

Macrofinance

Lecture 03: A Simple Real Macro Model with Heterogeneous Agents

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Course Overview

- 1 Intro
- 2 Portfolio & Consumption Choice

Real Macroeconomics Models with Heterogeneous Agents

- 3 Simple Real Macroeconomics Models
- 4 Endogenous (Price of) Risk Dynamics
- 5 Contrasting Financial Frictions

Immersion Chapters

Money Models

International Macroeconomics Models

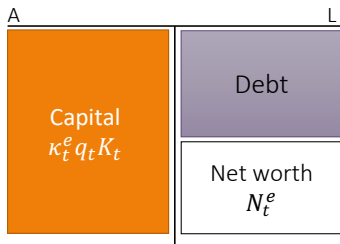
Real Macroeconomics Models with Heterogeneous Agents

A Two-Sector Macroeconomics Model

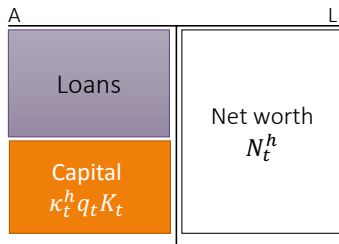
- Complete Markets Benchmark
 - No risk and no frictions
- Basak-Cuoco Model
 - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
 - No risk, two production sectors and leverage constraints

Two Sector Macroeconomics Model Setup

■ Expert sector (Farmers)



Household sector (Gatherers)



- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geq 0$
- Experts produce with capital with linear production function $a^e k_t^e (= a^e \kappa_t^e K_t)$.
- Households' production function $a^h(\kappa_t^h) k_t^h$ is fcn. of (aggregate) κ_t^h
 - Productivity $a^h(\kappa^h) \leq a^e$ with equality for $\kappa^h = 0$ and strictly decreasing in κ^h
- Experts can only issue debt with **collateral constraint**: $D_t^e \leq \ell \kappa_t^e q_t K_t$
 - $\frac{D_t^e}{N_t^e} \leq \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1 - \theta_t^{K,e}) \leq \ell \theta_t^{K,e} \Leftrightarrow (1 - \ell) \theta_t^{K,e} \leq 1$
- All experts' net worth $N_t^e = \int_0^1 n_t^{e,i} di = n_t^e$; all households' net worth $N_t^h = n_t^h$
- Assumption: aggregate physical capital evolves exogenously

$$\frac{dK_t}{K_t} = gdt + \sigma dZ_t$$

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Portfolio choice $\theta_t^{K,e} \equiv \kappa_t^e \frac{q_t K_t}{N_t^e}$

- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log(c_t^e) dt \right]$

Household Sector

- Output: $y_t^h = a^h (\kappa_t^h) k_t^h$
- Consumption rate: c_t^h
- Portfolio choice $\theta_t^{K,h} \equiv \kappa_t^h \frac{q_t K_t}{N_t^h}$

- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log(c_t^h) dt \right]$

Individual capital evolution: $\frac{d\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = g dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i}}$

Friction: Can only issue risk-free debt with collateral constraint $(1 - \ell)\theta_t^{K,e} \leq 1$

Sector Optimization

■ Expert's problem

$$\begin{aligned} & \max_{\{c_t^e, \theta_t^{K,e}\}_{t=0}^{\infty}} \mathbb{E}_t \left[\int_0^{\infty} e^{-\rho^e t} \log(c_t^e) dt \right] \\ \text{s.t.} \quad & dn_t^e = \left[-c_t^e dt + n_t^e \left(dr_t + \theta_t^{K,e} (dr_t^{K,e} - r_t) \right) \right] \\ & (1 - \ell) \theta_t^{K,e} \leq 1, \\ & \theta_t^{K,e} \geq 0, \\ & n_0^e \text{ given} \end{aligned}$$

■ Household's problem

$$\begin{aligned} & \max_{\{c_t^h, \theta_t^{K,h}\}_{t=0}^{\infty}} \mathbb{E}_t \left[\int_0^{\infty} e^{-\rho^h t} \log(c_t^h) dt \right] \\ \text{s.t.} \quad & dn_t^h = \left[-c_t^h dt + n_t^h \left(dr_t + \theta_t^{K,h} (dr_t^{K,h} - r_t) \right) \right] \\ & \theta_t^{K,h} \geq 0, \\ & n_0^h \text{ given} \end{aligned}$$

Market Clearing

- Goods market

$$C_t^e + C_t^h = a^e K_t^e + a^h(\kappa_t^h) K_t^h$$

- Capital market

$$K_t^e + K_t^h = K_t$$

- Debt market
clears by Walras law

Solving Macro Models Step-by-Step

0 Postulate aggregates, price processes and obtain return processes

1 For given $\check{\rho} := c^i/n^i$ -ratio and SDF^i processes for each i

finance block

Toolbox 1: HJB, Stochastic Maximum Principle, Martingale-approach

a Goods market clearing (*static*)

b Fisher separation theorem

Portfolio choice θ + asset market clearing

2 Evolution of state variable η (and K)

forward equation

3 Value functions

backward equation

a Value fcn. as fcn. of individual investment opportunities ω
Special case: log-utility $\check{\rho} := c/n = \rho$

4 Numerical model solution

5 KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

■ Capital aggregation:

- Within sector i : $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$

- Across sectors: $K_t = \sum_i K_t^i$ $dK_t/K_t = gdt + \sigma dZ_t$

- Capital share: $\kappa_t^i = K_t^i/K_t$

■ Net worth aggregation:

- Within sector i : $N_t^i \equiv \int n_t^{\tilde{i},i} d\tilde{i}$

- Across sectors: $N_t = \sum_i N_t^i$

- Net worth share: $\eta_t^i = N_t^i/N_t$

■ Value of capital stock: $q_t K_t$,

$$dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$$

■ Postulated SDF-process:

$$\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$$

0. Postulate Aggregates and Processes

■ Capital aggregation:

- Within sector i : $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$
 - Across sectors: $K_t = \sum_i K_t^i$
 - Capital share: $\kappa_t^i = K_t^i / K_t$
- $$dK_t / K_t = g dt + \sigma dZ_t$$

■ Net worth aggregation:

- Within sector i : $N_t^i \equiv \int n_t^{\tilde{i},i} d\tilde{i}$
- Across sectors: $N_t = \sum_i N_t^i$
- Net worth share: $\eta_t^i = N_t^i / N_t$

■ Value of capital stock: $q_t K_t$,

$$dq_t / q_t = \mu_t^q dt + \sigma_t^q dZ_t$$

■ Postulated SDF-process:

$$\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$$

Poll: Why drift of SDF equal is risk-free rate

- 1 no idio risk
- 2 $e^{-r_f} = \mathbb{E}[SDF]$
- 3 no jump in consumption

0. Return Processes

- Using Itô's product rule, the returns on capital follow

$$\begin{aligned} dr_t^{K,e} &= \overbrace{\frac{a^e}{q_t} dt}^{\text{Dividend Yield}} + \overbrace{\frac{d(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\ &= \left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t \\ dr_t^{K,h} &= \left(\frac{a^h(\kappa_t^h)}{q_t} + g + \mu_t^q + \sigma \sigma_t^q \right) dt + (\sigma + \sigma_t^q) dZ_t \end{aligned}$$

2. GE: Markov States and Equilibria

- Equilibrium is a **map**

Histories of shocks

$\{\mathbf{z}_{s \in [0, t]}\}$

→

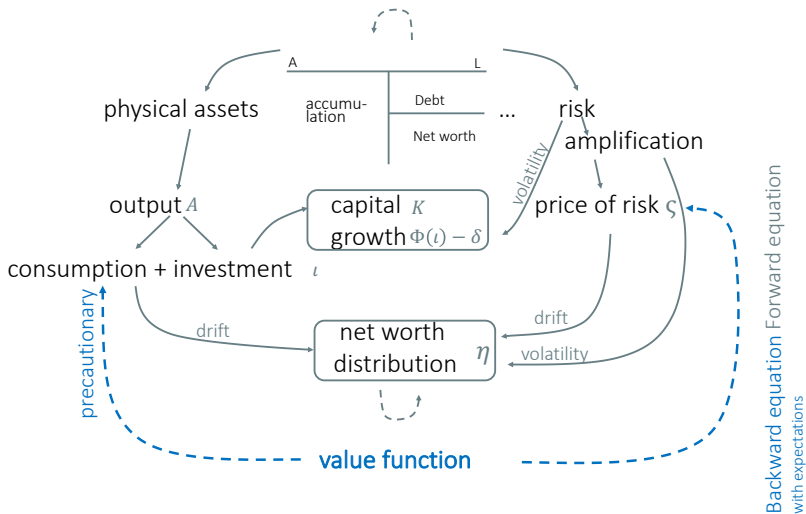
prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

net worth distribution

$$\eta_t^e = \frac{N^e}{q_t K_t} \in (0, 1)$$

- All agents maximize utility
 - Choose: consumption, portfolio, ...
- All markets clear
 - Consumption, capital, debt,

The Big Picture



Overview: This Lecture

- **Complete Markets Benchmark**
 - $\sigma = 0$ and ℓ unconstrained
- Basak Cuoco Model
 - $a^h \rightarrow -\infty$ and ℓ unconstrained
- Kiyotaki-Moore Model
 - $\sigma = 0$ and $\ell \leq 1$

1. Individual Agent Choice (Complete Markets Benchmark)

- $a^e \geq a^h(1 - \kappa_t) \Rightarrow \theta_t^{K,h} \geq 0$ must bind $\Rightarrow \kappa_t = 1$
- By log-utility $\check{\rho}^i := c_t^i/n_t^i = \rho^i \Rightarrow C_t^i = \rho^i N_t^i$ for $i \in \{e, h\}$.
Using $N_t^e = \eta_t q_t K_t$ and $N_t^h = (1 - \eta_t) q_t K_t$, market clearing implies

$$q_t = \frac{a^e \kappa_t + a^h(1 - \kappa_t)(1 - \kappa_t)}{\rho^e \eta_t + \rho^h(1 - \eta_t)}$$

- Using $\kappa_t = 1$ in equilibrium

$$q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1 - \eta_t)}.$$

- Since there is no leverage constraint, the experts instantly take out loans to finance the purchase of all available capital in the economy.
Capital market clearing condition

$$\theta_t^{K,e} \eta_t q_t K_t + \theta_t^{K,h} (1 - \eta_t) q_t K_t = q_t K_t,$$

implies that $\theta_t^{K,e} = 1/\eta_t$ instantaneously.

2. Evolution of state variable η_t

- Recall $\eta_t = N_t^e/N_t$ so that by Itô's lemma

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1 - \eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

since $\mu_t^N = \frac{1}{N_t} \frac{dN_t}{dt} = \eta_t \mu_t^{N^e} + (1 - \eta_t) \mu_t^{N^h}$.

- The evolution of experts' net worth

$$\mu_t^{N^e} = -\rho^e + r_t + \theta^{K,e}(r_t^{K,e} - r_t) = -\rho^e + r_t$$

since $r_t^{K,e} = r_t$.

- The evolution of households' net worth

$$\mu_t^{N^h} = -\rho^h + r_t + \theta^{K,h}(r_t^{K,h} - r_t) = -\rho^h + r_t$$

since $\theta^{K,h} = 0$.

- Hence, net worth share follows

$$\mu_t^\eta = -(1 - \eta_t)(\rho^e - \rho^h)$$

which yields the following closed-form expression for η_t

$$\eta_t = \frac{\eta_0 e^{-(\rho^e - \rho^h)t}}{1 - \eta_0 + \eta_0 e^{-(\rho^e - \rho^h)t}}$$

for initial condition η_0 .

Frictionless Case: Main Takeaways

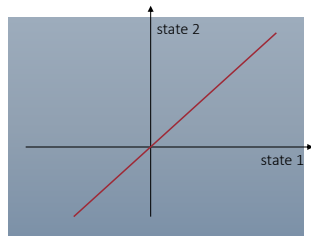
- 1 Capital Allocation:** always held by the most efficient sector
- 2 Consumption Allocation:** determined by the initial wealth distribution and wealth only moves due to differences in preferences for the timing of consumption
 - $\rho^e = \rho^h$: every initial condition leads to a steady state
 - $\rho^e > \rho^h$: converges in the long run to a boundary $\eta = 0$
 - $\rho^e < \rho^h$: converges in the long run to a boundary $\eta = 1$
- 3 Price of Capital:** $q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1 - \eta_t)}$
 - $\rho^e = \rho^h$: $q_t = a^e / \rho^e$ is constant
 - otherwise, it rises over time because the agents with the lower marginal propensity to consume become richer as time passes.

Overview: This Lecture

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- **Basak Cuoco Model**
 - $a^h \rightarrow -\infty$ and ℓ unconstrained
- Kiyotaki-Moore Model
 - $\sigma = 0$ and $\ell \leq 1$

Financial Frictions and Distortions

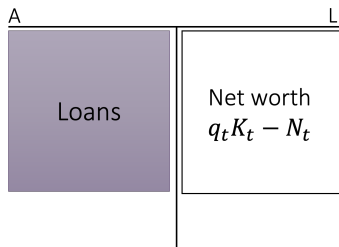
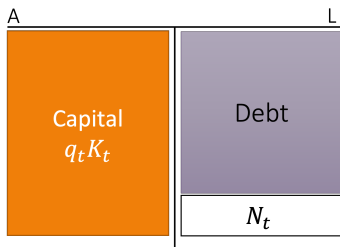
- **Incomplete markets:**
 - “natural” leverage constraint (*BruSan*)
 - Costly state verification (*BGG*)
- Leverage constraints
(no “liquidity creation”)
 - Exogenous limit (*Bewley/Ayagari*)
 - Collateral constraint
 - Current price $Rb_t \leq q_t k_t$
 - Next period’s price $Rb_t \leq q_{t+dt} k_t$ (*KM*)
 - Next period’s VaR $Rb_t \leq VaR_t(q_{t+dt}) k_t$ (*BruPed*)



Simple Two Sector Model: Basak Cuoco (1998)

- Special case with
 - risk $\sigma > 0$
 - no leverage constraint ℓ
 - unproductive household sector $a^h \rightarrow -\infty$
- Expert sector

Household sector



See Lecture Notes, Chapter 3 or Handbook of Macroeconomics 2017, Chapter 18

1. Given SDF processes, derive individual FOC

- Hamiltonian for the experts is given by

$$\begin{aligned}\mathcal{H}_t^e &= e^{-\rho^e t} \log c_t^e + \xi_t^e n_t^e \mu_t^{n^e} - \varsigma_t^e \xi_t^e n_t^e \sigma_t^{n^e} \\ &= e^{-\rho^e t} \log c_t^e + \xi_t^e \left[-c_t^e + n_t^e r_t + n_t^e \theta_t^{K,e} \left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q - r_t \right) \right] \\ &\quad - \varsigma_t^e \xi_t^e n_t^e \theta_t^{K,e} (\sigma + \sigma_t^q)\end{aligned}$$

- FOCs wrt c_t^e and $\theta_t^{K,e}$ are given by

$$\begin{aligned}e^{-\rho^e t} (c_t^e)^{-1} &= \xi_t^e \\ \frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q - r_t &= \varsigma_t^e (\sigma + \sigma_t^q)\end{aligned}$$

- The costate equation reads by virtue of the

$$\begin{aligned}d\xi_t^e &= -\frac{\partial \mathcal{H}_t^e}{\partial n_t^e} dt - \varsigma_t^e \xi_t^e dZ_t \\ &= -r_t \xi_t^e - \varsigma_t^e \xi_t^e dZ_t\end{aligned}$$

where the equality follows from the first-order condition for $\theta_t^{K,e}$.

1. Given SDF processes, derive individual FOC

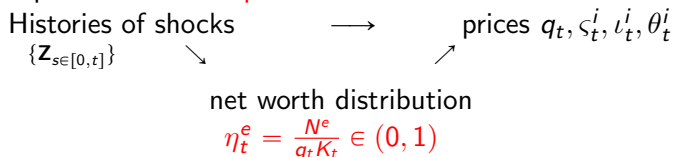
- Using the first-order condition for consumption, we also find by Itô's lemma that

$$d\xi_t^e = [-\rho^e - \mu_t^{c^e} + (\sigma_t^{c^e})^2] dt - \sigma_t^{c^e} dZ_t$$

- Since for log-utility $c_t^e/n_t^e = \rho^e$, $\mu_t^{c^e} = \mu_t^{n^e}$ and $\sigma_t^{c^e} = \sigma_t^{n^e}$.
- Then the price of risk is given by $\zeta_t^e = \sigma_t^{c^e} = \sigma_t^{n^e} = \theta_t^{K,e}(\sigma + \sigma_t^q)$.
- Similarly, the household sector will consume according to $c_t^h/n_t^h = \rho^h$.

2. GE: Markov States and Equilibria

- Equilibrium is a **map**



- All agents maximize utility
 - Choose consumption, portfolio, ...
- All markets clear
 - Consumption, capital, debt,

2. Evolution of state variable η_t

- Recall that $\eta_t = \frac{N_t^e}{q_t K_t}$
- The total wealth of experts N_t^e follows

$$\begin{aligned}\frac{dN_t^e}{N_t^e} &= \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e} dt + r_t dt + \theta_t^{K,e} \left[dr_t^K - r_t dt \right] \\ &= -\frac{c_t^e}{n_t^e} dt + r_t dt + \theta_t^{K,e} \left\{ \left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q - r_t \right] dt + (\sigma + \sigma_t^q) dZ_t \right\} \\ &= -\frac{c_t^e}{n_t^e} dt + r_t dt + \theta_t^{K,e} \left\{ \varsigma_t^e (\sigma + \sigma_t^q) dt + (\sigma + \sigma_t^q) dZ_t \right\}.\end{aligned}$$

- Also

$$\begin{aligned}\frac{d(q_t K_t)}{q_t K_t} &= [\mu_t^q + g + \sigma \sigma_t^q] dt + (\sigma + \sigma_t^q) dZ_t \\ &= \left[r_t - \frac{a^e}{q_t} + \varsigma_t^e (\sigma + \sigma_t^q) \right] dt + (\sigma + \sigma_t^q) dZ_t.\end{aligned}$$

- Apply Itô's quotient rule to $\eta_t = N_t^e / q_t K_t$:

$$\frac{d\eta_t}{\eta_t} = \left[-\frac{c_t^e}{n_t^e} + \frac{a^e}{q_t} - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)(\varsigma_t^e - (\sigma + \sigma_t^q)) \right] dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q) dZ_t$$

2. Evolution of state variable η_t

- Using $c_t^e = \rho^e n_t^e$ and $\zeta_t^e = \theta_t^{K,e}(\sigma + \sigma_t^q)$, we have

$$\frac{d\eta_t}{\eta_t} = \left[-\rho^e + \frac{a^e}{q_t} + (1 - \theta_t^{K,e})^2(\sigma + \sigma_t^q)^2 \right] dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t$$

- Goods market clearing

$$C_t = \rho^e \eta_t q_t K_t + \rho^h (1 - \eta_t) q_t K_t = a^e K_t \implies q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1 - \eta_t)}$$

- Itô's lemma implies

$$\sigma_t^q = -\frac{\rho^e - \rho^h}{\rho^e \eta_t + \rho^h (1 - \eta_t)} \sigma_t^\eta \eta_t$$

- Capital market clearing

$$\theta_t^{K,e} = \frac{q_t K_t}{N_t^e} = \frac{1}{\eta_t}$$

- Hence η_t follows

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left[-(\rho^e - \rho^h) + \frac{1 - \eta_t}{\eta_t^2} (\sigma + \sigma_t^q)^2 \right] dt + \frac{1 - \eta_t}{\eta_t} (\sigma + \sigma_t^q) dZ_t$$

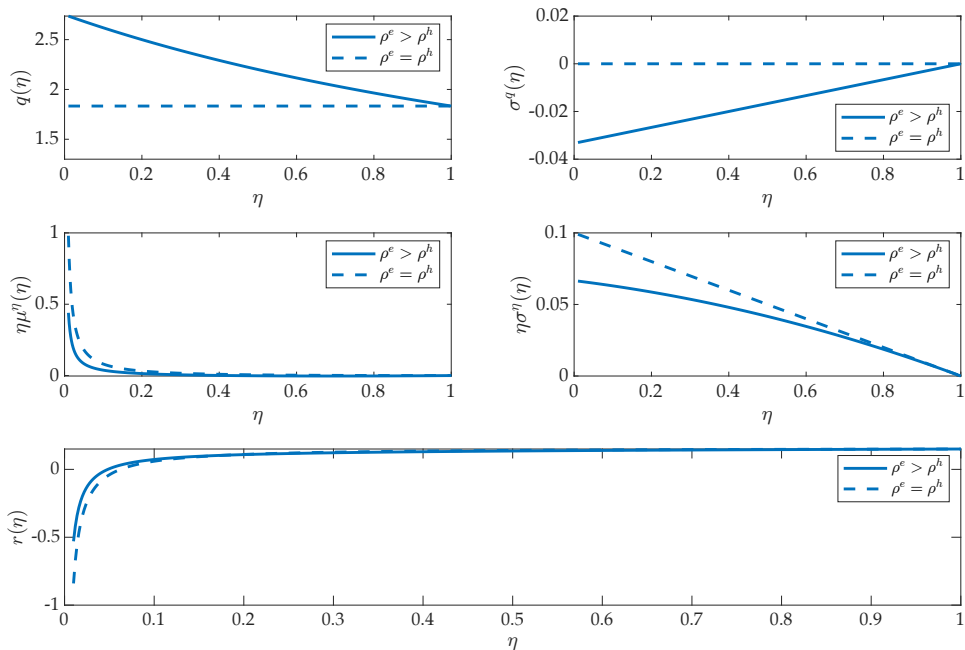
2. Evolution of state variable η_t

- Combining the above we get

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left[-(\rho^e - \rho^h) + \frac{1 - \eta_t [\rho^e \eta_t + \rho^h (1 - \eta_t)]^2}{\eta_t^2 (\rho^e)^2} \sigma^2 \right] dt + \frac{1 - \eta_t \rho^e \eta_t + \rho^h (1 - \eta_t)}{\eta_t \rho^e} \sigma dZ_t$$

a simple one-dimensional stochastic differential equation (SDE).

4. Numerical Example of Basak-Cuoco



$$\rho^e = 0.06, \rho^h = 0.04, a^e = 0.11, \sigma = 0.10, g=0.1$$

Observation of Basak-Cuoco Model

1 Changes in Risk-free Rate drives Risk Premium

Price of risk ζ^e , i.e., Sharpe ratio is:

$$\frac{1}{\eta_t}(\sigma + \sigma_t^q) = \frac{\eta_t \rho^e + (1 - \eta_t) \rho^h + g + \mu_t^q + \sigma \sigma_t^q - r_t}{\sigma + \sigma_t^q}$$

- Goes to ∞ as η_t goes to zero
- RHS can only go to ∞ if risk-free rate $r_t \rightarrow \infty$
 $r_t = \eta_t \rho^e + (1 - \eta_t) \rho^h + g + \mu_t^q + \sigma \sigma_t^q - (\sigma + \sigma_t^q)^2 / \eta_t \rightarrow -\infty$
- **Special case** $\rho^e = \rho^h$: since $q = \frac{a^e}{\rho^e}$ is constant ($\mu^q = \sigma^q = 0$), easy to see.

2 q_t Movements Mitigates rather than Amplifies Shocks, $\sigma^q \leq 0$

Stationary distribution requires $\rho^e > \rho^h \Rightarrow q_t$ appreciates after $dZ_t < 0$ -shock

- Otherwise for $\rho^e \leq \rho^h$, $\mu_t^\eta > 0 \forall \eta$, $\Rightarrow \eta^{SS} = 1$
in the long run HH-net worth share vanishes (degenerated stationary distribution)
- Alternative ways out to obtain a stationary distribution without $q(\eta)$ decreasing in η (instead of $\rho^e > \rho^h$):
 - Switching types (jumps) (BGG)
 - 2 types of experts (BruSan AEJ:Macro international paper)

Overview: This Lecture

- Complete Markets Benchmark
 - $\sigma = 0$ and ℓ unconstrained
- Basak Cuoco Model
 - $a^h \rightarrow -\infty$ and ℓ unconstrained
- **Kiyotaki-Moore Model**
 - $\sigma = 0$ and $\ell \leq 1$, and $a^h(\cdot)$ is a function of κ^h

Financial Frictions and Distortions

- Incomplete markets:

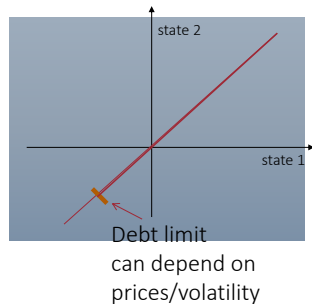
- “natural” leverage constraint (*BruSan*)
- Costly state verification (*BGG*)

- **Leverage constraints:**

(no “liquidity creation”)

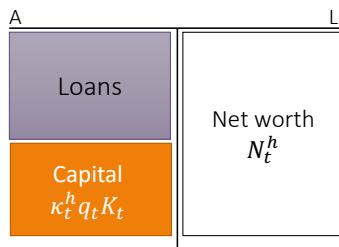
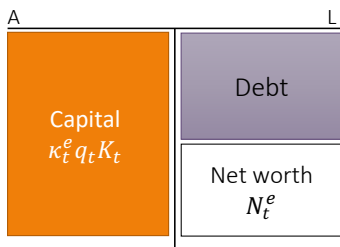
- Exogenous limit (*Bewley/Ayagari*)
- Collateral constraint

- Current price $Rb_t \leq q_t k_t$
- Next period's price $Rb_t \leq q_{t+dt} k_t$ (*KM*)
- Next period's VaR $Rb_t \leq \text{VaR}_t(q_{t+dt}) k_t$ (*BruPed*)



Kiyotaki Moore (1997) in Cts. Time

- No risk: $\sigma = 0$
- Leverage constraint: $\ell \leq 1$
- Households can produce: $a^h(\kappa^h)$ is a decreasing function of κ^h
- Expert sector (Farmers) Household sector (Gatherers)



Kiyotaki Moore (1997) in Cts. Time

Expert Sector (Farmers)

- Output: $y_t^e = a^e k_t^e = a^e \kappa_t^e K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^e t} \log(c_t^e) dt$

Household Sector (Gatherers)

- Output: $y_t^h = a^h(\kappa_t^h) k_t^h = a^h(\cdot) \kappa_t^h K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^h t} \log(c_t^h) dt$

Assumptions:

- Experts are more impatient $\rho^e > \rho^h$
 - Productivity $a^h(\kappa^h) \leq a^e$ with equality for $\kappa^h = 0$ and constant return to scale individually, but $a^h(\cdot)$ decreasing in (aggregate) κ^h
 - No equity issuance
 - Debt issuance with **collateral constraint**: $D_t^e \leq \ell \kappa_t^e q_t K_t$
 $\Leftrightarrow \frac{D_t^e}{N_t^e} \leq \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1 - \theta_t^{K,e}) \leq \ell \theta_t^{K,e}$
- Collateral constraint in KM97: $D_t^e(1 + r_{t+dt}) \leq \ell \kappa_{t+dt}^e q_{t+dt} K_t$

1. Portfolio choices: Hamiltonian Approach

- Experts' problem: $\max_{c_t^e, \theta_t^{K,e}} \int_s^\infty e^{-\rho^e t} u(c_t^e) dt$ s.t. $(1 - \ell)\theta_t^{K,e} \leq 1$, and

$$\frac{dn_t^e}{dt} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e} (r_t^{K,e} - r_t) \right) \right]$$

- Households' problem: $\max_{c_t^h, \theta_t^{K,h}} \int_s^\infty e^{-\rho^h t} u(c_t^h) dt$, s.t.

$$\frac{dn_t^h}{dt} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h} (r_t^{K,h} - r_t) \right) \right],$$

- The Hamiltonians can be constructed as

$$\mathcal{H}_t^e = e^{-\rho^e t} u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e} (r_t^{K,e} - r_t) \right) \right]}^{\mu_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left(1 - (1 - \ell)\theta_t^{K,e} \right)$$
$$\mathcal{H}_t^h = e^{-\rho^h t} u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h} (r_t^{K,h} - r_t) \right) \right]$$

- ξ_t^i multiplier on the budget constraint, $\xi_t^e n_t^e \lambda_t^\ell$ multiplier on leverage constraint
 - Later we show that co-state variable ξ_t^i equals SDF, which for log-utility = $e^{-\rho^i t} \frac{1}{\rho^i n_t^i}$
- Fisher Separation Theorem btw. consumption and portfolio choice

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i :

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{ log utility}$$

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i :

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{ log utility}$$

- FOC w.r.t $\theta_t^{K,i}$:

$$\begin{cases} r_t^{K,e} - r_t = (1 - \ell) \lambda_t^\ell \\ r_t^{K,h} - r_t = 0 \end{cases}$$

- Where capital returns are: (dividend + price drift)

$$\begin{cases} r_t^{K,e} = \frac{a^e}{q_t} + \frac{1}{q_t} \frac{dq_t}{dt} \\ r_t^{K,h} = \frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t} \frac{dq_t}{dt} \end{cases}$$

2. Net Worth Evolution

- Equilibrium objects are functions of state, net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t K}$
- Price dynamics: (No arbitrage for households)

$$\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t} \frac{dq_t}{dt} = r_t,$$

- State dynamics:

$$\mu_t^N dt = \frac{dN_t}{N_t} = \underbrace{\frac{N_t^e}{N_t}}_{\eta_t} \mu_t^{N^e} dt + \underbrace{\frac{N_t^h}{N_t}}_{(1-\eta_t)} \mu_t^{N^h} dt$$

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1 - \eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

$$\begin{aligned} &= (1 - \eta_t) \left[-(\rho^e - \rho^h) + \theta_t^{K,e} \left(\frac{a^e}{q_t} + \frac{1}{q_t} \frac{dq_t}{dt} - r_t \right) - \theta_t^{K,h} \left(\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t} \frac{dq_t}{dt} - r_t \right) \right] \\ &= (1 - \eta_t) \left[-(\rho^e - \rho^h) + \theta_t^{K,e} \left(\frac{a^e}{q_t} - \frac{a^h(\kappa_t^h)}{q_t} \right) \right] \\ & \quad \underbrace{\hspace{10em}}_{=r_t^{K,e} - r_t^{K,h}} \end{aligned}$$

=0 from above

2. Net Worth Evolution

- Equilibrium objects $(\kappa^e, \kappa^h, q, r)$ are functions of state, net worth share,

$$\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t K}$$

- pinned down by:

$$q_t K_t [\rho^e \eta_t + \rho^h (1 - \eta_t)] = [a^e \kappa_t^e + a^h (\kappa_t^h) \kappa_t^h] K_t \quad (\text{Goods market})$$

$$\underbrace{\theta_t^{Ke} \eta_t}_{=\kappa_t^e} q_t K_t + \underbrace{\theta_t^{Kh} (1 - \eta_t)}_{=\kappa_t^h} q_t K_t = q_t K_t \quad (\text{Capital market})$$

$$\kappa_t^e \leq \frac{\eta_t}{1 - \ell} \quad (\text{Collateral Constraint})$$

$$\mu_t^\eta = (1 - \eta_t) \left[-(\rho^e - \rho^h) + \theta_t^{K,e} \frac{a^e - a^h (\kappa_t^h)}{q_t} \right]$$

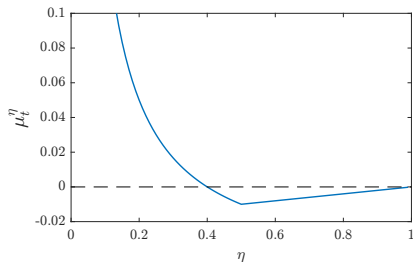
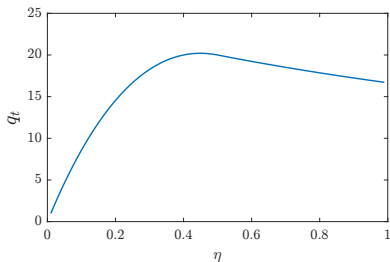
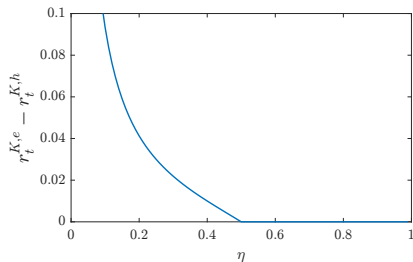
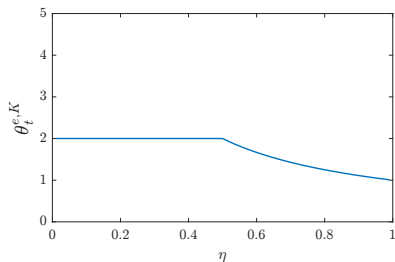
- simplified to (and define $\kappa_t := \kappa_t^e = 1 - \kappa_t^h$)

$$q_t [(\rho^e - \rho^h) \eta_t + \rho^h] = \kappa_t a^e + (1 - \kappa_t) a^h (1 - \kappa_t)$$

$$\kappa_t \leq \frac{\eta_t}{1 - \ell}$$

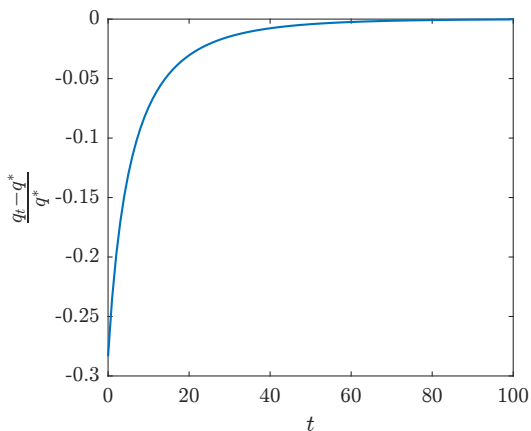
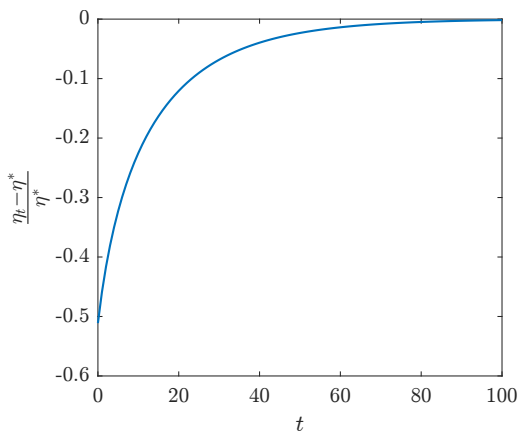
$$\mu_t^\eta = (1 - \eta_t) \left[-(\rho^e - \rho^h) + \frac{\kappa_t a^e - a^h (1 - \kappa_t)}{\eta_t q_t} \right]$$

Global Non-linear Solution



Parameters: $\rho^e = 0.06$, $\rho^h = 0.04$, $\ell = 0.05$, $a^e = 1.0$, $a^h(1 - \kappa) = a^e\kappa$

Impulse Responses



Impulse response function with 30% (of η) negative redistribution shock.

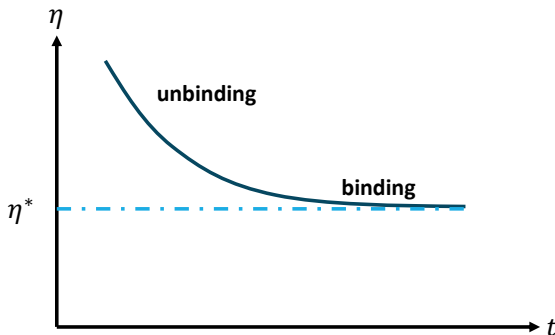
Parameters: $\rho^e = 0.06$, $\rho^h = 0.04$, $\ell = 0.5$, $a^e = 1.0$, $a^h(1 - \kappa) = a^e \kappa$

Log-linearization around Steady State

- 1 Derive steady state with $\mu^n = 0$
with its properties
- 2 Log-linearize around steady state
characterize dynamical system locally around the steady state

The Steady State: Binding Collateral Constraint

- The collateral constraint always binds in the steady state
 - If collateral constraint does not bind $\lambda_t^l = 0$ and hence $r^{K,e} = r^{K,h}$, i.e. $a^e = a^h(\cdot)$
- Note, the constraint does not need to bind only if $\kappa_t = 1$.
 - Then $\mu_t^\eta = (1 - \eta_t)(\rho^h - \rho^e)$
 - as $\rho^e > \rho^h \Rightarrow \mu_t^\eta < 0$, i.e. η declines
- Characterization of Steady State (Next Page)



Steady State

- Since Collateral constrained binds, steady state capital share

$$\kappa^{SS} = \frac{\eta^{SS}}{1-\ell}$$

- Expert sector's net worth share is $\eta_t := \frac{N_t^e}{q_t K}$, is constant, i.e. $\mu_t^\eta := \frac{d\eta_t}{dt} = 0$

$$q^{SS}[(\rho^e - \rho^h)\eta^{SS} + \rho^h] = \kappa^{SS} a^e + (1 - \kappa^{SS}) a^h (1 - \kappa^{SS})$$

$$(\rho^e - \rho^h) = \frac{\kappa^{SS} a^e - a^h (1 - \kappa^{SS})}{\eta^{SS} q^{SS}} \quad \text{for } \mu^\eta = 0$$

- Combine $\kappa^{SS} a^e - \kappa^{SS} a^h (1 - \kappa^{SS}) + q^{SS} \rho^h = \kappa^{SS} a^e + (1 - \kappa^{SS}) a^h (1 - \kappa^{SS})$
 $\Rightarrow q^{SS} = a^h (1 - \kappa^{SS}) / \rho^h$,

where the steady state κ^{SS} is implicitly given by:

$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell} \frac{a^e - a^h (1 - \kappa^{SS})}{a^h (1 - \kappa^{SS})}$$

- For specific functional form $a^h(\cdot) = a^e \kappa_t$:

$$\kappa^{SS} = \frac{1}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1} \Rightarrow \eta^{SS} = \frac{1-\ell}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1}$$

Steady State: Comparative Static

- For the specific example $a^h(\cdot) = a^e \kappa$:
- For higher leverage, ℓ , (i.e. less tight collateral constraint)
 - κ^{SS} , SS-capital share, is higher.
 - η^{SS} , SS-net worth share, is lower.
 - $q^{SS} = \frac{a^h}{\rho^h}$, price of capital, is higher.
 $q^{SS} \bar{K}$, total wealth in the economy, is higher too.
 - $N^{e,SS}$ experts' net worth in steady state, is higher (Check?)
 - Comparative Static = permanent (long-run) shift to new steady state
 - Next: Dynamics of how to return to the old steady state
(after an unanticipated shock)

Log-linearized Dynamics Around Steady State

- Analytical solutions to η_t, q_t dynamics are hard to obtain. Expansion around the steady state:

$$\log(\eta_t/\eta^{SS}) = \hat{\eta}_t$$

$$\log(q_t/q^{SS}) = \hat{q}_t$$

$$\log(r_t/r^{SS}) = \hat{r}_t$$

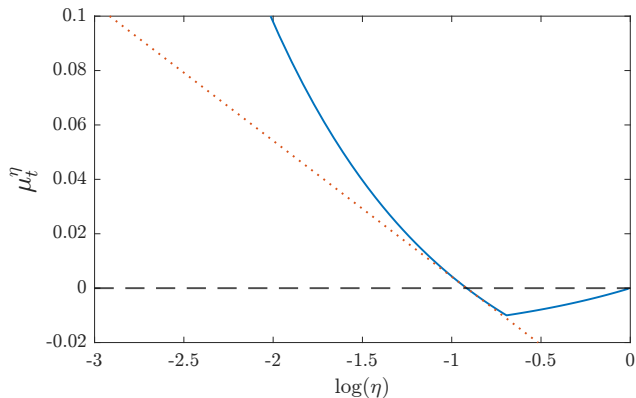
$$\log(a_t^h/a^{h,SS}) = \hat{a}_t^h$$

- Expression for \hat{a}_t^h, \hat{q}_t^h as a function of $\hat{\eta}_t$
- State dynamics and price dynamics become:

$$\frac{d\hat{\eta}_t}{dt} = \frac{1 - \eta^{SS}}{1 - \ell} \left(-\frac{a^{h,SS}}{q^{SS}} \hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}} \hat{q}_t \right)$$

$$\frac{d\hat{q}_t}{dt} = r^{SS} (\hat{r}_t + \hat{q}_t - \hat{a}_t^h)$$

Global vs. Log-linearized Solution for η -drift

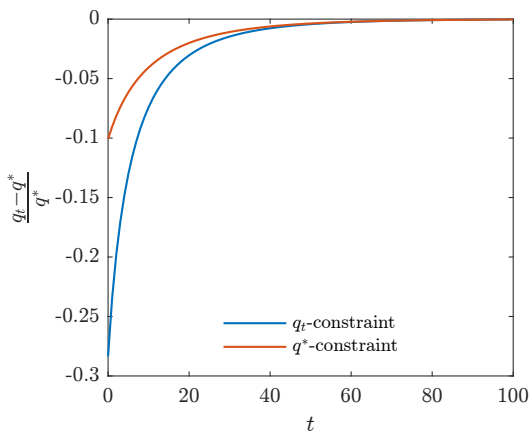
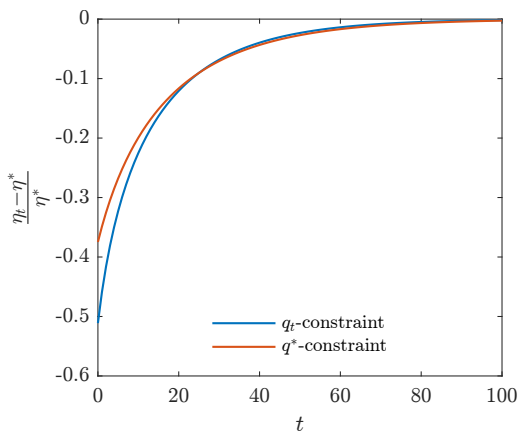


- Note: x-axis is $\log(\eta)$, since log-linearization

Decomposing Amplification Effects

- Start at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$
- Shock: redistribution of a fraction of experts' net worth share to households
 - In KM productivity shock lasts for one period (not for an instant), causes initial redistribution
- **Impulse response function** (with deterministic recovery)
- Immediate impact at $t = 0$
 - direct redistributive effect/shock
 - **price-net worth effect**
decline in q_t reduces experts' net worth share as they are levered \Rightarrow feedback
 - **price-collateral effect**
decline in q_t tightens collateral constraints \Rightarrow feeds back on price-net worth effect
- Subsequent impact $t > 0$ (which feeds back to immediate impact)
- **Decomposition:**
Switch off price-collateral effect by assuming that collateral constraint is determined by **SS-price** q^{SS} instead of **equilibrium price** q_t .
(Formally, collateral constraint, $\kappa_t \leq \frac{\eta_t}{1-\ell}$, becomes $\kappa_t \leq \frac{\eta_t}{1-\ell q^{SS}/q_t}$.)

Decomposition of Amplification: Impulse Response Fcn



Impulse response function with 30% (of η) negative redistribution shock.

Parameters: $\rho^e = 0.06$, $\rho^h = 0.04$, $\ell = 0.5$, $a^e = 1.0$, $a^h(1 - \kappa) = \kappa$

Decomposing Amplification at $t = 0$

- At time t , the economy is at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$.
- Negative initial/direct redistributive shock $\eta' = (1 - \epsilon)\eta^{SS}$, new price q' , and capital holding κ' solves:

$$q' = \frac{\kappa' a^e + (1 - \kappa') a^h (1 - \kappa')}{(\rho^e - \rho^h) \eta' + \rho^h} \quad (\text{Goods market})$$

$$\kappa' = \frac{\eta^{SS} (1 - \epsilon)}{1 - \ell} \quad (q_t\text{-constraint})$$

$$\kappa' = \frac{\eta^{SS} (1 - \epsilon)}{1 - \ell q^{SS} / q'} \quad (q^{SS}\text{-constraint})$$

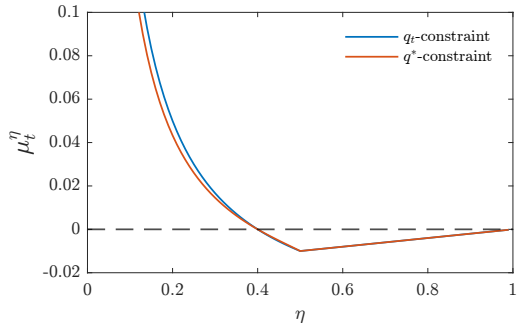
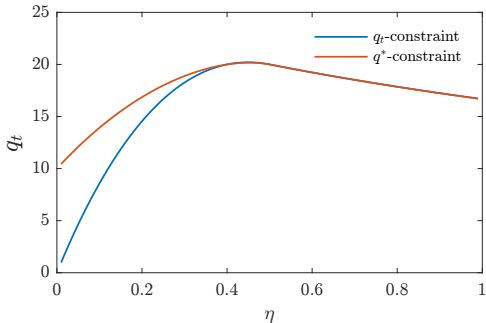
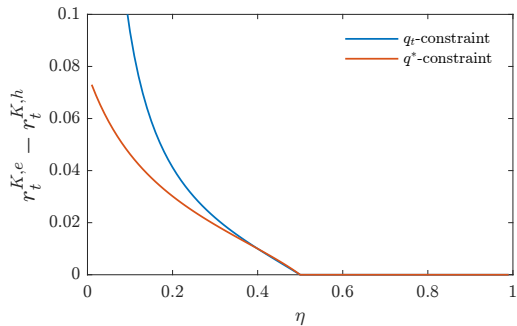
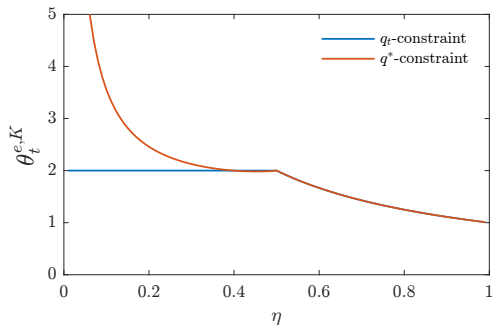
- However, debt contract was signed by old price $q^{SS} \Rightarrow \eta$ drops further
- Consider the balance sheet (first round effect):

$$\frac{\eta'}{1 - \ell} q' = \frac{\ell}{1 - \ell} \eta' q^{SS} + \eta'' q'$$

To get the convergence result, we need to do this procedure iteratively.

Decomposing Amplification for $t > 0$ (global solution)

$$\rho^e = 0.06, \rho^h = 0.04, \ell = 0.05, a^e = 1.0, a^h(1 - \kappa) = a^e \kappa$$



Decomposing Amplification for $t > 0$ (log-linearized sol.)

- Price dynamics:

$$\frac{d\hat{q}_t}{dt} = r^{SS} \hat{r}_t - r^{SS} \hat{a}_t^h + r^{SS} \hat{q}_t$$

- State dynamics with q_t -collateral constraint:

$$\frac{d\hat{\eta}_t}{dt} = \frac{1 - \eta^{SS}}{1 - \ell} \left(-\frac{a^{h,SS}}{q^{SS}} \hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}} \hat{q}_t \right)$$

- State dynamics with q^{SS} -collateral constraint:

$$\frac{d\hat{\eta}_t}{dt} = \frac{1 - \eta^{SS}}{1 - \ell} \left(-\frac{a^{h,SS}}{q^{SS}} \hat{a}_t^h - \frac{1}{1 - \ell} \frac{a^e - a^{h,SS}}{q^{SS}} \hat{q}_t \right)$$

$\hat{q}_t, \hat{a}_t^h, \hat{r}_t$ are different with different constraints.

“Single Shock Critique”

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
 - Length of slump is deterministic (and commonly known)
 - No safety cushion needed
- In reality an adverse shock may be followed by additional adverse shocks
 - Build-up extra safety cushion for an additional shock in a crisis
- Impulse response vs. volatility dynamics

Overview: This Lecture

Real Macroeconomics Models with Heterogeneous Agents

A Two-Sector Macroeconomics Model

- Complete Markets Benchmark
 - No risk and no frictions
- Basak-Cuoco Model
 - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
 - No risk, two production sectors and leverage constraints

- Physical Investment

Adding Investments/Physical Capital Formation

- Instead of exogenous capital stock, convert goods into physical capital (not buying it from others)
- Capital conversion function $\Phi(\iota)$ (increasing and concave)

$$dk_t^i = (\Phi(\iota_t^i)k_t^i - \delta k_t^i)dt + \Delta_t^{k,i}k_t^i$$

- ι_t^i is investment rate of **new** physical capital (real investment is $\iota_t^i k_t^i$)
 - occurs within the period (no “time-to-build”) \Rightarrow static problem
- $\Delta_t^{k,i}k_t^i$ is the purchase/sale of physical capital at q_t from/to others
 - Hint: $\Delta_t^{k,i}k_t^i$ doesn't impact return on capital $r_t^{K,i}$
 - δ is the depreciation rate of capital
- Optimal investment rate ι_t^i depends on price of physical capital q_t .
 - Tobin's Q:

$$q_t = 1/\Phi'(\iota_t)$$

- attractive functional form with adjustment cost ϕ :
$$\Phi(\iota) = \frac{1}{\phi} \log(\phi\iota + 1)$$
- K_t is a second state variable, which can be solved ex-post
- Homework: Redo continuous time KM analysis with ι -investment.

Bernanke, Gertler, Gilchrist 1999

- Fully fledged DSGE Model with price stickiness, idiosyncratic firm risk, ...
- Aggregate shocks are unanticipated zero-probability shocks (MIT shocks)
- No fire-sale to less productive household sector (unlike in KM97)
- Divestment: Convert physical capital back to consumption good at a cost (captured by $\Phi(\cdot)$ -adjustment cost function)
- Financial Frictions:
 - No equity issuance
 - Debt issues with costly state verification (instead of collateral constraint)
 - If firm defaults (after negative idiosyncratic shock), creditor has to pay cost to verify true (remaining) cash flow
 - Optimal contract is a debt contract (debt payoff is hockey stick function of cash flow)
 - De-facto borrowing firms pay verification costs in expectations (in form of higher interest rate/funding costs)
 - A negative aggregate shock, lowers firms' net worth \Rightarrow firm's default prob. rises \Rightarrow expected verification cost rise \Rightarrow Firms funding costs rise

Conclusion & Takeaways

■ Equity Issuance Friction

1 Incomplete Markets (Basak-Cuoco)

- No fire sales (since $a^h \rightarrow -\infty$)
- Asset Price constant (or appreciates after adverse shock $\sigma^q \leq 0$) \Rightarrow Fluctuating risk-free rate
- No net worth trap, fat tails, spillovers

■ Debt Issuance Friction (explicit)

2 Collateral Constraint (Kiyotaki-Moore)

- Fire sales (since $a^h(\cdot) > \infty$)
- Dynamic amplification
- Single zero-probability shock
- No risk dynamics, no net worth trap
- No Volatility Paradox, Minsky Hypothesis

■ Physical Investment/ Tobin's Q is within-period/static Decision

Extra Slides for KM97: Understanding Asset Prices

- Price dynamics (with some proper initial conditions):

$$\frac{1}{q_t} \frac{dq_t}{dt} + \frac{a^h(\kappa_t^h)}{q_t} = r_t,$$
$$q_t = \int_t^{\infty} e^{-\int_t^s r_u du} a^h(\kappa_s^h) ds$$

- Discrete time analogy:

$$\frac{q_{t+1} - q_t}{q_t} + \frac{a^h(\kappa_t^h)}{q_t} = r_t$$
$$q_t = \sum_{s=0}^{\infty} \left[\prod_{u=0}^s \frac{1}{(1 + r_{t+u})} \right] a^h(\kappa_{t+s}^h)$$

- Asset price = sum of discounted dividend flows.
- Asset prices are **solved backward**