, Redistributive Monetary Policy
- Price Rigidities
- Welfare analysis **Today**
  - Optimal Monetary Policy and Macroprudential Policy
- International Monetary Models

Optimal Policy

- Finding the Optimal Policy is generally Complicated, we Need
  1. Precise Definition of Policy Space
  2. Analytical Tools to Characterize the Optimum
- One Side: Inefficiencies / Tradeoffs
  - Insurance vs. Investment (One Sector/Type)
  - Allocation of Assets / Risk (across Sectors/Types)
- Other Side: "Large" Policy Space
  - Controlling Money Growth Rate
  - Macroprudential Tools / Wealth Redistribution
  - Risk Redistribution
- Approach
  - Start with Simple Model
  - Add Step-by-step More Model Elements

Roadmap

- Expected Utility/Value function with log-utility
- One Sector Model with Stochastic Idiosyncratic Risk
- Two Sector Model
    - With Exogenous Net Worth Share η
    - With Endogenous Net Worth Share η
    - I Theory (with Two Technologies)

Welfare with log utility

- The welfare for any agent $\tilde{i}$ of type i

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t^{\tilde{i}})\mathrm{d}t\right]$$

$$\tilde{\eta}_0^{\tilde{i}} = 1, \frac{\mathrm{d}\tilde{\eta}_t^{\tilde{i}}}{\tilde{\eta}_t^{\tilde{i}}} = \tilde{\sigma}_t^{\tilde{\eta}^{\tilde{i}}}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

- Recall from general model with log utility
  - $\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} = \rho$
  - $c_t^{\tilde{i}} = \eta_t^i(A(\boldsymbol{\kappa}) - \iota_t)K_t\tilde{\eta}_t^{\tilde{i}}$, using goods market clearing

Welfare with log utility

- The welfare of any agent $\tilde{i}$ is:

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t^{\tilde{i}})\mathrm{d}t\right] =& \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(\eta_t^{i}(A(\boldsymbol{\kappa}_t)-\iota_t)K_t\tilde{\eta}_t^{\tilde{i}})\mathrm{d}t\right] \\
=& \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(\eta_t^{\tilde{i}})\mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(A(\boldsymbol{\kappa}_t)-\iota_t)\mathrm{d}t\right] \\
&+ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log K_t\mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\tilde{\eta}_t^{\tilde{i}}\mathrm{d}t\right]
\end{aligned}
$$

Aside: Expected Integral

- Consider X_t that follows $\frac{dX_t}{X_t} = \mu_t^X dt + \sigma_t^X dZ_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t$
- Recall $\quad \log X_t - \log X_0 = \int_0^t \mathrm{d} \log X_s$
- Integral by parts:

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log X_t \mathrm{d}t\right] = \mathbb{E}\left[-\frac{1}{\rho}\int_0^\infty \log X_t \mathrm{d}e^{-\rho t}\right]$$
$$= \mathbb{E}\left[-\frac{1}{\rho}\Big(-\log X_0 - \int_0^\infty e^{-\rho t} \mathrm{d} \log X_t\Big)\right]$$

- Apply to Itô's lemma:

$$\mathrm{d} \log X_t = \left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right) dt + \sigma_t^X dZ_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t$$

- Plug into expected integral

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log X_t \mathrm{d}t\right] = \frac{1}{\rho}\log(X_0) + \frac{1}{\rho}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right)\mathrm{d}t\right]$$

Welfare with log utility

- The welfare of any agent $\tilde{i}$ is: ignoring constant $\frac{\log \rho}{\rho}$

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t^{\tilde{i}})\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(\eta_t^i(A(\boldsymbol{\kappa}_t)-\iota_t)K_t\tilde{\eta}_t^{\tilde{i}})\mathrm{d}t\right]$$

$$= \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(\eta_t^{\tilde{i}})\mathrm{d}t\right]}_{\frac{\log \eta_0^i}{\rho}+\mathbb{E}[\int_0^\infty e^{-\rho t}(\frac{\mu_t^{\eta^i}}{\rho}-\frac{(\sigma_t^{\eta^i})^2}{2\rho})]} + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(A(\boldsymbol{\kappa}_t)-\iota_t)\mathrm{d}t\right]$$

$$+ \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log K_t\mathrm{d}t\right]}_{\frac{\log K_0}{\rho}+\mathbb{E}[\int_0^\infty e^{-\rho t}(\frac{\Phi(\iota_t)-\delta}{\rho}-\frac{(\sigma_t^K)^2}{2\rho})]} + \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log \tilde{\eta}_t^{\tilde{i}}\mathrm{d}t\right]}_{-\mathbb{E}[\int_0^\infty e^{-\rho t}\frac{(\sigma_t^{\tilde{\eta}^{\tilde{i}}})^2}{2\rho}\mathrm{d}t]}$$

Welfare of Intermediaries I and HH h

- Intermediaries (Pareto weight $\underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log \eta_t + \log(A(\boldsymbol{\kappa}) - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{(\sigma^K)^2}{2\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{\kappa^2\varphi^2\tilde{\sigma}_t^2}{\eta^2}\right)\mathrm{d}t\right]$$

- Households (Pareto weight $1 - \underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(1-\eta_t) + \log(A(\boldsymbol{\kappa}) - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{(\sigma^K)^2}{2\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{(1-\kappa)^2\tilde{\sigma}_t^2}{(1-\eta)^2}\right)\mathrm{d}t\right]$$

Overview

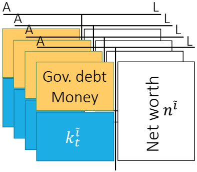
- Expected Utility/Value function with log-utility
- One Sector Model with Stochastic Idiosyncratic Risk
- Two Sector Model
  - With Exogenous Net Worth Share η
  - With Endogenous Net Worth Share η
  - I Theory (with Two Technologies)

One Sector Model with Stochastic Idiosyncratic Risk

- Each heterogenous citizen $\tilde{i} \in [0,1]$:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log(c_t^{\tilde{i}}) \right) \mathrm{d}t \right] \text{ where } K_t := \int k_t^{\tilde{i}} \mathrm{d}\tilde{i}, \text{and } \sigma^K = 0$$

$$s.t. \frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}\mathrm{d}t + \mathrm{d}r_t^{\mathcal{B}} + (1-\theta_t^{\tilde{i}})(\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - \mathrm{d}r_t^{\mathcal{B}}) \text{ \& No Ponzi}$$



- Each citizen operates physical capital $k_t^{\tilde{i}}$
  - Output (net investment): $y_t^{\tilde{i}}\mathrm{d}t = (a_t k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}})\mathrm{d}t$
  - $\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta \right) \mathrm{d}t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}} + \mathrm{d}\Delta_t^{k,\tilde{i}}, \quad \sigma^K = 0,$

    ($\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ idiosyncratic Brownian)
- Output tax $\tau a_t k_t^{\tilde{i}} \mathrm{d}t$
  - Financial Friction: Incomplete markets: no $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ claims
  - Aggregate risk $\tilde{\sigma}_t, a_t, g_t$ exogenous process by aggregate Brownian $\mathrm{d}Z_t$
    - E.g. Heston model: $\mathrm{d}\tilde{\sigma}_t^2 = -\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) - \sigma\tilde{\sigma}_t \mathrm{d}Z_t$ CIR-ensures $\tilde{\sigma}_t$ stays positive
    - $a_t = a(\tilde{\sigma}_t), g_t = g(\tilde{\sigma}_t)$
  - Money/bond issuing policy: $\mathrm{d}\mathcal{B}_t/\mathcal{B}_t = \mu_t^{\mathcal{B}}\mathrm{d}t + \sigma_t^{\mathcal{B}}\mathrm{d}Z_t$

One Sector Model with Money/Gov. Bond

- Dynamics of $\tilde{\eta}_t^{\tilde{i}}$:

$$\frac{\mathrm{d}\tilde{\eta}_t^{\tilde{i}}}{\tilde{\eta}_t^{\tilde{i}}} = \frac{\mathrm{d}(n_t^{\tilde{i}}/N_t^i)}{\tilde{\eta}_t^{\tilde{i}}} = (1-\vartheta_t)\tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

- Total wealth as numeraire has return ρ, $dr_t^N = \rho dt$
- Money as return:

$$dr_t^{\vartheta/\mathcal{B}} = \frac{\mathrm{d}(\vartheta_t/\mathcal{B}_t)}{\vartheta_t/\mathcal{B}_t} = \underbrace{\left(\mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})\right)}_{\mu_t^{\vartheta/\mathcal{B}}}\mathrm{d}t + \underbrace{(\sigma_t^{\vartheta} - \sigma_t^{\mathcal{B}})}_{\sigma_t^{\vartheta/\mathcal{B}}}\mathrm{d}Z_t$$

- Money/bond valuation (FTPL) equation:

$$\rho - \mu_t^{\vartheta/\mathcal{B}} = \left(\tilde{\sigma}_t^{\tilde{\eta}^{\tilde{i}}}\right)^2 = (1-\vartheta_t)^2\tilde{\sigma}_t^2$$

Without policy, equation:

$$\rho - \mu_t^{\vartheta} = (1-\vartheta_t)^2\tilde{\sigma}_t^2$$

has a unique solution $\vartheta(\tilde{\sigma}) \in (0,1)$ (if $\tilde{\sigma}_t$ sufficiently large)

Recall Equilibrium

- Price of physical capital

$$q_t^K = (1-\vartheta_t)\frac{1+\phi a}{(1-\vartheta_t)+\phi\rho}$$

- Price of nominal capital

$$q_t^{\mathcal{B}} = \vartheta_t\frac{1+\phi a}{(1-\vartheta_t)+\phi\rho}$$

- Optimal investment rate

$$\iota_t = \frac{(1-\vartheta_t)a-\rho}{(1-\vartheta_t)+\phi\rho}$$

- Fraction of nominal wealth ϑ_t:

$$1-\vartheta_t = \frac{\sqrt{\rho+\mu_t^{\mathcal{B}}-(\sigma_t^{\mathcal{B}})^2-\mu_t^{\vartheta}+\sigma_t^{\vartheta}\sigma_t^{\mathcal{B}}}}{\tilde{\sigma}_t}$$

Welfare

- Welfare for one sector model (no intermediaries): $\eta = 0, \kappa = 0$ is:

$$\frac{\log K_0}{\rho} - \frac{\delta}{\rho^2} + \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(\overbrace{A(\boldsymbol{\kappa}_t)}^{=a} - \iota_t)\mathrm{d}t\right]}_{\mathbb{E}[\int_0^\infty e^{-\rho t}[\log(\rho \frac{\phi a+1}{\rho\phi+1-\vartheta_t})]\mathrm{d}t]}$$

$$+ \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{\Phi(\iota_t)}{\rho}\mathrm{d}t\right]}_{\frac{1}{\rho\phi}\mathbb{E}[\int_0^\infty e^{-\rho t}\log(\frac{(a\phi+1)(1-\vartheta_t)}{\rho\phi+1-\vartheta_t})]} - \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{(1-\vartheta_t)^2\tilde{\sigma}_t^2}{2\rho}\mathrm{d}t\right]$$

Optimal Policy

- Welfare is:

$$\frac{\log K_0}{\rho} - \frac{\delta}{\rho^2} + \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big[\log\big(\rho\frac{\phi a + 1}{\rho\phi + 1 - \vartheta_t}\big) + \frac{1}{\rho\phi}\log\big(\frac{(a\phi+1)(1-\vartheta_t)}{\rho\phi + 1 - \vartheta_t}\big) - \frac{(1-\vartheta_t)^2\tilde{\sigma}_t^2}{2\rho}\big]\mathrm{d}t\Big]$$

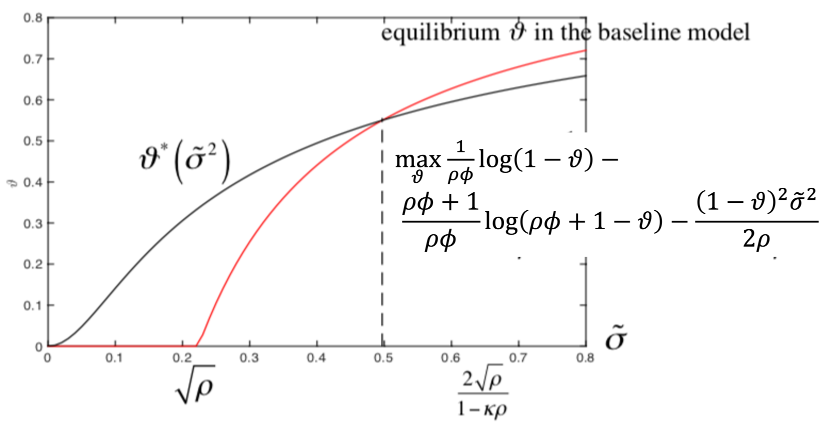
- **Lemma**: Problem collapses to a static problem for each t.

- Let $\vartheta^*(\tilde{\sigma}_t^2)$ be be the maximizer of welfare (optimal policy).

$$\boxed{\vartheta^*(\tilde{\sigma}_t^2) := \arg\max_{\vartheta_t} \frac{1}{\rho\phi}\log(1-\vartheta_t) - \frac{\rho\phi+1}{\rho\phi}\log(\rho\phi + 1 - \vartheta_t) - \frac{(1-\vartheta_t)^2\tilde{\sigma}_t^2}{2\rho}}$$

Optimal Policy

Red: equilibrium ϑ in the baseline model
Black: optimal policy ϑ^*



Pecuniary Externality Explanation

- Money/bond growth $\mu^{\mathcal{B}}$ affects
  - Shadow risk-free rate
  - (Steady state) inflation in two ways

$$\pi = \mu^{\mathcal{B}} + i - \underbrace{(\Phi(\iota(\mu^{\mathcal{B}})) - \delta)}_{g}$$

- Proposition
  - For sufficiently large $\tilde{\sigma}$ and $\phi < \infty$ welfare maximizing $\mu^{\mathcal{B}^*} > 0$.
    - Laissez-faire Market outcome is not even **constrained Pareto efficient**
    - Economic growth rate g is also higher
  - Growth maximizing $\mu^{g*} \geq \mu^{\mathcal{B}^*}$, s.t. $p^{g^*} = 0$, Tobin (1965)
- Corollary: No super-neutrality of money
  - i : Super-neutrality only w.r.t. part of money growth rate that is used to pay interest on money (Neo-Fisherian)
  - $\mu^{\mathcal{B}}$: Nominal money growth rate affects real economic growth by distorting portfolio choice if $\phi < \infty$
    (No price/wage rigidity, no monopolistic competition)

Optimal Policy

- If the planner can control ϑ_t directly, she would set $\vartheta_t = \vartheta^*(\tilde{\sigma}_t^2)$

$$\mathrm{d}\tilde{\sigma}_t = \mu^{\tilde{\sigma}}(\tilde{\sigma}_t)\tilde{\sigma}_t \mathrm{d}t + \sigma^{\tilde{\sigma}}(\tilde{\sigma}_t)\tilde{\sigma}_t \mathrm{d}Z_t$$
$$\mathrm{d}\vartheta_t = \mu^{\vartheta}(\tilde{\sigma}_t)\vartheta_t \mathrm{d}t + \sigma^{\vartheta}(\tilde{\sigma}_t)\vartheta_t \mathrm{d}Z_t$$

- The planner can choose instruments $\mu^{\mathcal{B}}(\tilde{\sigma}_t), \sigma^{\mathcal{B}}(\tilde{\sigma}_t)$ to achieve any function ϑ_t
  - How to find the instruments $\mu^{\mathcal{B}}(\tilde{\sigma}_t), \sigma^{\mathcal{B}}(\tilde{\sigma}_t)$ that achieve $\vartheta^*(\tilde{\sigma}_t)$?
    — solving money/bond valuation (FTPL) equation

$$\rho - \underbrace{(\mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta}))}_{=\mu_t^{\vartheta/\mathcal{B}}} = (1-\vartheta_t)^2\tilde{\sigma}_t^2$$

- Optimal policy is easier to find than equilibrium outcome.
  - differentiation vs. integration (or solve PDEs)
- If we choose optimal policy, risk-free rate

$$r_t^F = \Phi(\iota_t) - \delta + \mu_t^{\vartheta/\mathcal{B}} = \rho - (1-\vartheta_t^*)^2\tilde{\sigma}_t^2 + \Phi(\iota_t) - \delta$$

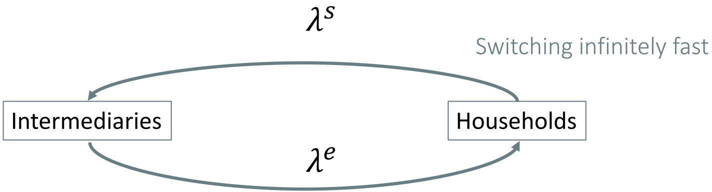
  declines as $\tilde{\sigma}_t^2$ increases
- Nice relationship between baseline and dynamic model.

Overview

- Expected Utility/Value function with log-utility
- One Sector Model with Stochastic Idiosyncratic Risk
- Two Sector Model
    - With Exogenous Net Worth Share η
    - With Endogenous Net Worth Share η
    - I Theory (with Two Technologies)

Two Switching Sectors with Exogenous η

- Model setup with infinitely fast switching types kills dynamic.



Agents	Intermediaries	Householdss
Networth Share	η	$1-\eta$
Idio risk of capital	$\varphi\tilde{\sigma}, \varphi \in (0,1)$	$\tilde{\sigma}$
Output per unit of capital	$a^I = a$	$a^h = a$

- Policy marker can choose the money/bond growth rate $\mu_t^{\mathcal{B}}, \sigma_t^{\mathcal{B}}$.

Remark

- Policy-marker cannot affect the wealth shares
- Welfare Pareto weights
  - $\underline{\lambda} = \eta$ for intermediaries and
  - $1 - \underline{\lambda} = 1 - \eta$ for households from the setup
- Optimal monetary
  (with or without macroprudential policy – controlling capital allocation)
  - Perfect commitment – Ramsey problem

Equilibrium Capital Allocation

- Fraction χ of risk (κ of capital) is held by the intermediaries ($\chi = \kappa$)
- Capital allocation must be such that

$$\underbrace{\varphi\tilde{\sigma}_t}_{\text{idio risk of } I} \underbrace{\frac{(1-\vartheta)\kappa\varphi\tilde{\sigma}_t}{\eta}}_{I\text{'s price of idio risk}} = \underbrace{\tilde{\sigma}_t}_{\text{idio risk of } h} \underbrace{\frac{(1-\vartheta)(1-\kappa)\tilde{\sigma}_t}{1-\eta}}_{h\text{'s price of idio risk}}$$

$$\Rightarrow \kappa = \frac{\eta}{\varphi^2(1-\eta)+\eta}$$

- Policy marker may try to affect κ ...

Welfare of Intermediaries I and HH h

- Recall $A(\kappa) = a \; \forall \kappa$, $\sigma^K = 0$
- Intermediaries (Pareto weight $\underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log \eta_t + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{\kappa^2 \varphi^2 \tilde{\sigma}_t^2}{\eta^2}\right)\mathrm{d}t\right]$$

- Households (Pareto weight $1 - \underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{(1-\kappa)^2 \tilde{\sigma}_t^2}{(1-\eta)^2}\right)\mathrm{d}t\right]$$

Welfare

- Law of large numbers: switching risk does not matter.
  Everyone's wealth growth averages out to $\Phi(\iota_t) - \delta$ and idiosyncratic risk exposure to (ignoring t-subscripts)

$$\eta(\tilde{\sigma}^l)^2 + (1-\eta)(\tilde{\sigma}^h)^2 = (1-\vartheta)^2 \underbrace{\tilde{\sigma}^2 \left(\underline{\lambda} \frac{\kappa^2 \varphi^2}{\eta^2} + (1-\underline{\lambda}) \frac{(1-\kappa)^2}{(1-\eta)^2} \right)}_{:=(\tilde{\sigma}^{avg})^2}$$

$$\tilde{\sigma}^I = \frac{(1-\vartheta)\kappa\varphi\tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = \frac{(1-\vartheta)(1-\kappa)\tilde{\sigma}}{1-\eta}$$

- Welfare

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(a - \iota(\vartheta_t)) \mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho} \mathrm{d}t\right]$$
$$- \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{(1-\vartheta_t)^2 (\tilde{\sigma}^{avg})^2}{2\rho} \mathrm{d}t\right]$$

- Given $\tilde{\sigma}^{average}$, optimal to set $\vartheta = \vartheta^*((\tilde{\sigma}^{avg})^2)$
- Set $\underline{\lambda} = \eta$, (Pareto weight is population share)

Money/Bond Valuation (FTPL) Equation

- Money/bond valuation (FTPL) equation

$$\rho - \underbrace{\left(\mu_t^{\vartheta} - \mu_t^{\mathcal{B}} + \sigma_t^{\mathcal{B}}(\sigma_t^{\mathcal{B}} - \sigma_t^{\vartheta})\right)}_{\mu_t^{\vartheta/\mathcal{B}}} = \underbrace{\eta(\tilde{\sigma}_t^{l})^2 + (1-\eta)(\tilde{\sigma}_t^{h})^2}_{(1-\vartheta_t)^2(\tilde{\sigma}_t^{avg})^2}$$

Macroprudential Tools

- Average idiosyncratic risk of capital

$$\tilde{\sigma}^2 \left(\frac{\kappa^2 \varphi^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta} \right)$$

is minimized when:

$$\frac{\kappa \varphi^2}{\eta} = \frac{1-\kappa}{1-\eta} \Rightarrow \kappa = \frac{\eta}{\varphi^2(1-\eta)+\eta}$$

- This is the same as the equilibrium κ allocation!
- **Lemma**: Optimal not to use macroprudential tools.
  assuming $\underline{\lambda} = \eta$

Remarks

- Same trade-off between insurance and investment
- Equilibrium allocation is efficient, minimizes the cost of risk exposure
- Policy space
  - (1) money/bond growth and
  - (1) + (2) (money growth + macroprudential tools) leads to the same outcome

Overview

- Expected Utility/Value function with log-utility
- One Sector Model with Stochastic Idiosyncratic Risk
- Two Sector Model
  - With Exogenous Net Worth Share η
  - With Endogenous Net Worth Share η
  - I Theory (with two Technologies)

Endogenous η-Law of Motion

- Wealth distribution can change endogenously with
  - Risk exposure of intermediaries and households
  - Risk premia
  - Consumption rates
- Consider the following model

Fixed Types (No Switching)

- Model Setup: Types fixed (no switching)
  You have already seen this model except here $\bar{\kappa} = 1$ (or $\simeq \bar{\chi} = 1$)

Agents	Intermediaries	Households
Welfare weights	$\underline{\lambda}$	$1 - \underline{\lambda}$
Net worth Share	η	$1 - \eta$
Aggregate risk	σ	σ
Idio risk of capital	$\varphi\tilde{\sigma}, \varphi \in (0, 1)$	$\tilde{\sigma}$
Output per unit of capital	$a^I = a$	$a^h = a$

- Two policy settings:
  (1) money growth rate $\mu_t^{\mathcal{B}}$ only
  (1) + (2) also choose allocation (macroprudential)
  and transfer wealth between group
  (why/how?)

Welfare of Intermediaries I and HH h

- Recall $A(\kappa) = a \ \forall \kappa$
- Intermediaries (Pareto weight $\underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log \eta_t + \log(a-\iota_t) + \frac{\Phi(\iota_t)-\delta}{\rho} - \frac{(\sigma^K)^2}{2\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{\kappa^2\varphi^2\tilde{\sigma}^2}{\eta^2}\right)\mathrm{d}t\right]$$

- Households (Pareto weight $1-\underline{\lambda}$)

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(1-\eta_t) + \log(a-\iota_t) + \frac{\Phi(\iota_t)-\delta}{\rho} - \frac{(\sigma^K)^2}{2\rho} - \frac{(1-\vartheta_t)^2}{2\rho}\frac{(1-\kappa)^2\tilde{\sigma}^2}{(1-\eta)^2}\right)\mathrm{d}t\right]$$

Optimal policy: (1) MoPo + (2) MacroPru

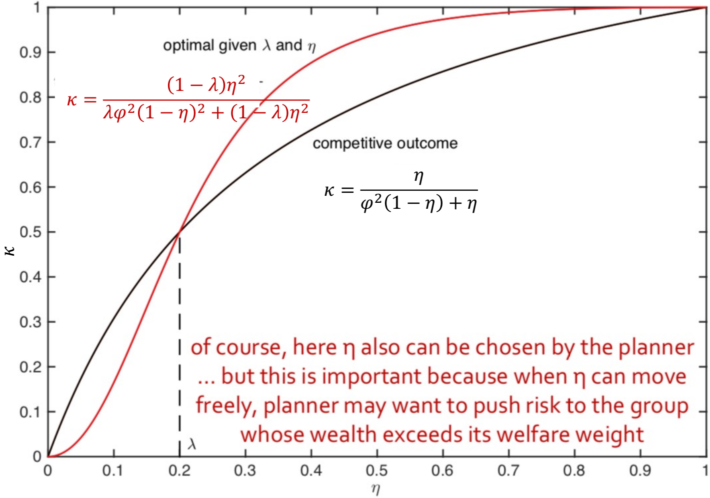
- Planner chooses ϑ, κ and η to maximize discount integral of:

$$\underline{\lambda} \log \eta_t + (1-\underline{\lambda}) \log(1-\eta_t) + \log(a - \iota(\vartheta_t)) + \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho} - \frac{(\sigma^K)2}{2\rho}$$
$$- \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\underline{\lambda} \frac{\kappa^2 \varphi^2}{\eta^2} + (1-\underline{\lambda}) \frac{(1-\kappa)^2}{(1-\eta)^2} \right)}_{= \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}}$$

Given the optimal choice of $\kappa = \frac{(1-\underline{\lambda})\eta^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}$
not the competitive allocation (unless $\eta = \underline{\lambda}$)

Optimal Policy: (1) MoPo + (2) MacroPru

- Step 1: Solve optimal κ (or χ) for a given η and $\underline{\lambda}$
  Competitive κ vs. minimizing cost of risk



Optimal Policy: (1) MoPo + (2) MacroPru

- Planner chooses ϑ, κ and η to maximize discount integral of:

$$\underline{\lambda} \log \eta_t + (1-\underline{\lambda}) \log(1-\eta_t) + \log(a - \iota(\vartheta_t)) + \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho}$$

$$- \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\underline{\lambda} \frac{\kappa^2 \varphi^2}{\eta^2} + (1-\underline{\lambda}) \frac{(1-\kappa)^2}{(1-\eta)^2} \right)}_{= \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}}$$

  Given the optimal choice of $\kappa = \frac{(1-\underline{\lambda})\eta^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}$
  not the competitive allocation (unless $\eta = \underline{\lambda}$)
- Step 2: solve $\vartheta_t = \vartheta^*(\cdot)$ (having used optimal κ_t) for each given η
- Given κ and η, optimal to set ϑ to:

$$\vartheta = \vartheta^*(\tilde{\sigma}^2 \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2})$$

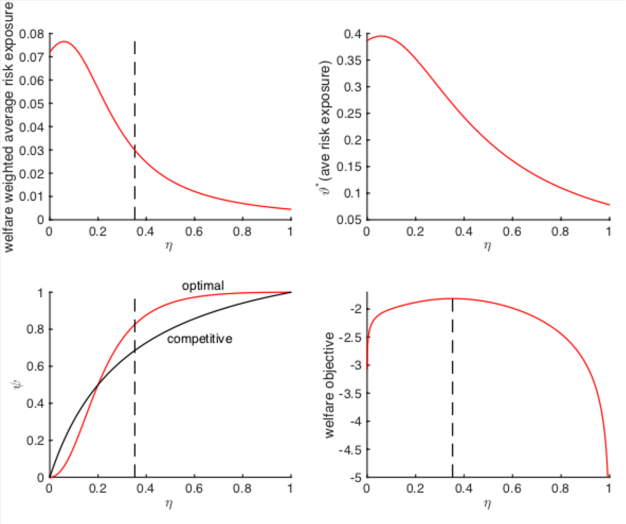
Optimal Policy: (1) MoPo + (2) MacroPru

- Step 3: Optimal η (given ϑ)
- Let's look at terms containing η
- Given κ and η,

$$\max_{\eta} \underline{\lambda} \log \eta_t + (1-\underline{\lambda}) \log(1-\eta_t) - \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}$$

- Hence it is optimal to set $\eta > \underline{\lambda}$
  (unfortunately no closed-form expression for the optimal η)
- Push more risk onto intermediaries than they'd take under competitive outcome
- Relative to previous infinite switching model
  - It is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  - Overall risk is reduced and the value of money/bond is lower (more intermediation)

Optimizing over η



$$\rho = 0.05, \phi = 2, \tilde{\sigma} = 0.3, \varphi = 0.5, \underline{\lambda} = 0.2$$

Optimal Policy: (1) MoPo only

- Planner cannot alter competitive alloc. $\kappa_t = \frac{\eta_t}{\varphi^2(1-\eta_t)+\eta_t}$
- Welfare is the discount integral of

$$\underline{\lambda} \log \eta_t + (1-\underline{\lambda}) \log(1-\eta_t) + \log(a - \iota(\vartheta_t)) + \frac{\Phi(\iota(\vartheta_t)) - \delta}{\rho}$$
$$-\frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\underline{\lambda}\frac{\kappa_t^2 \varphi^2}{\eta_t^2} + (1-\underline{\lambda})\frac{(1-\kappa_t)^2}{(1-\eta_t)^2}\right)}_{=\frac{\underline{\lambda}\varphi^2 + (1-\underline{\lambda})\varphi^4}{(\varphi^2(1-\eta_t)+\eta_t)^2}}$$

s.t. $\frac{\mathrm{d}\eta_t}{\eta_t} = (1-\eta_t)\left((\tilde{\sigma}_t^l)^2 - (\tilde{\sigma}_t^h)^2\right) dt = (1-\eta_t)\frac{(1-\vartheta_t)^2 \tilde{\sigma}^2 \varphi^2 (1-\varphi^2)}{(\varphi^2(1-\eta_t)+\eta_t)^2}\mathrm{d}t$

- Planner can not choose κ_t or η_t but has some control over μ_t^η
- Now, fully dynamic problem!

Optimal Policy: (1) MoPo only

- Payoff flow: $f(\eta_t, \vartheta_t) = \underline{\lambda} \log \eta_t + (1-\underline{\lambda}) \log(1-\eta_t) + \frac{\log(1-\vartheta_t)}{\rho\phi}$

$$-\frac{\rho\phi+1}{\rho\phi}\log(\rho\phi+1-\vartheta_t) - \frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}\left(\underline{\lambda}\frac{\kappa_t^2\varphi^2}{\eta_t^2} + (1-\underline{\lambda})\frac{(1-\kappa_t)^2}{(1-\eta_t)^2}\right)$$

  with $\kappa = \frac{\eta}{\varphi^2(1-\eta)+\eta}$
- HJB equation:

$$\rho V(\eta) = \max_{\vartheta} f(\eta, \vartheta) + V'(\eta)\mu^\eta \eta + \frac{1}{2}V''(\eta)(\sigma^\eta \eta)^2$$

- Law of motion of η:

$$\frac{\mathrm{d}\eta_t}{\eta_t} = (1-\eta_t)\frac{(1-\vartheta_t)^2\tilde{\sigma}^2\varphi^2(1-\varphi^2)}{(\varphi^2(1-\eta_t)+\eta_t)^2}\mathrm{d}t + 0\mathrm{d}Z_t$$

Optimal Policy: (1) MoPo only

- Payoff flow: $f(\eta_t, \vartheta_t) = \underline{\lambda} \log \eta_t + (1 - \underline{\lambda}) \log(1 - \eta_t) + \frac{\log(1-\vartheta_t)}{\rho\phi}$

$$-\frac{\rho\phi + 1}{\rho\phi} \log(\rho\phi + 1 - \vartheta_t) - \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \left(\underline{\lambda} \frac{\kappa_t^2 \varphi^2}{\eta_t^2} + (1 - \underline{\lambda}) \frac{(1-\kappa_t)^2}{(1-\eta_t)^2} \right)$$

  with $\kappa = \frac{\eta}{\varphi^2(1-\eta)+\eta}$

- HJB equation:

$$\rho V(\eta) = \max_{\vartheta} f(\eta, \vartheta) + V'(\eta)\mu^\eta \eta + \cancel{\frac{1}{2} V''(\eta)(\sigma^\eta \eta)^2}$$

- Law of motion of η:

$$\frac{\mathrm{d}\eta_t}{\eta_t} = (1 - \eta_t) \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2 \varphi^2 (1 - \varphi^2)}{(\varphi^2(1-\eta_t) + \eta_t)^2} \mathrm{d}t + 0 \mathrm{d}Z_t$$

Optimal Policy: (1) MoPo only

- Optimal ϑ^*
- HJB equation:

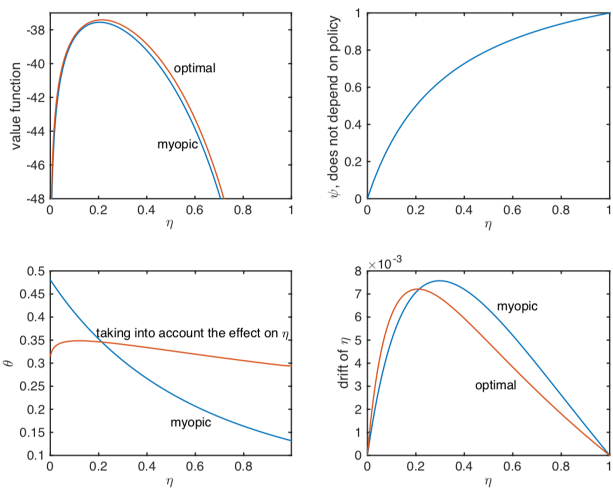
$$\max_{\vartheta} \frac{\log(1-\vartheta)}{\rho\phi} - \frac{\rho\phi+1}{\rho\phi}\log(\rho\phi+1-\vartheta) - \frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}\left(\underline{\lambda}\frac{\kappa^2\varphi^2}{\eta^2} + (1-\underline{\lambda})\frac{(1-\kappa)^2}{(1-\eta)^2}\right)$$
$$+ V'(\eta)(1-\vartheta_t)^2\frac{\eta(1-\eta)\tilde{\sigma}^2\varphi^2(1-\varphi^2)}{(\varphi^2(1-\eta)+\eta)^2}$$

- ϑ affets the drift of η, it is optimal to choose

$$\vartheta^*\left(\tilde{\sigma}^2\left(\underline{\lambda}\frac{\kappa^2\varphi^2}{\eta^2} + (1-\underline{\lambda})\frac{(1-\kappa)^2}{(1-\eta)^2}\right) - 2\rho V'(\eta)\frac{\eta(1-\eta)\tilde{\sigma}^2\varphi^2(1-\varphi^2)}{(\varphi^2(1-\eta)+\eta)^2}\right)$$

- Speed up η when $V' > 0$, slow down when $V' < 0$.

Example: using ϑ to push η



$$\rho = 0.05, \phi = 2, \tilde{\sigma} = 0.3, \varphi = 0.5, \underline{\lambda} = 0.2$$

Optimal policy: (1) MoPo only

- Using MoPo ϑ to push η (to recapitalize banks via risk premia)
- Using screwdriver as hammer

Overview

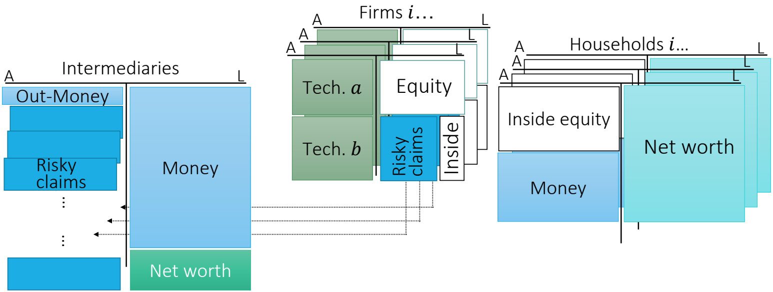
- Expected Utility/Value function with log-utility
- One Sector Model with Stochastic Idiosyncratic Risk
- Two Sector Model
  - With Exogenous Net Worth Share η
  - With Endogenous Net Worth Share η
  - I Theory (with Two Technologies)

I Theory of Money

- Aim: intermediary sector is not perfectly hedged
- Idiosyncratic risk that HH have to bear is time-varying
- Needed: Intermediaries' aggregate risk $\neq$ aggregate risk of economy
  - One way to model: 2 technologies a and b

Technology	**a**	**b**
Capital share (Leontief)	$1-\bar{\kappa}$	$\bar{\kappa}$
Risk	$\frac{dk_t}{k_t} = (\cdot)dt + \sigma^a dZ_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t$	$\frac{dk_t}{k_t} = (\cdot)dt + \sigma^b dZ_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t$
Intermediaries	No	Yes, reduce $\tilde{\sigma}$ to $\varphi\tilde{\sigma}$
Excess risk	$-\bar{\kappa}\sigma - \frac{\sigma^{\vartheta}-\sigma^{\mathcal{B}}}{1-\vartheta}$	$(1-\bar{\kappa})\sigma - \frac{\sigma^{\vartheta}-\sigma^{\mathcal{B}}}{1-\vartheta}$

I Theory: Balance Sheets



- Frictions:
  - Household cannot diversify idio risk
  - Limited risky claims issuance
  - Only nominal deposits

Model with Intermediaries – New Policy

- Model Setup:

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma^K dZ_t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t$$

  - Intermediaries can hold equality share up to $\bar{\kappa}$
  - can diversify some idiosyncratic risk, reduce it to $\varphi\tilde{\sigma}$
  - Intermediaries' wealth share $\eta_t = N_t/(q_t^{\mathcal{B}} + q_t K_t)$
  - Welfare weight $\underline{\lambda}$ on the intermediaries, $1 - \underline{\lambda}$ for HH
- Two policy settings:
  - (1) money growth rate $\mu_t^{\mathcal{B}}$ only
  - (1) + (2) also choose allocation (macroprudential) and transfer wealth between group (why/how?)

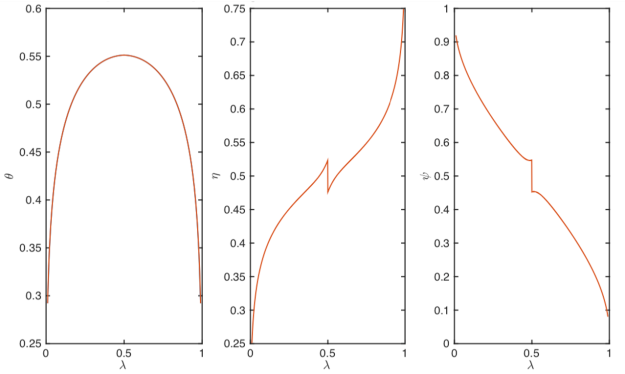
Optimal Policy: (1) MoPo + (2) MacroPru

- Same steps as above
- Step 1: Optimal $\kappa = \min\left\{\frac{(1-\underline{\lambda})\eta^2}{\underline{\lambda}\varphi^2(1-\eta)^2+(1-\underline{\lambda})\eta^2}, \bar{\kappa}\right\}$ given η
- Step 2: Optimal $\vartheta = \vartheta^*(\underbrace{\tilde{\sigma}^2 \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2+(1-\underline{\lambda})\eta^2}}_{\text{welfare weighted average risk exposure}})$
- Step 3: Optimal η (given ϑ) as a function of Pareto weight $\underline{\lambda}$

Optimal Policy: (1) MoPo + (2) MacroPru

- Step 3: Optimal η (given ϑ) – let's look at terms containing η

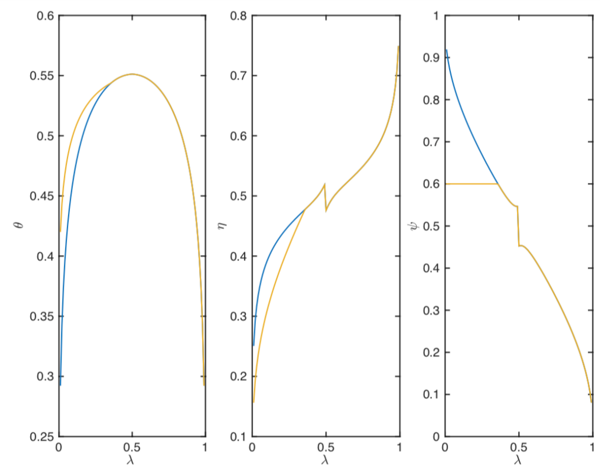
$$\max_{\eta} \quad \underbrace{\underline{\lambda} \log \eta + (1-\underline{\lambda}) \log(1-\eta)}_{\text{concave, max at } \eta=\underline{\lambda}\text{, goes to } \infty \text{ at } 0\&1} - \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}$$



For $\varphi = 1$, the optimal policy as a function of $\underline{\lambda}$ is

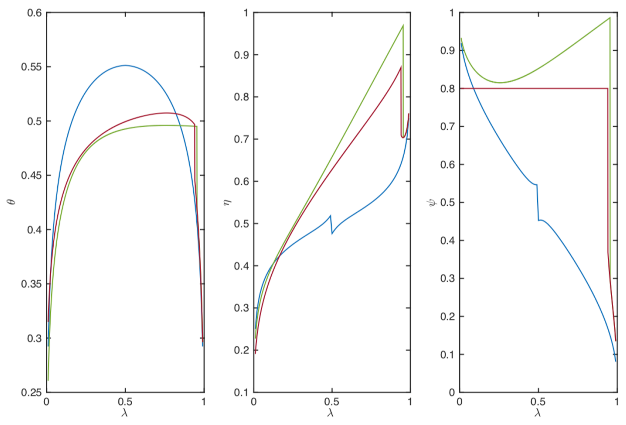
Optimal Policy: (1) MoPo + (2) MacroPru

- For $\varphi = 1$, and $\bar{\kappa} = 0.6$ (intermediaries' risk taking is constrained)



Optimal policy: (1) MoPo + (2) MacroPru

- For $\varphi = 0.8, \bar{\kappa} = 1$, and $\varphi = 0.8, \bar{\kappa} = 0.8$
  (Intermediaries given a lot more risk when they can diversify it)



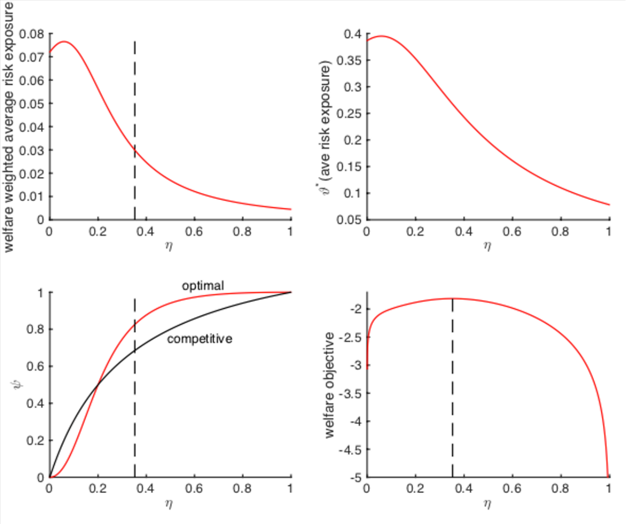
Optimal Policy: (1) MoPo + (2) MacroPru

- Step 3: Optimal η (given ϑ) – let's look at terms containing η
- Same as above
- Given κ and η,

$$\max_{\eta} \quad \underbrace{\underline{\lambda} \log \eta + (1-\underline{\lambda}) \log(1-\eta)}_{\text{concave, max at } \eta = \underline{\lambda}, \text{ goes to } \infty \text{ at } 0\&1} - \frac{(1-\vartheta_t)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\frac{\underline{\lambda}(1-\underline{\lambda})\varphi^2}{\underline{\lambda}\varphi^2(1-\eta)^2 + (1-\underline{\lambda})\eta^2}}_{\text{concave also, max at } \frac{\underline{\lambda}\varphi^2}{\underline{\lambda}\varphi^2 + 1 - \underline{\lambda}} < \underline{\lambda}}$$

- Assuming FOC holds uniquely, it is optimal to set $\eta > \underline{\lambda}$
- push more risk to intermediaries and they'd take under competitive outcome
- relative to previous infinite switching model
  - it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  - overall risk is reduced and the value of money is lower (more intermediation)

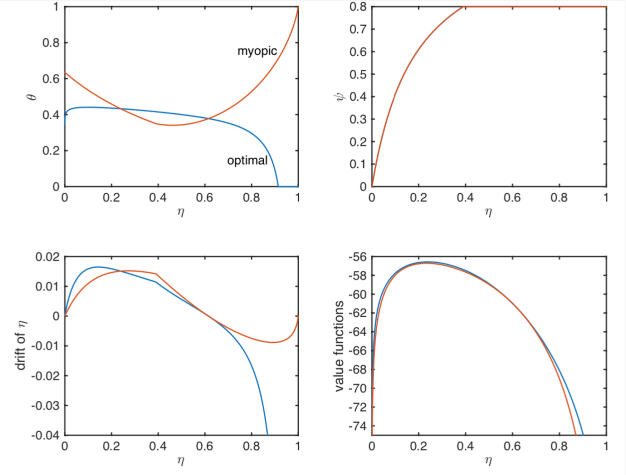
Optimizing over η



$$\rho = 0.05, \phi = 2, \tilde{\sigma} = 0.3, \varphi = 0.5, \underline{\lambda} = 0.2$$

Optimal Policy, (1) MoPo only

- Using ϑ to push η – Same analytical steps as before



$$\rho = 0.05, \phi = 2, \tilde{\sigma} = 0.3, \varphi = 0.5, \underline{\lambda} = 0.2$$

Take-aways of Optimal Policy

- Baseline (one-sector) model
  - Trade-off: insurance vs. investment (growth)
- Multi-sector model
  - Allocation of risk/assets
- Money is not super-neutral
  - since it affect portfolio choice, risk allocation
  - Price of risk (risk premia), η-drift
- (1) MoPo + (2) MacroPru
  - Static problem – 3 steps maximization
    Always $\vartheta^*(\cdot)$-function
- (1) MoPo only
  - Using screwdriver as hammer to push η