

Eco529: Macroeconomics

Lecture 11: One Sector Monetary Model with Time-varying Idiosyncratic Risk and $\beta < 0$ Safe Asset

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Course Overview

1 Intro

Real Macroeconomics Models with Heterogeneous Agents

Immersion Chapters

Money Models

- 10 Single Sector: Money Model with Store of Value and Medium of Change
- 11 Safe Asset with Time-varying Idiosyncratic Risk
- 12 Multi-sector: Money Model with Redistributive Monetary Policy
- 13 Price Stickiness (New Keynesian)
- 14 Welfare and Optimal Policies

International Macroeconomics Models

Overview of Money Lectures

- One Sector and No Aggregate Risk
 - Store of Value Role: Safe Asset and Service Flows
 - Medium of Exchange Role
 - Bubble (mining) or not
 - Price Level Determination
- Store of Value Monetary Model with Time-varying Idiosyncratic Risk
 - Safe Asset, Flight-to-Safety and negative CAPM- β
 - Flight-to-Safety and Equity Excess Volatility
 - Debt valuation puzzle, Debt Laffer Curve,
 - Safe Asset and Bubble Complementarity
 - Policies to Maintain Safe Asset Privilege on Gov. Bond
- Multiple Sector Model
 - Redistributive Monetary Policy

Main Takeaways

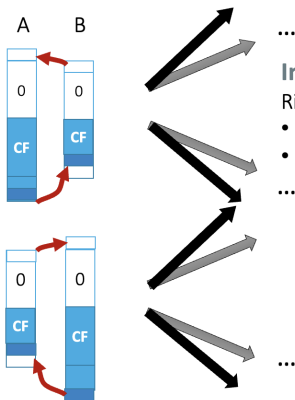
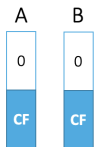
- Aggregate Risk in Form of Time-Varying Idiosyncratic Risk, N_t -Numeraire Analysis
- Stationary Monetary Equilibrium with Bubble on Bonds
 - Safe Asset, Flight-to-Safety and Negative CAPM- β
 - Exorbitant Privilege, Laffer Curve
- Non-uniqueness of Equilibrium
 - Safe Asset \neq Bubble but Complementarity
 - Loss of Safe Asset Status
 - Bubble Bursts or Jumps to Other Asset, Which? (Ponzi-Right-Assignment)
- How to Ensure Uniqueness
 - Elimination Non-stationary Equilibria
 - Elimination of Bubbles on Other Assets
- Modern Debt Sustainability Analysis
 - Debt Valuation Puzzles
 - Off-equilibrium Fiscal Capacity

Safety \neq risk free
 \neq liquidity
 \neq bubble

What's a Safe Asset? What is its Service Flow?

$$\frac{B_t}{P_t} = \mathbb{E}_t[PV_{\xi^{**}}(\text{cash flow})] + \mathbb{E}_t[PV_{\xi^{**}}(\text{service flow})]$$

- Value come from **re-trading**
- Insures by partially completing markets



- Can be “bubbly” = fragile

In recessions:

Risk is higher

- Service flow is more valuable
- Cash flows are lower
(depends on fiscal policy)

...

...

What's a Safe Asset?

- In incomplete markets setting (Bewley, Aiyagari, BruSan, ...)
- Good friend analogy (Brunnermeier Haddad, 2012)
 - When one needs funds, one can sell at stable price ... since others buy
 - **Idiosyncratic shock:** *Partial insurance through **retrading**, low bid-ask spread*
 - **Aggregate (volatility) shock:** *Appreciate in value in times of crises*
- Safe asset definition
 - Tradeable: no asymmetric info – info insensitive
 - $\beta < 0$ relative to individual net worth:

$$\text{Cov}_t \left[d\xi_t^i / \xi_t^i, dr_t^{\text{safe}} - dr_t^{n^i} \right] > 0 \Leftrightarrow \beta_t^i = - \frac{\text{Cov}_t \left[d\xi_t^i / \xi_t^i, dr_t^{\text{safe}} - dr_t^{n^i} \right]}{\text{Var}_t \left[d\xi_t^i / \xi_t^i \right]} < 0,$$

where ξ_t^i is SDF of agent i .

Note: $-\text{Cov}_t[d\xi_t^i / \xi_t^i, dr_t] = \zeta_t^i \sigma_t^r + \tilde{\zeta}_t^i \tilde{\sigma}_t^{r,i}$, where $d\xi_t^i / \xi_t^i = -r_t^f dt - \zeta_t^i dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t^i$

Model with Capital + Safe Asset

- Each heterogenous citizen $\tilde{i} \in [0, 1]$:

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{(c_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} + f(\mathcal{G}_t K_t) \right) dt \right] \text{ where } K_t := \int k_t^{\tilde{i}} d\tilde{i}, \text{ and } \sigma = 0$$

$$\text{s.t. } \frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} dt + dr_t^B + (1 - \theta_t^{\tilde{i}})(dr_t^{K, \tilde{i}}(\iota_t^{\tilde{i}}) - dr_t^B) \text{ \& No Ponzi}$$

- Each citizen operates physical capital $k_t^{\tilde{i}}$

- Output (net investment): $y_t^{\tilde{i}} dt = (a_t k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}}) dt$

$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta \right) dt + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}} + d\Delta_t^{k, \tilde{i}},$$

($d\tilde{Z}_t^{\tilde{i}}$ idiosyncratic Brownian)

- Output tax $\tau a_t k_t^{\tilde{i}} dt$

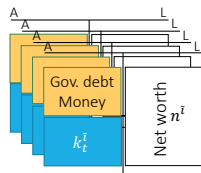
- Financial Friction: Incomplete markets: no $d\tilde{Z}_t^{\tilde{i}}$ claims

- Aggregate risk $\tilde{\sigma}_t, a_t, \mathcal{G}_t$ exogenous process by aggregate Brownian dZ_t

- E.g. Heston model: $d\tilde{\sigma}_t^2 = -\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)dt - \sigma^{\tilde{\sigma}} \tilde{\sigma}_t dZ_t$ CIR-ensures $\tilde{\sigma}_t$ stays positive

- $a_t = a(\tilde{\sigma}_t), \mathcal{G}_t = \mathcal{G}(\tilde{\sigma}_t)$

- Money/bond issuing policy: $d\mathcal{B}_t/\mathcal{B}_t = \mu_t^B dt + \sigma_t^B dZ_t$



Government: Taxes, Bond/Money Supply, Gov. Budget

- $d\mathcal{B}_t/\mathcal{B}_t = \mu_t^{\mathcal{B}}dt + \sigma_t^{\mathcal{B}}dZ_t$, initial debt \mathcal{B}_0 (state variable)
- $\sigma_t^{\mathcal{B}} \neq 0$ leads to stochastic (state contingent) “seigniorage revenue”
- Relabel tax revenue process to: $\frac{d\tau_t}{\tau_t} = \mu_t^{\tau}dt + \sigma_t^{\tau}dZ_t$
 - we can also label s_t (primary surplus) as a process
- Government budget constraint (BC)

$$d\mathcal{B}_t - i_t\mathcal{B}_t + \mathcal{P}_t K_t (d\tau_t a_t - \mathcal{G}_t dt) = 0$$

- Return on Gov. Bond/Money: in output/consumption numeraire

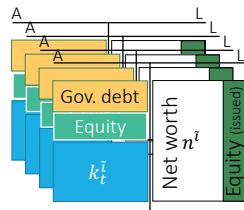
$$\begin{aligned} dr_t^{\mathcal{B}} &= i_t dt + \underbrace{\frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t}}_{-\text{inflation}} = i_t dt + \underbrace{\frac{d(q_t^{\mathcal{B}} K_t / \mathcal{B}_t)}{q_t^{\mathcal{B}} K_t / \mathcal{B}_t}}_{-\text{inflation}} \\ &= \frac{d(q_t^{\mathcal{B}} K_t)}{q_t^{\mathcal{B}} K_t} - \check{\mu}_t^{\mathcal{B}} dt - \sigma_t^{\mathcal{B}} dZ_t + \sigma_t^{\mathcal{B}} (\sigma_t^{\mathcal{B}} - \sigma - \sigma_t^{q, \mathcal{B}}) dt \end{aligned}$$

- In this lecture, we restrict attention to the case where $\sigma_t^{\mathcal{B}} = 0$.

Introduce Outside Equity and Mutual Funds

■ Equity Market

- Each citizen \tilde{i} can sell off a fraction $(1 - \bar{\chi})$ of capital risk to outside equity holders
- Return $dr_t^{E, \tilde{i}}$
 - Same risk as $dr_t^{K, \tilde{i}}$
 - But $\mathbb{E}[dr_t^{E, \tilde{i}}] < \mathbb{E}[dr_t^{K, \tilde{i}}]$... due to insider premium



Proposition

equations as before but replace $\tilde{\sigma}$ with $\bar{\chi}\tilde{\sigma}$

Equilibrium (before solving for portfolio choice)

Equilibrium:

$$\begin{aligned}q_t^B &= \vartheta_t \frac{1 + \phi \check{a}}{(1 - \vartheta_t) + \phi \check{\rho}_t} \\q_t^K &= (1 - \vartheta_t) \frac{1 + \phi \check{a}}{(1 - \vartheta_t) + \phi \check{\rho}_t} \\l_t &= \frac{(1 - \vartheta_t) \check{a} - \check{\rho}_t}{(1 - \vartheta_t) + \phi \check{\rho}_t}\end{aligned}$$

$$\check{a} = a - \mathcal{E}$$

For log utility

$$\check{\rho}_t = \rho$$

- Moneyless equilibrium with $q_t^B = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.

Solving Macro Models Step-by-Step

0 Postulate aggregates, price processes and obtain return processes

1 For given $\check{\rho}^i := C^i/N^i$ -ratio and $\xi^i = SDF^i$ processes for each i

finance block

Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach

Fisher separation theorem

a Real investment ι + Goods market clearing (*static*)

b Portfolio choice θ + asset market clearing or

Asset allocation κ & risk allocation χ

Toolbox 2: "Price-taking" social planner approach

Toolbox 3: Change in numeraire to total wealth (including SDF)

- "Gov. Liability Evaluation/FTPL equation" ϑ

2 Evolution of state variable η (and K)

forward equation

3 Solve $\check{\rho}^i := C^i/N^i$ -ratio and $\xi^i = SDF^i$ processes

backward equation

a Investment opportunities ω and K_t and $\tilde{\eta}^i$ -descaled v_t^i -process

b Derive C^i/N^i -ratio and ζ^i price of risk

c Derive BSDEs

d Separating risk aversion from intertemporal substitution

4 Numerical model solution

a Inner loop: For given $\check{\rho}^i := C^i/N^i$'s and ζ^i 's solve ODE for $q(\eta)$

b Outer loop: Transform BSDE into PDE and **iterate** functions $v^i(\eta, t)$

5 KFE: Stationary distribution, fan charts

1. Portfolio choice θ : Bond Evaluation/FTPL Equation

■ Recall martingale method

- Excess expected return of risky asset A to risky asset B :

$$\mu_t^A - \mu_t^B = \zeta_t^i(\sigma_t^A - \sigma_t^B) + \tilde{\zeta}_t^i(\tilde{\sigma}_t^A - \tilde{\sigma}_t^B)$$

■ 4 alternative derivations:

■ In consumption numeraire

- i. Expected excess return of capital w.r.t. bond return

Note: With $\sigma_t^B \neq 0$ seigniorage is stochastic. As it lowers capital taxes it complicates capital return to:

$$dr_t^{K,\tilde{i}} = \left(\frac{a_t - \mu_t^{\mathcal{F}} - \tilde{i}_t}{q_t^K} + \Phi(\tilde{i}_t) - \delta + \mu_t^{q^K} + \frac{q_t^B}{q_t^K} (\tilde{\mu}_t^B + (\sigma + \sigma_t^{q^B} - \sigma_t^B)\sigma_t^B) \right) dt + \left(\sigma + \sigma_t^{q^K} + \frac{q_t^B \sigma_t^B - \sigma_t^{\mathcal{F}}}{q_t^K} \right) dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}}$$

- ii. Expected excess return of net worth (portfolio) w.r.t. bond return

■ In total net worth numeraire

- iii. Expected excess return of capital w.r.t. bond return

- iv. Expected excess return of individual net worth (=net worth share)
w.r.t. bond return (per bond)

Note: even with $\sigma_t^B \neq 0$ equation stay tractable

1. Portfolio choice θ : N_t -numeraire, N_t to single bond

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[dr_t^{\tilde{\eta}^i}]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(l_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\zeta}_t(1 - \theta_t)\tilde{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[dr_t^{\vartheta/B}]}{dt} = i_t + \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(l_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\text{price of risk in } N_t\text{-numeraire}}$$

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\zeta}_t(1 - \theta_t)\tilde{\sigma}\tilde{\chi}$$

1. Portfolio choice θ : N_t -numeraire, N_t to single bond

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[dr_t^{\tilde{\eta}^i}]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\zeta}_t(1 - \theta_t)\tilde{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[dr_t^{\vartheta/B}]}{dt} = i_t + \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\text{price of risk in } N_t\text{-numeraire}}$$

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\zeta}_t(1 - \theta_t)\tilde{\sigma}\tilde{\chi}$$

- Remark:

- Value of a single bond/coin in N_t -numeraire

$$\begin{aligned} \frac{d(\vartheta_t/B_t)}{\vartheta_t/B_t} &= \mu_t^{\vartheta}dt + \sigma_t^{\vartheta}dZ_t - \mu_t^Bdt - \sigma_t^BdZ_t + \sigma_t^B(\sigma_t^B - \sigma_t^{\vartheta})dt \\ &= (\mu_t^{\vartheta} - \mu_t^B)dt + \sigma_t^{\vartheta}dZ_t \quad (\text{for our case with } \sigma_t^B = 0) \\ &= \mu_t^{\vartheta/B}dt + \sigma_t^{\vartheta/B}dZ_t \quad (\text{defining return-drift and volatility}) \end{aligned}$$

- Terms are shifted into risk-free rate in N_t -numeraire, which drop out when differencing

1. Portfolio choice θ : N_t -numeraire, N_t to single bond

- Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[dr_t^{\tilde{\eta}^i}]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\zeta}_t(1 - \theta_t)\tilde{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[dr_t^{\vartheta/B}]}{dt} = i_t + \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\text{price of risk in } N_t\text{-numeraire}}$$

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\zeta}_t(1 - \theta_t)\tilde{\sigma}\tilde{\chi}$$

- Capital market clearing: $1 - \theta = 1 - \vartheta$
- Price of risk: $\varsigma_t = \text{Step 3}$

- for log-utility:

$$\check{\rho}_t = \rho, \varsigma_t = \sigma_t^N, \tilde{\zeta}_t = (1 - \vartheta_t)\tilde{\chi}\tilde{\sigma}_t$$

$$\rho - i_t - \mu_t^{\vartheta/B} = (1 - \vartheta_t)^2\tilde{\chi}^2\tilde{\sigma}^2$$

- for CRRA utility:

De-scale individual value functions to v^i (here only one sector)
solve 2 PDE of $v(\tilde{\sigma}^2)$ and $\vartheta(\tilde{\sigma}^2)$

Recall:

$$\mu_t^{\vartheta/B} = \mu_t^{\vartheta} - \mu_t^B, \quad \sigma_t^{\vartheta/B} = \sigma_t^{\vartheta}$$

Log-utility: 4. Numerical Steps

- Recall: $\mu_t^{\vartheta/\mathcal{B}} = \mu_t^{\vartheta} - \mu_t^{\mathcal{B}}$, $\sigma_t^{\vartheta/\mathcal{B}} = \sigma_t^{\vartheta}$
 - $\rho = (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2 + \mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}}$
 - Multiply with ϑ and write ϑ as function of $\tilde{\sigma}^2$:
 - $\rho\vartheta(\tilde{\sigma}^2) = (1 - \vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}^2) + \mu_t^{\vartheta} \vartheta(\tilde{\sigma}^2) - \check{\mu}_t^{\mathcal{B}} \vartheta(\tilde{\sigma}^2)$

- Drift $\vartheta(\tilde{\sigma}^2)$ from Itô's Lemma:

$$\mu_t^{\vartheta} \vartheta_t = \vartheta'(\tilde{\sigma}^2) \mu_t^{\tilde{\sigma}} \tilde{\sigma}_t + \frac{1}{2} \vartheta''(\tilde{\sigma}^2) (\sigma^{\tilde{\sigma}} \tilde{\sigma}_t)^2$$

- Equate drift and add time-derivative

$$\rho\vartheta(\tilde{\sigma}^2) = [(1 - \vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2 - \check{\mu}^{\mathcal{B}}] \vartheta(\tilde{\sigma}^2) - \psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) \vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}} \tilde{\sigma}_t)^2}{2} \vartheta''(\tilde{\sigma}^2)$$

$$\rho\vartheta_t(\tilde{\sigma}^2) = \partial_t \vartheta_t(\tilde{\sigma}^2) + \underbrace{[(1 - \vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2 \vartheta(\tilde{\sigma}^2) - \check{\mu}^{\mathcal{B}} \vartheta(\tilde{\sigma}^2)]}_{u\vartheta} - \underbrace{\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) \vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}} \tilde{\sigma}_t)^2}{2} \vartheta''(\tilde{\sigma}^2)}_{M\vartheta}$$

- Solve for $\vartheta(\tilde{\sigma}^2)$ with function iteration – see numerical videos

CRRA: 3. Deriving price of idiosyncr. risk ζ^i , C/N-ratio ρ

- Recall CRRA-value function:

$$V_t^i = \frac{1}{\rho} \frac{(\omega_t^i \tilde{n}_t^i)^{1-\gamma}}{1-\gamma} = \frac{1}{\rho} \underbrace{(\omega_t^i n_t^i / K_t)^{1-\gamma}}_{v_t^i :=} \underbrace{\left(\frac{\tilde{n}_t^i}{n_t^i} \right)^{1-\gamma}}_{(\tilde{n}_t^i)^{1-\gamma}} \frac{K_t^{(1-\gamma)}}{1-\gamma}$$

- Recall value function envelop condition

$$\begin{aligned} \frac{\partial V_t^i}{\partial \tilde{n}_t^i} &= \frac{1}{\rho} \underbrace{(\omega_t^i)^{1-\gamma}}_{=v_t^i (K_t/n_t^i)^{1-\gamma}} \underbrace{(\tilde{n}_t^i)^{-\gamma}}_{=(\tilde{n}_t^i)^{-\gamma} (n_t^i)^{-\gamma}} = (c_t^i)^{-\gamma} = \frac{\partial u}{\partial c_t^i} \\ &= v_t^i K_t^{1-\gamma} (n_t^i)^{-1} (\tilde{n}_t^i)^{-\gamma} \\ &= v_t^i K_t^{-\gamma} (q_t^B + q_t^K)^{-1} (\tilde{n}_t^i)^{-\gamma} \quad (\text{after noting that } n_t^i = N_t = (q_t^B + q_t^K) K_t) \end{aligned}$$

- For aggregate price of risk (recall: $\sigma_t = 0$, no aggregate risk for K)

- $\sigma_t^v - \cancel{\gamma\sigma} - \sigma_t^{q^B+q^K} = -\gamma\sigma_t^{c^i} = -\varsigma_t$

- For idiosyncratic price of risk (recall: $\sigma_t^{\tilde{n}^i} = \sigma_t^{\tilde{n}^i}$)

- $\zeta_t^i = \gamma \tilde{\sigma}_t^{\tilde{n}^i} = \gamma(1-\vartheta)\bar{\chi}\tilde{\sigma}_t$

CRRA: 1. θ -choice: N_t -numeraire, N_t to single bond

■ Asset pricing equation (martingale method)

$$\frac{\mathbb{E}[dr_t^{\tilde{r}}]}{dt} = \check{\rho}_t = \left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right) + (\varsigma_t - \sigma_t^N)0 + \tilde{\zeta}_t(1 - \theta_t)\tilde{\chi}\tilde{\sigma}$$

$$\frac{\mathbb{E}[dr_t^{\vartheta/B}]}{dt} = i_t + \mu_t^{\vartheta/B} = \underbrace{\left(r_t^f - (\Phi(\iota_t) - \delta) - \mu_t^{q^K+q^B} - \sigma\sigma_t^{q^K+q^B} + \varsigma_t\sigma_t^{q^K+q^B} \right)}_{\text{risk-free rate in } N_t\text{-numeraire}} + \underbrace{(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B}}_{\text{price of risk in } N_t\text{-numeraire}}$$

$$\check{\rho}_t - i_t - \mu_t^{\vartheta/B} = -(\varsigma_t - \sigma_t^N)\sigma_t^{\vartheta/B} + \tilde{\zeta}_t(1 - \theta_t)\tilde{\sigma}\tilde{\chi}$$

- Capital market clearing: $1 - \theta = 1 - \vartheta$
- Price of risk: $\varsigma_t = -\sigma_t^v + \sigma_t^{q^B+q^K} + \cancel{\gamma\sigma}$, $\tilde{\zeta}_t = \gamma\tilde{\sigma}_t^n = \gamma(1 - \theta_t)\tilde{\chi}\tilde{\sigma}$

$$\check{\rho}_t - \mu_t^{\frac{\vartheta}{B}} = (\sigma_t^v - \cancel{(\gamma-1)\sigma})\sigma_t^{\frac{\vartheta}{B}} + \gamma(1 - \vartheta_t)^2\tilde{\chi}^2\tilde{\sigma}^2$$

Recall:

$$\mu_t^{\vartheta/B} = \mu_t^{\vartheta} - \mu_t^B$$

$$\sigma_t^{\vartheta/B} = \sigma_t^{\vartheta}$$

CRRA: 3. BSDE for v_t^i

$$\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} = \frac{d\left(v_t^i(\tilde{\eta}_t^i)^{1-\gamma}K_t^{1-\gamma}\right)}{v_t^i(\tilde{\eta}_t^i)^{1-\gamma}K_t^{1-\gamma}}$$

- By Itô's product rule:

$$= \left[\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^{\tilde{z}} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2) + (1-\gamma)\sigma\sigma_t^v \right] dt + \text{volatility}$$

- Recall by consumption optimality $\frac{dV_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho dt + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$ follows a martingale

- Hence, drift above = $\rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$

- Equate drift terms to obtain BSDE:

$$\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)(\sigma^{\tilde{z}} + (\tilde{\sigma}_t^{n^{\tilde{i}}})^2) + (1-\gamma)\sigma\sigma_t^v = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

CRRA: 4. Numerical Steps

1 Generalize PDE for ϑ : (now with $\gamma \neq 1$ and σ_t^ν)

- Recall $\mu_t^{\vartheta/B} = \mu_t^\vartheta - \mu_t^B, \sigma_t^{\vartheta/B} = \sigma_t^\vartheta$
 - Plug them into $\check{\rho}_t - i_t - \mu_t^{\vartheta/B} = (\sigma_t^\nu - (\gamma-1)\sigma) \sigma_t^{\vartheta/B} + \gamma(1-\theta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2$
 - $\check{\rho}_t = \mu_t^\vartheta - \check{\mu}_t^B + (\sigma_t^\nu - (\gamma-1)\sigma) \sigma_t^\vartheta + \gamma(1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}^2$
 - Multiply with ϑ and write $\vartheta, \check{\rho}$, and σ^ν as functions of $\tilde{\sigma}^2$
 - $\check{\rho}_t(\tilde{\sigma}^2) \vartheta(\tilde{\sigma}^2) = [\mu_t^\vartheta - \check{\mu}_t^B + \sigma_t^\nu(\tilde{\sigma}^2) \sigma_t^\vartheta + \gamma(1-\vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2] \vartheta(\tilde{\sigma}^2)$
- Drift and volatility of $\vartheta(\tilde{\sigma}^2)$ from Itô's Lemma:

$$\mu_t^\vartheta \vartheta_t = \vartheta'(\tilde{\sigma}^2) \mu_t^{\tilde{\sigma}} \tilde{\sigma}_t + \frac{1}{2} \vartheta''(\tilde{\sigma}^2) (\sigma^{\tilde{\sigma}} \tilde{\sigma}_t)^2, \quad \sigma_t^\vartheta \vartheta_t = -\vartheta'(\tilde{\sigma}^2) \sigma^{\tilde{\sigma}} \tilde{\sigma}_t$$

- Equate drift and add time-derivative

$$\begin{aligned} \check{\rho}(\tilde{\sigma}^2) \vartheta(\tilde{\sigma}^2) &= [\gamma(1-\vartheta(\tilde{\sigma}))^2 \bar{\chi}^2 \tilde{\sigma}^2 - \check{\mu}_t^B] \vartheta(\tilde{\sigma}^2) - \psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) \vartheta'(\tilde{\sigma}^2) \\ &\quad + \frac{(\sigma^{\tilde{\sigma}} \tilde{\sigma}_t^2)}{2} \vartheta''(\tilde{\sigma}^2) - \sigma_t^\nu(\tilde{\sigma}^2) \vartheta'(\tilde{\sigma}^2) \sigma^{\tilde{\sigma}} \tilde{\sigma}_t \\ \check{\rho}_t(\tilde{\sigma}^2) \vartheta_t(\tilde{\sigma}^2) &= \partial_t \vartheta_t(\tilde{\sigma}^2) + \underbrace{[\gamma(1-\vartheta(\tilde{\sigma}^2))^2 \bar{\chi}^2 \tilde{\sigma}^2 - \check{\mu}_t^B] \vartheta(\tilde{\sigma}^2)}_{u\vartheta} \\ &\quad \underbrace{[-\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2) - \sigma^{\tilde{\sigma}} \tilde{\sigma}_t \sigma_t^\nu(\tilde{\sigma}^2)] \vartheta'(\tilde{\sigma}^2) + \frac{(\sigma^{\tilde{\sigma}} \tilde{\sigma}_t^2)}{2} \vartheta''(\tilde{\sigma}^2)}_{M\vartheta} \end{aligned}$$

2 Derive PDE for v :

CRRA: 4. Numerical Steps

1 Generalize PDE for ϑ : (now with $\gamma \neq 1$ and σ_t^y)

2 Derive PDE for v :

- $\mu_t^y + (1 - \gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1 - \gamma)(\sigma^z + (\tilde{\sigma}_t^{n^i})^2) + (1 - \gamma)\sigma\sigma_t^y = \rho - \frac{c_t^i}{n_t^i}$

- Itô's Lemma for $v(\tilde{\sigma}^2)$:

$$dv(\tilde{\sigma}^2) = \underbrace{\left(-\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)v'(\tilde{\sigma}^2) + \frac{1}{2}(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t^2)v''(\tilde{\sigma}^2) \right)}_{=v\mu_t^y} dt - \underbrace{\sigma^{\tilde{\sigma}}\tilde{\sigma}_t v'(\tilde{\sigma}^2)}_{=v\sigma_t^y} dZ_t$$

- PDE for v :

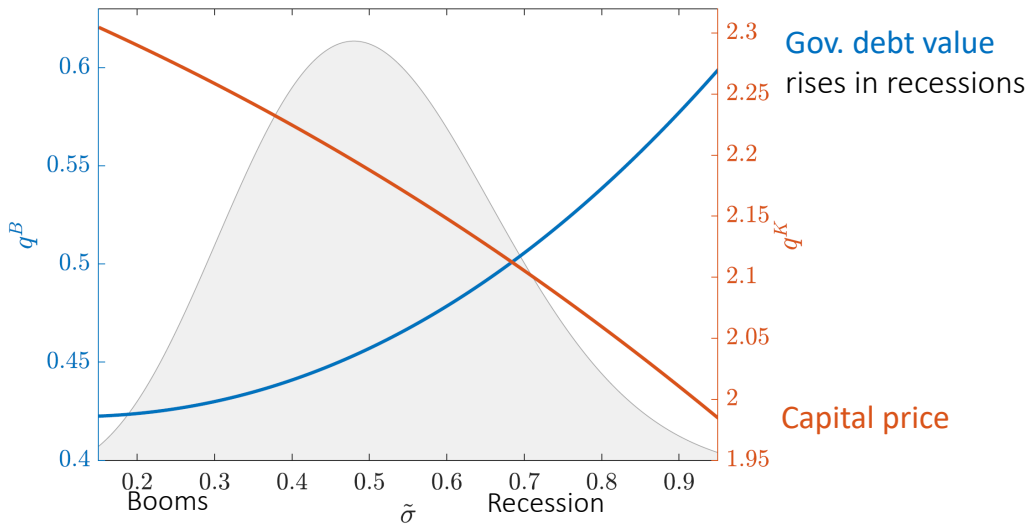
$$\rho v(\tilde{\sigma}^2) = \overbrace{\partial_t v(\tilde{\sigma}^2) + \left(\frac{c_t^i}{n_t^i} + (1 - \gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1 - \gamma)(\sigma^z + (\tilde{\sigma}_t^{n^i})^2) + (1 - \gamma)\sigma\sigma_t^y \right) v}_{uv} - \underbrace{\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)v'(\tilde{\sigma}) + \frac{(\sigma^{\tilde{\sigma}}\tilde{\sigma}_t)^2}{2}v''(\tilde{\sigma}^2)}_{Mv}$$

- 2 PDEs: Solve both by iterating simultaneously (outer loop)

- No inner loop since trivial (since κ, χ within sector have no macro-implications)

Bond & Capital Value for Time-varying Idiosyncratic Risk

- Comparative static w.r.t. idiosyncratic risk $\tilde{\sigma}$



FTPL Equation with Bubble: 2 Perspectives

- Agent \tilde{i} 's SDF, $\xi_t^{\tilde{i}}$: $d\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t^f dt - \varsigma_t dZ_t - \tilde{\zeta}_t^{\tilde{i}} d\tilde{Z}_t^{\tilde{i}}$
- Buy and Hold Perspective:

$$\frac{B_0}{P_0} = \lim_{T \rightarrow \infty} \left(\mathbb{E} \left[\int_0^T \xi_t^{\tilde{i}} s_t K_t dt \right] + \mathbb{E} \left[\xi_T^{\tilde{i}} \frac{B_T}{P_T} \right] \right)$$

Bubble is possible: $\lim_{T \rightarrow \infty} \mathbb{E} \left[\bar{\xi}_T \frac{B_T}{P_T} \right] > 0$ if $r_t^f + \varsigma_t \sigma_t^{q,B} \leq g_t$ (on average)
 $g - \tilde{\mu}^B =$ discount rate

- Dynamic Trading Perspective:

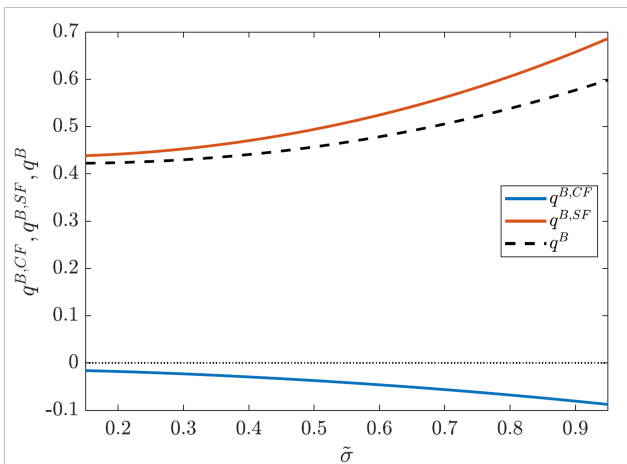
- Value cash flow from individual bond portfolios, including trading cash flows
- Integrate over citizens weighted by net worth share $\eta_t^{\tilde{i}}$
- Bond as part of a dynamic trading strategy

$$\frac{B_0}{P_0} = \mathbb{E} \left[\int_0^\infty \underbrace{\left(\int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} d\tilde{i} \right)}_{\xi_t^{**}} s_t K_t dt \right] + \mathbb{E} \left[\int_0^\infty \underbrace{\left(\int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} d\tilde{i} \right)}_{\xi_t^{**}} \gamma (\tilde{\sigma}_t)^2 \frac{B_t}{P_t} dt \right]$$

- Discount rate: $\mathbb{E}[dr_t^{\eta}] / dt = r^f + \tilde{\zeta} \tilde{\sigma}$

Dynamic Trading Perspective Decomposition

$$\frac{B_0}{P_0} = \overbrace{\mathbb{E} \left[\int_0^\infty \xi_t^{**} s_t K_t dt \right]}^{\text{EPV(cash flow)}} + \overbrace{\mathbb{E} \left[\int_0^\infty \xi_t^{**} \gamma (\tilde{\sigma}_t^c)^2 \frac{B_t}{P_t} dt \right]}^{\text{EPV(service flow)}}$$



Excess Stock Market Volatility due to Flight to Safety

- “Aggregate Intertemporal Budget Constraint

$$\underbrace{q_t^K K_t + q_t^B K_t}_{\text{total (net) wealth}} = \mathbb{E}_t \left[\int_t^\infty \frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di} C_s ds \right] \quad (*)$$

- Lucas-type models: $q^B = 0$ (also $C_t = Y_t$, no idiosyncratic risk)
 - Value of equity (Lucas tree) = PV of consumption claim
 - Volatility equity values require volatile RHS of (*)
- This model: even for constant RHS of (*), $q_t^K K_t$ can be volatile due to flight to safety: ($\text{Cov}[q^K, q^B] < 0$)
 - increase in $\tilde{\sigma}_t \Rightarrow$ Portfolio reallocation from capital to bonds, $q_t^K K_t \downarrow, \mathcal{B}_t/P_t \uparrow$,
- Outside equity is linked to $q_t^K K_t$ and even more volatile due to countercyclical insider equity premium. (see below)
- Quantitatively relevant? Yes
Excess return volatility
 - 2.9% in equivalent bondless model ($s = 0$ and no bubble)
 - 12.9% in out model

Calibration

- Exogenous processes:
Recessions feature high idiosyncratic risk and low consumption

- $\tilde{\sigma}_t$: Heston (1993) model of stochastic volatility:

$$d\tilde{\sigma}_t^2 = -\psi(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2)dt - \sigma^{\tilde{\sigma}}\tilde{\sigma}_t dZ_t$$

- a_t : $a_t = a(\tilde{\sigma}_t)$:

$$a_t(\tilde{\sigma}) = a^0 - \alpha^a(\tilde{\sigma}_t - \tilde{\sigma}^0)$$

- $\mathcal{G}_t = 0$

- Government (bubble-mining policy)

$$\check{\mu}_t^B = \check{\mu}_t^{B,0} + \alpha^B(\tilde{\sigma} - \tilde{\sigma}^0)$$

- Calibration to US data (1970-2019, period length is one year)

Calibration: Parameters

parameter	description	value	parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.54	g	gov. expenditures	0.138
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67	$\check{\mu}_t^{B,0}$	$\check{\mu}_t^B$ stoch. steady state	0.0026
$\sigma^{\tilde{\sigma}}$	$\tilde{\sigma}_t^2$ volatility	0.4	a^a	a_t slope	0.072
$\tilde{\chi}$	undiversifiable risk	0.3	a^B	$\check{\mu}_t^B$ slope	0.12
γ	risk aversion	6	ϕ	capital adj. cost	8.1
ρ	time preference	0.138	l^0	capital adj. intercept	-0.022
a^0	a_t stoch. steady state	0.625	δ	depreciation rate	0.055

Quantitative Model Fit

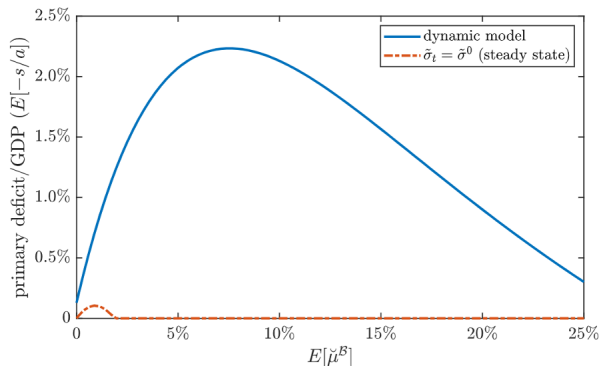
symbol	moment description	model	data
(1) targeted moments			
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.61	0.64
$\sigma(I)/\sigma(Y)$	relative investment volatility	3.35	3.38
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.58	0.56
$\mathbb{E}[G/Y]$	average government expenditures-output ratio	0.22	0.22
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0005	-0.0005
$\mathbb{E}[I/K]$	average investment rate	0.12	0.12
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[dr^E - dr^B]$	average (unlevered) equity premium	3.59%	3.40%
$\frac{\mathbb{E}[dr^E - dr^B]}{\sigma(dr^E - dr^B)}$	equity sharpe ratio	0.31	0.31
(2) untargeted moments			
$\rho(Y, C)$	correlation of output and consumption	0.98	0.92
$\rho(Y, I)$	correlation of output and investment	0.99	0.94
$\rho(Y, S/Y)$	correlation of output and surpluses	0.98	0.60
$\sigma(q^B K/Y)$	volatility of debt-output ratio	4.8%	2.0%
$\mathbb{E}[r^f]$	average risk-free rate	5.18%	0.64%
$\sigma(r^f)$	volatility of risk-free rate	5.47%	2.25%

Two Debt Valuation Puzzles

- Properties of US primary surpluses
 - Average surplus ≈ 0
 - Procyclical surplus (> 0 in booms, < 0 in recessions)
- Two valuation puzzles from standard perspective:
(Jiang, Lustig, van Nieuwerburgh, Xiaolan, 2019, 2020)
 1. “Public Debt Valuation Puzzle”
 - Empirical $\mathbb{E}[PV(\text{surpluses})] < 0$, yet $B/P > 0$
 - Our model: bubble/service flow component overturns results
 2. “Gov. Debt Risk Premium Puzzle”
 - Debt should be positive β asset, but market doesn't price it this way
 - Our model: can be rationalized with countercyclical bubble/service flow

Debt Laffer Curve

- Issue bonds at a faster rate $\check{\mu}^B$ (esp. in recessions)
 - \Rightarrow tax precautionary self insurance \Rightarrow tax rate \uparrow
 - \Rightarrow real value of bonds: $B/P \downarrow \Rightarrow$ “tax base” \downarrow
 - Less so in recession due to **flight-to-safety**



Sizeable revenue only if Gov. debt has negative β

Roadmap

- Stationary Monetary Equilibrium with Bubble on Bonds
 - Safe asset, Flight-to-Safety and Negative CAPM- β
 - Exorbitant Privilege, Laffer Curve
- Non-uniqueness of Equilibrium
 - Safe Asset \neq Bubble but Complementarity
 - Loss of Safe Asset Status
 - Bubble Bursts or Jumps to Other Asset, Which?
(Ponzi-Right-Assignment)
- How to Ensure Uniqueness
 - Elimination Non-stationary Equilibria
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 - Off-equilibrium Fiscal Capacity

Safety \neq risk free
 \neq liquidity
 \neq bubble

Non-uniqueness of Equilibria

- Among the stationary monetary equilibria with bubble on government bonds analyzed equilibrium is unique
 - See Appendix A.2 of Safe Asset Paper BruMerSan (2023)
- However, there might be
 - Non-bubble equilibrium
 - Bubble can be Associated with/jump to a Different Asset Than Gov. bond
 - Non-stationary Equilibria, in which Bubble Decays over time
- Safe Asset is related to concepts of Bubbles and Liquidity

Recall Safe Asset Definition (time & individual specific)

- Re-tradable
- Good friend:

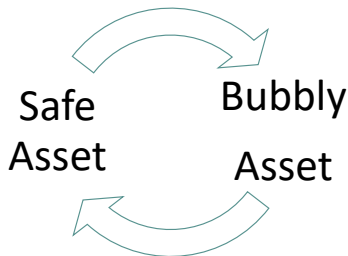
$$\text{Cov}_t \left[d\xi_t^i / \xi_t^i, dr_t^{\text{safe}} - dr_t^{n^i} \right] > 0 \Leftrightarrow \beta_t^i = - \frac{\text{Cov}_t \left[d\xi_t^i / \xi_t^i, dr_t^{\text{safe}} - dr_t^{n^i} \right]}{\text{Var}_t \left[d\xi_t^i / \xi_t^i \right]} < 0 \text{ w.r.t.}$$

idiosyncratic & aggregate risk (relative to own net worth return $dr_t^{n^i}$)

- Safe asset return : $dr_t^{\text{safe}} = \mu_t^{\text{safe}} dt + \sigma_t^{\text{safe}} dZ_t + \int \tilde{\sigma}_t^{\text{safe}, \tilde{i}} d\tilde{Z}_t^{\tilde{i}}$
- Net worth return of agent $\tilde{i} \in i$: $dr_t^{n^{\tilde{i}}} = \mu_t^{n^{\tilde{i}}} dt + \sigma_t^{n^{\tilde{i}}} dZ_t + \tilde{\sigma}_t^{n^{\tilde{i}}, \tilde{i}} d\tilde{Z}_t^{\tilde{i}}$
- SDF of agent $\tilde{i} \in i$: $\frac{d\xi_t^{\tilde{i}}}{\xi_t^{\tilde{i}}} = -r_t^f dt - \varsigma_t^{\tilde{i}} dZ_t - \tilde{\zeta}_t^{\tilde{i}} d\tilde{Z}_t^{\tilde{i}}$
- Typical safe asset doesn't load on $d\tilde{Z}_t^{\tilde{i}}$
- $\text{Cov}_t \left[d\xi_t^i / \xi_t^i, dr_t^{\text{safe}} - dr_t^{n^i} \right] = \varsigma_t^i \left(-\sigma_t^{\text{safe}} + \sigma_t^{n^i} \right) > 0$

Complementarity btw Safe Asset & Bubble

- bubble \Rightarrow easier to satisfy safe asset definition: negative β
 - Bubble raises q_t^B by keeping $q_t^{B, \text{cash flow}}$ unaffected $\Rightarrow \alpha_t^{cf}$ declines $\Rightarrow \beta$ is lower (partial equilibrium argument)
- safe asset \Rightarrow bubble condition easier: $\lim_{T \rightarrow \infty} \mathbb{E} \left[\xi_T \frac{B_T}{P_t} \right] > 0 \Leftrightarrow r^{safe} < g$
 - Negative β (or $\sigma^{safe} < 0$) $\Rightarrow r^{safe}$ is lower



Complementarity btw Safe Asset & Bubble

- bubble \Rightarrow easier to satisfy safe asset definition: negative β
 - Bubble raises q_t^B by keeping $q_t^{B, \text{cash flow}}$ unaffected $\Rightarrow \alpha_t^{cf}$ declines $\Rightarrow \beta$ is lower (partial equilibrium argument)
- safe asset \Rightarrow bubble condition easier: $\lim_{T \rightarrow \infty} \mathbb{E} \left[\xi_T \frac{B_T}{P_t} \right] > 0 \Leftrightarrow r^{\text{safe}} < g$
 - Negative β (or $\sigma^{\text{safe}} < 0$) $\Rightarrow r^{\text{safe}}$ is lower
- Split up aggregate risk (covariance with dZ_t)

$$\sigma_t^{\text{safe}} = \alpha_t^{cf} \sigma_t^{\text{cash flows}} + (1 - \alpha_t^{cf}) \overbrace{\sigma_t^{\text{service flows}}}^{<0}, \text{ where } \alpha_t^{cf} = \frac{q_t^{B, \text{cash flow}}}{q_t^B}$$

$$r_t^{\text{safe}} = r_t^f + \underbrace{\varsigma_t \sigma_t^{\text{safe}}}_{<0} r_t^{\text{safe}} = r_t^f + \underbrace{\beta_t^{\text{safe}, C}}_{\frac{\sigma_t^{\text{safe}} \sigma_t^C}{\sigma_t^C \sigma_t^C}} \underbrace{(r_t^C - r_t^f)}_{=\varsigma_t \sigma_t^C} \quad (C = \text{aggr. consumption claim})$$

- Note, for $\sigma_t^{\text{cash flows}}$ sufficiently high, bubble condition is violated

Safe Asset-Bubble on Gov Debt or Equity Mutual Fund

- For Gov. Debt:

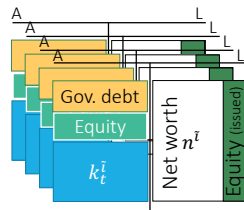
$$\sigma_t^{\text{cash flows}} = \sigma_t^{\text{primary surplus}}$$

- < 0 if procyclical (austerity) fiscal policy
- > 0 if countercyclical (stimulus) fiscal policy

- For Equity Mutual Fund:

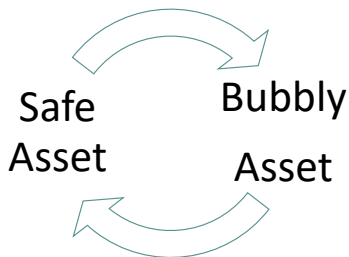
$\tilde{\sigma}_t^{\text{cash flows}}$ diversified stock portfolio is free of idiosyncratic risk

- can also self-insure idiosyncratic risk
- \Rightarrow **Good friend** in **idiosyncratically** bad times
- $\sigma_t^{\text{cash flows}} > 0$ poor hedge against aggregate risk, losses value in recessions
 - \Rightarrow **Bad friend** in **aggregate** bad times
 - Equity are claims to capital, but marginal capital holder is insider
 - Insider bears idiosyncratic risk, must be compensated
 - $\tilde{\sigma}_t \uparrow \Rightarrow$ insider premium $\mathbb{E}_t[dr_t^K] - \mathbb{E}_t[dr_t^E] \uparrow \Rightarrow$ payouts to outside stockholders fall



Aside: Complementarity btw Safety & Liquidity (but \neq)

- If high market liquidity (low bid-ask spread) \Rightarrow better safe asset
- Safe asset \Rightarrow high trading volume and better market liquidity



- Simply assumed in our model
 - all assets perfect liquidity (highlights safety \neq liquidity)
- How to maintain safe asset status?
 - Central Banks as Market Maker of Last Resort
 - Example: 10 year US Treasury in March 2020

Overview

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Safety \neq risk free
 \neq liquidity
 \neq bubble

Policies to Prevent Loss of Safe Asset Status

- Are there policies to prevent a loss of safe asset status?
 - Raise (positive) surpluses to generate safe cash flow component $q_t^{B,CF}$ and possibly also $\sigma_t^{cash\ flows} < 0$
 - If surpluses always exceed a (positive) fraction of total output, no bubble
 - But: gives up revenues from bubble mining
- Off-equilibrium tax backing
 - Sufficient to (credibly) promise policy above off-equilibrium
 - See “FTPL with a Bubble” paper

Uniqueness of Stationary Bubble Equilibria

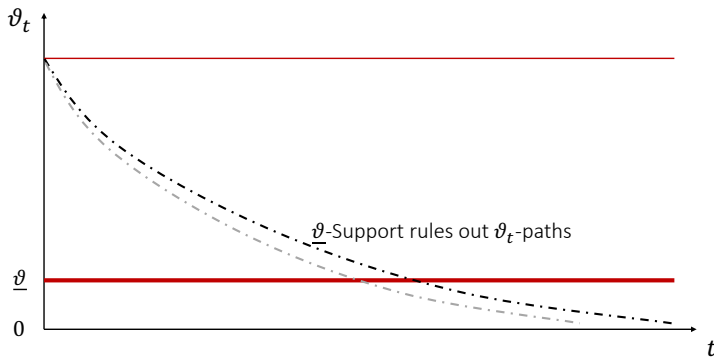
- Assume bubble is possibly only on gov. debt

$$\underbrace{\vartheta_t \mu_t^\vartheta}_{\dot{\vartheta}_t} = \vartheta_t (\rho - (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}^2 + \check{\mu}_t^B)$$

1. ϑ^* steady state level
 2. $\vartheta = 0$
 3. Continuum of $\vartheta_0 \in [0, \vartheta^*]$ equilibria, in which ϑ_t converges to 0.
- Fiscal rule: whenever ϑ_t drops below $\underline{\vartheta}$,
support $\underline{\vartheta}$ by switching to positive surplus $s > 0$
 - Off-equilibrium backing eliminates No-bubble and decaying bubble equilibria (2. and 3.)
⇒ equilibrium is stationary (Obstfeld-Rogoff Analogy)
- Uniqueness among stationary equilibria (see “Safe Asset Paper”, Appendix A.2)

Uniqueness of Stationary Bubble Equilibria

- So far, unique equilibrium among stationary bubble equilibria (Safe Asset Paper)
- Rule out declining bubble and no-bubble equilibrium



Uniqueness of Stationary Bubble Equilibria on Gov. Debt

- Bubble can be on other assets, e.g. crypto asset

Who can issue bubble assets in increasing quantity, run a Ponzi scheme?

Who owns “Exorbitant privilege” *is an equilibrium selection*

- $q_t^C K_t :=$ real value of crypto coin
- $\hat{v}_t = v_t^B + v_t^C$
- Two ODEs: one for \hat{v}_t and one for v_t^B .
- As before $v_t^B \geq \underline{v}$ all the time, where \underline{v} is supported by off-equilibrium $s > 0$.
- If $\underline{v} < v^*$
 - If $\check{\mu}_t^C > \check{\mu}_t^B$ crypto bubble can't exist
(as crypto dilution rate exceeds gov debt dilution rate, v_t^B must shrink, but sum of bubbles is bounded)
 - If $\check{\mu}_t^C \leq \check{\mu}_t^B$ gov. debt and cryptocurrency bubble can co-exist
 - But government can
 - Impose solvency law, taxes ...

Other Policy Tools to Keep Bubble on Gov. Debt

- If $\check{\mu}_t^C \leq \check{\mu}_t^B$ gov. debt and cryptocurrency bubble can co-exist
- But government can
 - Impose solvency law \Rightarrow private institutions cannot run Ponzi scheme
 - Impose taxes on crypto holdings \Rightarrow same as increasing $\check{\mu}_t^C$
 - Impose trading restrictions
 - Financial repression

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Fiscal Debt Sustainability (DSA): A Modern Perspective

- Debt valuation/FTPL equation with a bubble
with service flow using representative agent SDF ξ^{**}
- Exorbitant privilege
 - Debt Laffer Curve (tax on self-insurance) – only sizable with negative β
 - Ponzi scheme/mining the bubble
- Credible Off-equilibrium Fiscal Capacity
 - Bubble (incl. possibly safe asset status) can
 - Burst
 - Jump to foreign safe asset
 - Jump to crypto asset

Overview Across Lectures 10-14

- Store of Value Monetary Model with One Sector and No Aggregate Risk
 - Safe Asset and Service Flows
 - Bubble (mining) or not
 - 2 Different Asset Pricing Perspectives/SDFs
- Store of Value Monetary Model with Time-varying Idiosyncratic Risk
 - Safe asset, Flight-to-Safety and negative CAPM- β
 - Flight-to-Safety and Equity Excess Volatility
 - Debt valuation puzzle, Debt Laffer Curve,
 - Safe Asset and Bubble Complementarity
 - Policies to Maintain Safe Asset Privilege on Gov. Bond
- Medium of Exchange Role and Different “Monetary Theories”