Macrofinance

Lecture 03: A Simple Real Macro Model with Heterogeneous Agents

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Course Overview

Class Overview

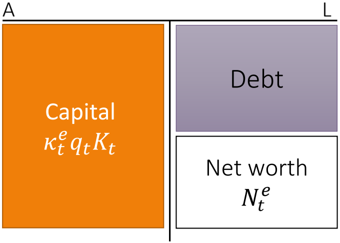
Real Macrofinance Models with Heterogeneous Agents

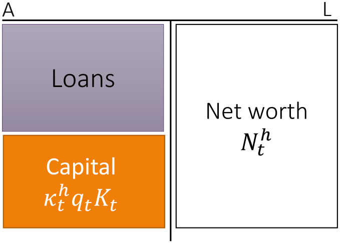
A Two-Sector Macrofinance Model

- Complete Markets Benchmark
  - No risk and no frictions
- Basak-Cuoco Model
  - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
  - No risk, two production sectors and leverage constraints

Two Sector Macrofinance Model Setup

- Expert sector (Farmers) Household sector (Gatherers)





- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- Experts produce with capital with linear production function $a^e k_t^e (= a^e \kappa_t^e K_t)$.
- Households' production function $a^h(\kappa_t^h) k_t^h$ is fcn. of (aggregate) κ_t^h
  - Productivity $a^h(\kappa^h) \leqslant a^e$ with equality for $\kappa^h = 0$ and strictly decreasing in κ^h
- Experts can only issue debt with collateral constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$
  - $\frac{D_t^e}{N_t^e} \leqslant \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1-\theta_t^{K,e}) \leqslant \ell \theta_t^{K,e} \Leftrightarrow (1-\ell)\theta_t^{K,e} \leqslant 1$
- All experts' net worth $N_t^e = \int_0^1 n_t^{e,i} di = n_t^e$; all households' net worth $N_t^h = n_t^h$
- Assumption: aggregate physical capital evolves exogenously

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t$$

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Portfolio choice $\theta_t^{K,e} \equiv \kappa_t^e \frac{q_t K_t}{N_t^e}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log\left(c_t^e\right) \mathrm{d}t\right]$

Household Sector

- Output: $y_t^h = a^h(\kappa_t^h) k_t^h$
- Consumption rate: c_t^h
- Portfolio choice $\theta_t^{K,h} \equiv \kappa_t^h \frac{q_t K_t}{N_t^h}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log\left(c_t^h\right) \mathrm{d}t\right]$

Individual capital evolution: $\frac{\mathrm{d}\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = g\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},i}$

Friction: Can only issue risk-free debt with collateral constraint $(1-\ell)\theta_t^{K,e} \leqslant 1$

Sector Optimization

- Expert's problem

$$\max_{\{c_t^e, \theta_t^{K,e}\}_{t=0}^{\infty}} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho^e t}\log(c_t^e)\mathrm{d}t\right]$$

$$\begin{aligned}
s.t.\quad & \mathrm{d}n_t^e = \left[-c_t^e \mathrm{d}t + n_t^e\left(\mathrm{d}r_t + \theta_t^{K,e}(\mathrm{d}r_t^{K,e} - r_t)\right)\right] \\
& (1-\ell)\theta_t^{K,e} \leqslant 1, \\
& \theta_t^{K,e} \geqslant 0, \\
& n_0^e \text{ given}
\end{aligned}$$

- Household's problem

$$\max_{\{c_t^h, \theta_t^{K,h}\}_{t=0}^{\infty}} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho^h t}\log(c_t^h)\mathrm{d}t\right]$$

$$\begin{aligned}
s.t.\quad & \mathrm{d}n_t^h = \left[-c_t^h \mathrm{d}t + n_t^h\left(\mathrm{d}r_t + \theta_t^{K,h}(\mathrm{d}r_t^{K,h} - r_t)\right)\right] \\
& \theta_t^{K,h} \geqslant 0, \\
& n_0^h \text{ given}
\end{aligned}$$

Market Clearing

- Goods market

$$C_t^e + C_t^h = a^e K_t^e + a^h(\kappa_t^h) K_t^h$$

- Capital market

$$K_t^e + K_t^h = K_t$$

- Debt market
  clears by Walras law

Solving Macro Models Step-by-Step

0 Postulate aggregates, price processes and obtain return processes

1 For given $\check{\rho} := c^i/n^i$-ratio and SDF^i processes for each i **finance block**

Toolbox 1: ~~HJB~~, Stochastic Maximum Principle, ~~Martingale approach~~

- a Goods market clearing *(static)*
- b Fisher separation theorem
  Portfolio choice θ + asset market clearing

2 Evolution of state variable η (and K) **forward equation**

3 Value functions **backward equation**

- a Value fcn. as fcn. of individual investment opportunities ω
  Special case: log-utility $\check{\rho} := c/n = \rho$

4 Numerical model solution

5 KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$ $\qquad \mathrm{d}K_t/K_t = g\mathrm{d}t + \sigma \mathrm{d}Z_t$
  - Capital share: $\kappa_t^i = K_t^i/K_t$
- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $N_t = \Sigma_i N_t^i$
  - Net worth share: $\eta_t^i = N_t^i/N_t$,
- Value of capital stock: $q_t K_t$, $\qquad dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\qquad \frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$

0. Postulate Aggregates and Processes

- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$ $\quad dK_t/K_t = g dt + \sigma dZ_t$
  - Capital share: $\kappa_t^i = K_t^i / K_t$
- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $N_t = \Sigma_i N_t^i$
  - Net worth share: $\eta_t^i = N_t^i / N_t$,
- Value of capital stock: $q_t K_t$, $\quad dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$

Poll: Why drift of SDF equal is risk-free rate

1. no idio risk
2. $e^{-r_f} = \mathbb{E}[SDF]$
3. no jump in consumption

0. Return Processes

- Using Itô's product rule, the returns on capital follow

$$\mathrm{d}r_t^{K,e} = \overbrace{\frac{a^e}{q_t}\mathrm{d}t}^{\text{Dividend Yield}} + \overbrace{\frac{\mathrm{d}(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}}$$

$$= \left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t$$

$$\mathrm{d}r_t^{K,h} = \left(\frac{a^h(\kappa_t^h)}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t$$

2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks $\longrightarrow$ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

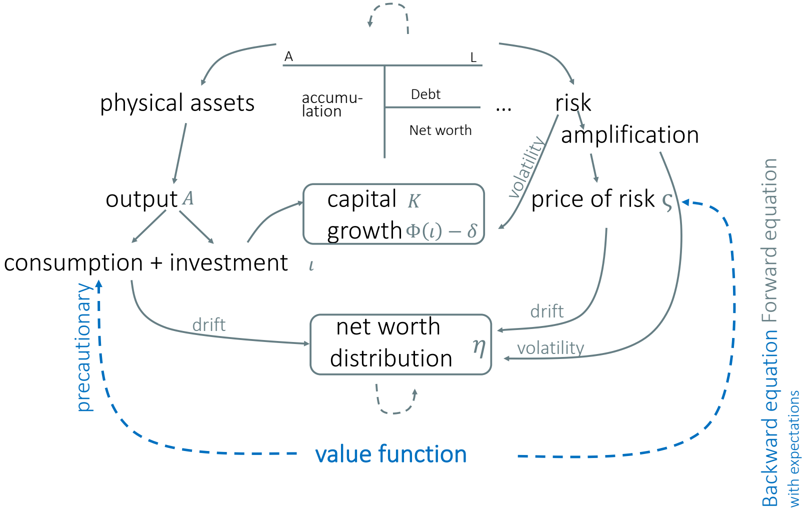
  $\{Z_{s\in[0,t]}\}$ $\searrow$ $\nearrow$

  net worth distribution

  $\eta_t^e = \frac{N^e}{q_t K_t} \in (0,1)$

- All agents maximize utility
  - Choose: consumption, portfolio, ...
- All markets clear
  - Consumption, capital, debt,

The Big Picture



Overview: This Lecture

- **Complete Markets Benchmark**
  - $\sigma = 0$ and ℓ unconstrained
- Basak Cuoco Model
  - $a^h \to -\infty$ and ℓ unconstrained
- Kiyotaki-Moore Model
  - $\sigma = 0$ and $\ell \leqslant 1$

1. Individual Agent Choice (Complete Markets Benchmark)

- $a^e \geqslant a^h(1-\kappa_t) \Rightarrow \theta_t^{K,h} \geqslant 0$ must bind $\Rightarrow \kappa_t = 1$
- By log-utility $\check{\rho}^i := c_t^i/n_t^i = \rho^i \implies C_t^i = \rho^i N_t^i$ for $i \in \{e, h\}$.
  Using $N_t^e = \eta_t q_t K_t$ and $N_t^h = (1-\eta_t) q_t K_t$, market clearing implies

$$q_t = \frac{a^e \kappa_t + a^h(1-\kappa_t)(1-\kappa_t)}{\rho^e \eta_t + \rho^h(1-\eta_t)}$$

- Using $\kappa_t = 1$ in equilibrium

$$q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

- Since there is no leverage constraint, the experts instantly take out loans to finance the purchase of all available capital in the economy.
  Capital market clearing condition

$$\theta_t^{K,e} \eta_t q_t K_t + \theta_t^{K,h}(1-\eta_t) q_t K_t = q_t K_t,$$

  implies that $\theta_t^{K,e} = 1/\eta_t$ instantaneously.

2. Evolution of state variable η_t

- Recall $\eta_t = N_t^e / N_t$ so that by Itô's lemma

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

  since $\mu_t^N = \frac{1}{N_t}\frac{\mathrm{d}N_t}{\mathrm{d}t} = \eta_t \mu_t^{N^e} + (1-\eta_t)\mu_t^{N^h}$.
- The evolution of experts' net worth

$$\mu_t^{N^e} = -\rho^e + r_t + \theta^{K,e}(r_t^{K,e} - r_t) = -\rho^e + r_t$$

  since $r_t^{K,e} = r_t$.
- The evolution of households' net worth

$$\mu_t^{N^h} = -\rho^h + r_t + \theta^{K,h}(r_t^{K,h} - r_t) = -\rho^h + r_t$$

  since $\theta^{K,h} = 0$.
- Hence, net worth share follows

$$\mu_t^\eta = -(1-\eta_t)(\rho^e - \rho^h)$$

  which yields the following closed-form expression for η_t

$$\eta_t = \frac{\eta_0 e^{-(\rho^e - \rho^h)t}}{1 - \eta_0 + \eta_0 e^{-(\rho^e - \rho^h)t}}$$

  for initial condition η_0.

Frictionless Case: Main Takeaways

1. **Capital Allocation:** always held by the most efficient sector
2. **Consumption Allocation:** determined by the initial wealth distribution and wealth only moves due to differences in preferences for the timing of consumption
   - $\rho^e = \rho^h$: every initial condition leads to a steady state
   - $\rho^e > \rho^h$: converges in the long run to a boundary $\eta = 0$
   - $\rho^e < \rho^h$: converges in the long run to a boundary $\eta = 1$
3. **Price of Capital:** $q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1 - \eta_t)}$
   - $\rho^e = \rho^h$: $q_t = a^e / \rho^e$ is constant
   - otherwise, it rises over time because the agents with the lower marginal propensity to consume become richer as time passes.

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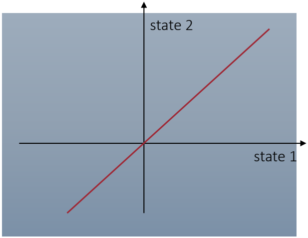
  - $a^h \to -\infty$ and ℓ unconstrained
- Kiyotaki-Moore Model
  - $\sigma = 0$ and $\ell \leqslant 1$

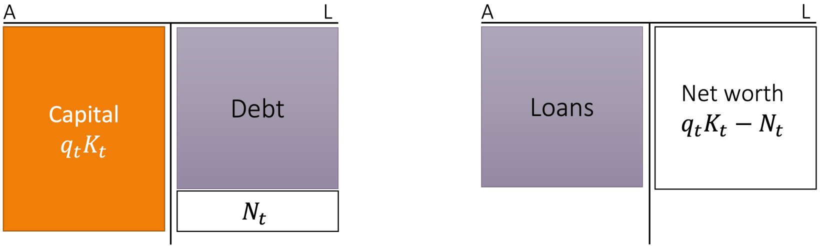
Financial Frictions and Distortions

- **Incomplete markets:**
  - **“natural” leverage constraint** (*BruSan*)
  - Costly state verification (*BGG*)
- Leverage constraints
  (no “liquidity creation”)
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period’s price $Rb_t \leqslant q_{t+dt} k_t$ (*KM*)
    - Next period’s VaR $Rb_t \leqslant VaR_t(q_{t+dt}) k_t$ (*BruPed*)



Simple Two Sector Model: Basak Cuoco (1998)

- Special case with
  - risk $\sigma > 0$
  - no leverage constraint ℓ
  - unproductive household sector $a^h \to -\infty$
- Expert sector Household sector



See Lecture Notes, Chapter 3 or Handbook of Macroeconomics 2017, Chapter 18

1. Given SDF processes, derive individual FOC

- Hamiltonian for the experts is given by

$$\begin{aligned}\mathcal{H}_t^e &= e^{-\rho^e t}\log c_t^e + \xi_t^e n_t^e \mu_t^{n^e} - \varsigma_t^e \xi_t^e n_t^e \sigma_t^{n^e} \\ &= e^{-\rho^e t}\log c_t^e + \xi_t^e \left[-c_t^e + n_t^e r_t + n_t^e \theta_t^{K,e}\left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right)\right] \\ &\quad - \varsigma_t^e \xi_t^e n_t^e \theta_t^{K,e}(\sigma + \sigma_t^q)\end{aligned}$$

- FOCs wrt c_t^e and $\theta_t^{K,e}$ are given by

$$e^{-\rho^e t}\left(c_t^e\right)^{-1} = \xi_t^e$$

$$\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t = \varsigma_t^e(\sigma + \sigma_t^q)$$

- The costate equation reads by virtue of the

$$\begin{aligned}d\xi_t^e &= -\frac{\partial H_t^e}{\partial n_t^e} dt - \varsigma_t^e \xi_t^e dZ_t \\ &= -r_t \xi_t^e - \varsigma_t^e \xi_t^e dZ_t\end{aligned}$$

where the equality follows from the first-order condition for $\theta_t^{K,e}$.

1. Given SDF processes, derive individual FOC

- Using the first-order condition for consumption, we also find by Itô's lemma that

$$\frac{d\xi_t^e}{\xi_t^e} = \left[-\rho^e - \mu_t^{c^e} + (\sigma_t^{c^e})^2\right] dt - \sigma_t^{c^e} dZ_t$$

- Since for log-utility $c_t^e/n_t^e = \rho^e$, $\mu_t^{c^e} = \mu_t^{n^e}$ and $\sigma_t^{c^e} = \sigma_t^{n^e}$.
- Then the price of risk is given by $\varsigma_t^e = \sigma_t^{c^e} = \sigma_t^{n^e} = \theta_t^{K,e}(\sigma + \sigma_t^q)$.
- Similarly, the household sector will consume according to $c_t^h/n_t^h = \rho^h$.

2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks $\longrightarrow$ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

  $\{Z_{s\in[0,t]}\}$ $\searrow$ $\nearrow$

  net worth distribution

  $\eta_t^e = \frac{N^e}{q_t K_t} \in (0,1)$

- All agents maximize utility
  - Choose consumption, portfolio, ...
- All markets clear
  - Consumption, capital, debt,

2. Evolution of state variable η_t

- Recall that $\eta_t = \frac{N_t^e}{q_t K_t}$
- The total wealth of experts N_t^e follows

$$\frac{dN_t^e}{N_t^e} = \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left[dr_t^K - r_t dt\right]$$

$$= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right]dt + (\sigma + \sigma_t^q)dZ_t\right\}$$

$$= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\varsigma_t^e(\sigma + \sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t\right\}.$$

- Also

$$\frac{d(q_t K_t)}{q_t K_t} = \left[\mu_t^q + g + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t$$

$$= \left[r_t - \frac{a^e}{q_t} + \varsigma_t^e(\sigma + \sigma_t^q)\right]dt + (\sigma + \sigma_t^q)dZ_t.$$

- Apply Itô's quotient rule to $\eta_t = N_t^e / q_t K_t$:

$$\frac{d\eta_t}{\eta_t} = \left[-\frac{c_t^e}{n_t^e} + \frac{a^e}{q_t} - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)\left(\varsigma_t^e - (\sigma + \sigma_t^q)\right)\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t$$

2. Evolution of state variable η_t

- Using $c_t^e = \rho^e n_t^e$ and $\varsigma_t^e = \theta_t^{K,e}(\sigma + \sigma_t^q)$, we have

$$\frac{d\eta_t}{\eta_t} = \left[-\rho^e + \frac{a^e}{q_t} + (1 - \theta_t^{K,e})^2(\sigma + \sigma_t^q)^2\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t$$

- Goods market clearing

$$C_t = \rho^e \eta_t q_t K_t + \rho^h (1 - \eta_t) q_t K_t = a^e K_t \implies q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1 - \eta_t)}$$

- Itô's lemma implies

$$\sigma_t^q = -\frac{\rho^e - \rho^h}{\rho^e \eta_t + \rho^h (1 - \eta_t)} \sigma_t^\eta \eta_t$$

- Capital market clearing

$$\theta_t^{K,e} = \frac{q_t K_t}{N_t^e} = \frac{1}{\eta_t}$$

- Hence η_t follows

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t)\left[-(\rho^e - \rho^h) + \frac{1 - \eta_t}{\eta_t^2}(\sigma + \sigma_t^q)^2\right]dt + \frac{1 - \eta_t}{\eta_t}(\sigma + \sigma_t^q)dZ_t$$

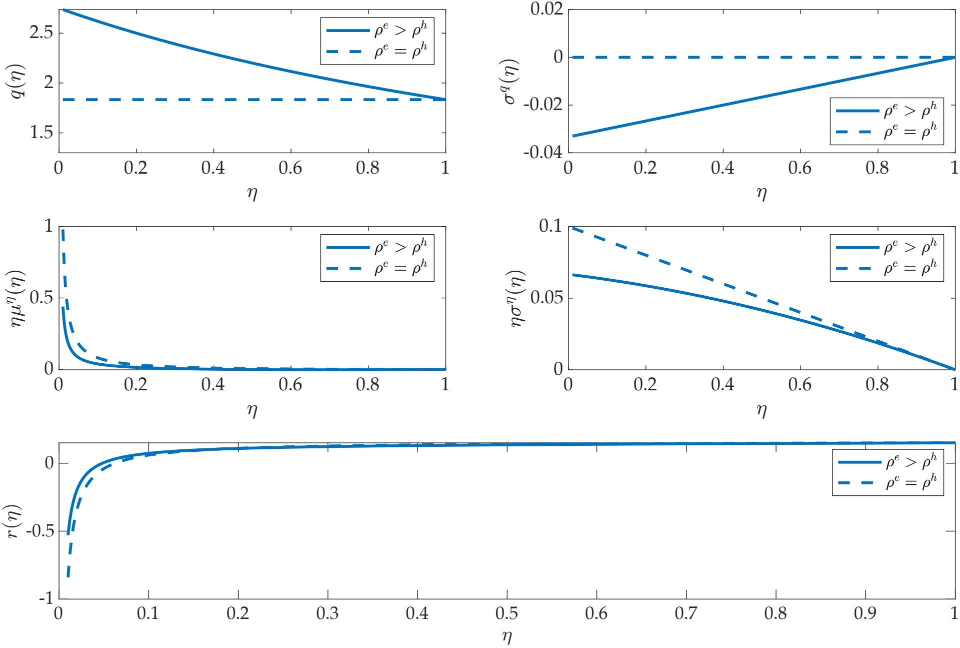
2. Evolution of state variable η_t

- Combining the above we get

$$
\begin{aligned}
\frac{\mathrm{d}\eta_t}{\eta_t} =& (1-\eta_t)\left[-(\rho^e-\rho^h)+\frac{1-\eta_t}{\eta_t^2}\frac{\left[\rho^e\eta_t+\rho^h(1-\eta_t)\right]^2}{(\rho^e)^2}\sigma^2\right]\mathrm{d}t \\
&+\frac{1-\eta_t}{\eta_t}\frac{\rho^e\eta_t+\rho^h(1-\eta_t)}{\rho^e}\sigma\mathrm{d}Z_t
\end{aligned}
$$

a simple one-dimensional stochastic differential equation (SDE).

4. Numerical Example of Basak-Cuoco



$$\rho^e = 0.06,\ \rho^h = 0.04,\ a^e = 0.11,\ \sigma = 0.10,\ g{=}0.1$$

Observation of Basak-Cuoco Model

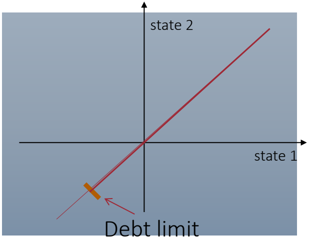
1. **Changes in Risk-free Rate drives Risk Premium**
   Price of risk ς^e, i.e., Sharpe ratio is:
   $$\frac{1}{\eta_t}(\sigma+\sigma_t^q)=\frac{\eta_t\rho^e+(1-\eta_t)\rho^h+g+\mu_t^q+\sigma\sigma_t^q-r_t}{\sigma+\sigma_t^q}$$
   - Goes to ∞ as η_t goes to zero
   - RHS can only go to ∞ if risk-free rate $r_t\to\infty$
     $r_t=\eta_t\rho^e+(1-\eta_t)\rho^h+g+\mu_t^q+\sigma\sigma_t^q-(\sigma+\sigma_t^q)^2/\eta_t\to-\infty$
   - **Special case** $\rho^e=\rho^h$: since $q=\frac{a^e}{\rho^e}$ is constant $(\mu^q=\sigma^q=0)$, easy to see.
2. q_t **Movements Mitigates rather than Amplifies Shocks,** $\sigma^q\leqslant 0$
   Stationary distribution requires $\rho^e>\rho^h\Rightarrow q_t$ appreciates after $dZ_t<0$-shock
   - Otherwise for $\rho^e\leqslant\rho^h$, $\mu_t^\eta>0\ \forall\eta,\Rightarrow\eta^{SS}=1$
     in the long run HH-net worth share vanishes (degenerated stationary distribution)
   - Alternative ways out to obtain a stationary distribution
     without $q(\eta)$ decreasing in η (instead of $\rho^e>\rho^h$):
     - Switching types (jumps) (BGG)
     - 2 types of experts (BruSan AEJ:Macro international paper)

Overview: This Lecture

- Complete Markets Benchmark
  - $\sigma = 0$ and ℓ unconstrained
- Basak Cuoco Model
  - $a^h \to -\infty$ and ℓ unconstrained
- **Kiyotaki-Moore Model**
  - $\sigma = 0$ and $\ell \leqslant 1$, and $a^h(\cdot)$ is a function of κ^h

Financial Frictions and Distortions

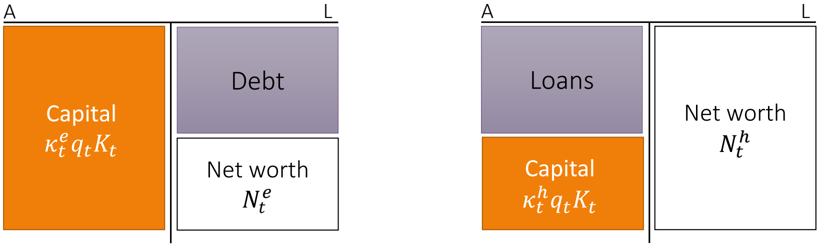
- Incomplete markets:
  - "natural" leverage constraint (*BruSan*)
  - Costly state verification (*BGG*)
- **Leverage constraints:**
  (no "liquidity creation")
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - **Current price** $Rb_t \leqslant q_t k_t$
    - **Next period's price** $Rb_t \leqslant q_{t+dt} k_t$ **(KM)**
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+dt})k_t$ (*BruPed*)



Debt limit
can depend on
prices/volatility

Kiyotaki Moore (1997) in Cts. Time

- No risk: $\sigma = 0$
- Leverage constraint: $\ell \leqslant 1$
- Households can produce: $a^h(\kappa^h)$ is a decreasing function of κ^h
- Expert sector (Farmers) Household sector (Gatherers)



Kiyotaki Moore (1997) in Cts. Time

Expert Sector (Farmers)

- Output: $y_t^e = a^e k_t^e = a^e \kappa_t^e K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^e t} \log(c_t^e) \mathrm{d}t$

Household Sector (Gatherers)

- Output: $y_t^h = a^h(\kappa_t^h) k_t^h = a^h(\cdot)\kappa_t^h K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^h t} \log(c_t^h) \mathrm{d}t$

Assumptions:

- Experts are more impatient $\rho^e > \rho^h$
- Productivity $a^h(\kappa^h) \leqslant a^e$ with equality for $\kappa^h = 0$ and constant return to scale individually, but $a^h(\cdot)$ decreasing in (aggregate) κ^h
- No equity issuance
- Debt issuance with collateral constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$

  $$\Leftrightarrow \frac{D_t^e}{N_t^e} \leqslant \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1-\theta_t^{K,e}) \leqslant \ell \theta_t^{K,e}$$

  Collateral constraint in KM97: $D_t^e(1 + r_{t+dt}dt) \leqslant \ell \kappa_t^e q_{t+dt} K_t$

1. Portfolio choices: Hamiltonian Approach

- Experts' problem: $\max_{c_t^e, \theta_t^{K,e}} \int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t$ s.t. $(1-\ell)\theta_t^{K,e} \leqslant 1$, and

$$\frac{\mathrm{d}n_t^e}{\mathrm{d}t} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]$$

- Households' problem: $\max_{c_t^h, \theta_t^h} \int_s^\infty e^{-\rho^h t} u(c_t^h) \mathrm{d}t$, s.t.

$$\frac{\mathrm{d}n_t^h}{\mathrm{d}t} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right],$$

- The Hamiltonians can be constructed as

$$\begin{aligned} \mathcal{H}_t^e =& e^{-\rho^e t} u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left(1 - (1-\ell)\theta_t^{K,e}\right) \\ \mathcal{H}_t^h =& e^{-\rho^h t} u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right] \end{aligned}$$

- ξ_t^i multiplier on the budget constraint, $\xi_t^e n_t^e \lambda_t^\ell$ multiplier on leverage constraint
  - Later we show that co-state variable ξ_t^i equals SDF, which for log-utility $= e^{-\rho^i t} \frac{1}{\rho^i n_t^i}$
- Fisher Separation Theorem btw. consumption and portfolio choice

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i:

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{log utility}$$

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i:

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{ log utility}$$

- FOC w.r.t $\theta_t^{K,i}$:

$$\begin{cases} r_t^{K,e} - r_t = (1-\ell)\lambda_t^\ell \\ r_t^{K,h} - r_t = 0 \end{cases}$$

- Where capital returns are: (dividend + price drift)

$$\begin{cases} r_t^{K,e} = \dfrac{a^e}{q_t} + \dfrac{1}{q_t}\dfrac{\mathrm{d}q_t}{\mathrm{d}t} \\ r_t^{K,h} = \dfrac{a^h(\kappa_t^h)}{q_t} + \dfrac{1}{q_t}\dfrac{\mathrm{d}q_t}{\mathrm{d}t} \end{cases}$$

2. Net Worth Evolution

- Equilibrium objects are functions of state, net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t \bar{K}}$
- Price dynamics: (No arbitrage for households)

$$\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} = r_t,$$

- State dynamics:

$$\mu_t^N dt = \frac{dN_t}{N_t} = \underbrace{\frac{N_t^e}{N_t}}_{\eta_t} \mu_t^{N^e} dt + \underbrace{\frac{N_t^h}{N_t}}_{(1-\eta_t)} \mu_t^{N^h} dt$$

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

$$= (1-\eta_t)[-(\rho^e - \rho^h) + \theta_t^{K,e}(\frac{a^e}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} - r_t) - \theta_t^{K,h}\overbrace{(\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} - r_t)}^{=0 \text{ from above}}]$$

$$= (1-\eta_t)[-(\rho^e - \rho^h) + \theta_t^{K,e}(\underbrace{\frac{a^e}{q_t} - \frac{a^h(\kappa_t^h)}{q_t}}_{=r_t^{K,e} - r_t^{K,h}})]$$

2. Net Worth Evolution

- Equilibrium objects $(\kappa^e, \kappa^h, q, r)$ are functions of state, net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t \bar{K}}$
- pinned down by:

$$q_t K_t[\rho^e \eta_t + \rho^h(1-\eta_t)] = [a^e \kappa_t^e + a^h(\kappa_t^h)\kappa_t^h]K_t \qquad \text{(Goods market)}$$

$$\underbrace{\theta_t^{Ke}\eta_t}_{=\kappa_t^e} q_t K_t + \underbrace{\theta_t^{Kh}(1-\eta_t)}_{=\kappa_t^h} q_t K_t = q_t K_t \qquad \text{(Capital market)}$$

$$\kappa_t^e \leqslant \frac{\eta_t}{1-\ell} \qquad \text{(Collateral Constraint)}$$

$$\mu_t^\eta = (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}\frac{a^e - a^h(\kappa_t^h)}{q_t}\right]$$

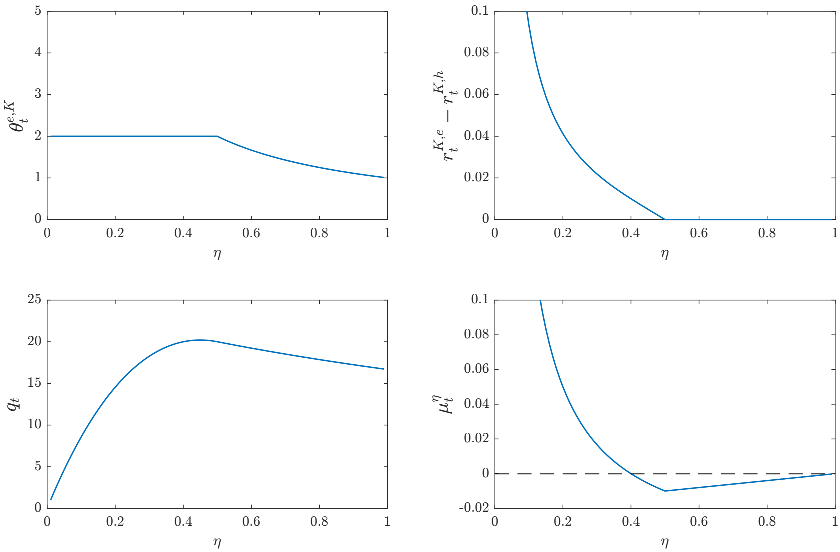
- simplified to (and define $\kappa_t := \kappa_t^e = 1 - \kappa_t^h$)

$$q_t[(\rho^e - \rho^h)\eta_t + \rho^h] = \kappa_t a^e + (1-\kappa_t)a^h(1-\kappa_t)$$

$$\kappa_t \leqslant \frac{\eta_t}{1-\ell}$$

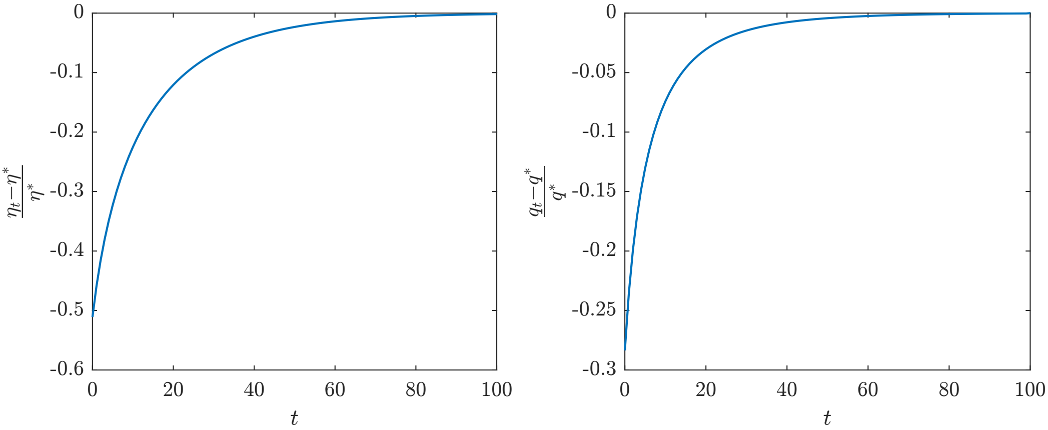
$$\mu_t^\eta = (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{\kappa_t}{\eta_t}\frac{a^e - a^h(1-\kappa_t)}{q_t}\right]$$

Global Non-linear Solution



Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.05, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

Impulse Responses



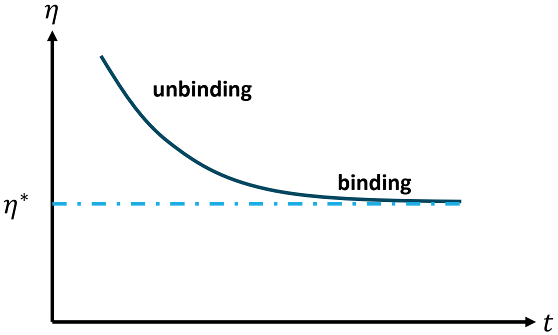
Impulse response function with 30% (of η) negative redistribution shock.
Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

Log-linearization around Steady State

1. Derive steady state with $\mu^\eta = 0$
   with its properties
2. Log-linearize around steady state
   characterize dynamical system locally around the steady state

The Steady State: Binding Collateral Constraint

- The collateral constraint always binds in the steady state
  - If collateral constraint does not bind $\lambda_t^\ell = 0$ and hence $r^{K,e} = r^{K,h}$, i.e. $a^e = a^h(\cdot)$
- Note, the constraint does not need to bind only if $\kappa_t = 1$.
  - Then $\mu_t^\eta = (1 - \eta_t)(\rho^h - \rho^e)$
  - as $\rho^e > \rho^h \Rightarrow \mu_t^\eta < 0$, i.e. η declines
- Characterization of Steady State (Next Page)



Steady State

- Since Collateral constrained binds, steady state capital share

$$\kappa^{SS} = \frac{\eta^{SS}}{1-\ell}$$

- Expert sector's net worth share is $\eta_t := \frac{N_t^e}{q_t \bar{K}}$, is constant, i.e. $\mu_t^\eta := \frac{\mathrm{d}\eta_t}{\mathrm{d}t} = 0$

$$q^{SS}[(\rho^e - \rho^h)\eta^{SS} + \rho^h] = \kappa^{SS} a^e + (1-\kappa^{SS})a^h(1-\kappa^{SS})$$

$$(\rho^e - \rho^h) = \frac{\kappa^{SS}}{\eta^{SS}} \frac{a^e - a^h(1-\kappa^{SS})}{q^{SS}} \quad \text{for } \mu^\eta = 0$$

- Combine

$$\kappa^{SS} a^e - \kappa^{SS} a^h(1-\kappa^{SS}) + q^{SS}\rho^h = \kappa^{SS} a^e + (1-\kappa^{SS})a^h(1-\kappa^{SS})$$

$$\Rightarrow \quad q^{SS} = a^h(1-\kappa^{SS})/\rho^h,$$

where the steady state κ^{SS} is implicitly given by:

$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell} \frac{a^e - a^h(1-\kappa^{SS})}{a^h(1-\kappa^{SS})}.$$

- For specific functional form $a^h(\cdot) = a^e \kappa_t$:

$$\kappa^{SS} = \frac{1}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1} \quad \Rightarrow \eta^{SS} = \frac{1-\ell}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1}$$

Steady State: Comparative Static

- For the specific example $a^h(\cdot) = a^e \kappa$:

- For higher leverage, ℓ, (i.e. less tight collateral constraint)
  - κ^{SS}, SS-capital share, is higher.
  - η^{SS}, SS-net worth share, is lower.
  - $q^{SS} = \frac{a^h}{\rho^h}$, price of capital, is higher.
    $q^{SS}\bar{K}$, total wealth in the economy, is higher too.
  - $N^{e,SS}$ experts' net worth in steady state, is higher (Check?)
  - Comparative Static = permanent (long-run) shift to new steady state
  - Next: Dynamics of how to return to the old steady state
    (after an unanticipated shock)

Log-linearized Dynamics Around Steady State

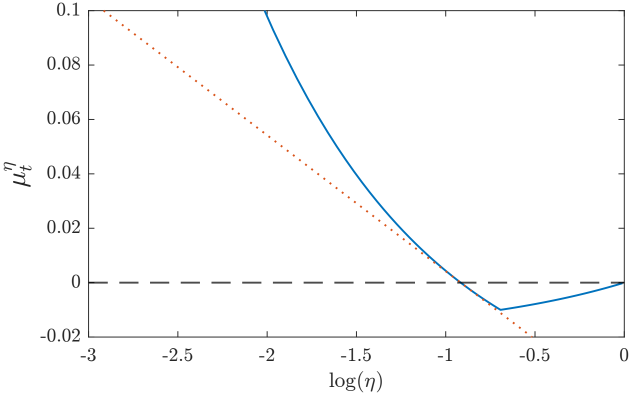
- Analytical solutions to η_t, q_t dynamics are hard to obtain. Expansion around the steady state:

$$\begin{aligned}
\log(\eta_t/\eta^{SS}) &= \hat{\eta}_t \\
\log(q_t/q^{SS}) &= \hat{q}_t \\
\log(r_t/r^{SS}) &= \hat{r}_t \\
\log(a^h_t/a^{h,SS}) &= \hat{a}^h_t
\end{aligned}$$

- Expression for $\hat{a}^h_t, \hat{q}^h_t$ as a function of $\hat{\eta}_t$
- State dynamics and price dynamics become:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} &= \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}^h_t - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right) \\
\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} &= r^{SS}(\hat{r}_t + \hat{q}_t - \hat{a}^h_t)
\end{aligned}$$

Global vs. Log-linearized Solution for η-drift

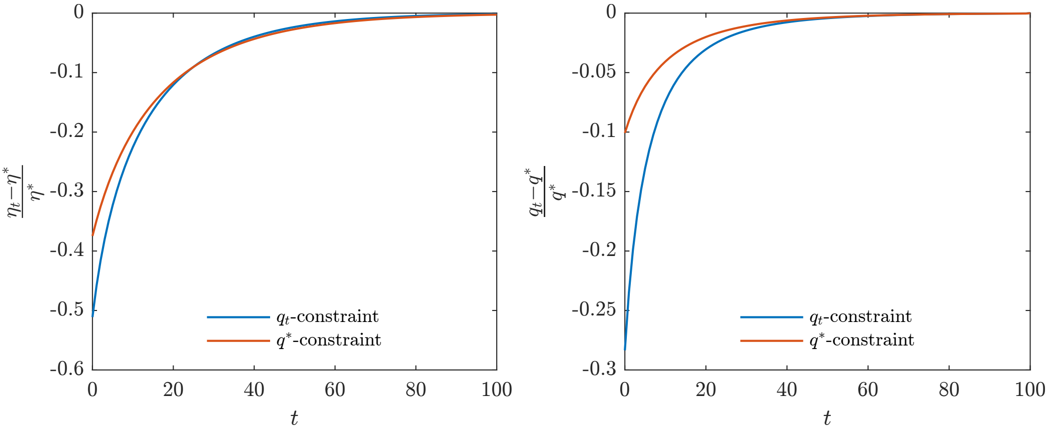


- Note: x-axis is $\log(\eta)$, since log-linearization

Decomposing Amplification Effects

- Start at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$
- Shock: redistribution of a fraction of experts' net worth share to households
  - In KM productivity shock lasts for one period (not for an instant), causes initial redistribution
- **Impulse response function** (with deterministic recovery)
- Immediate impact at $t = 0$
  - direct redistributive effect/shock
  - **price-net worth effect**
    decline in q_t reduces experts' net worth share as they are levered $\Rightarrow$ feedback
  - **price-collateral effect**
    decline in q_t tightens collateral constraints $\Rightarrow$ feeds back on price-net worth effect
- Subsequent impact $t > 0$ (which feeds back to immediate impact)
- **Decomposition:**
  Switch off price-collateral effect by assuming that
  collateral constraint is determined by SS-price q^{SS} instead of equilibrium price q_t.
  (Formally, collateral constraint, $\kappa_t \leqslant \frac{\eta_t}{1-\ell}$, becomes $\kappa_t \leqslant \frac{\eta_t}{1-\ell q^{SS}/q_t}$.)

Decomposition of Amplification: Impulse Response Fcn



Impulse response function with 30% (of η) negative redistribution shock.
Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = \kappa$

Decomposing Amplification at $t = 0$

- At time t, the economy is at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$.
- Negative initial/direct redistributive shock $\eta' = (1-\epsilon)\eta^{SS}$, new price q', and capital holding κ' solves:

$$q' = \frac{\kappa' a^e + (1-\kappa')a^h(1-\kappa')}{(\rho^e - \rho^h)\eta' + \rho^h} \qquad \text{(Goods market)}$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell} \qquad (q_t\text{-constraint})$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell q^{SS}/q'} \qquad (q^{SS}\text{-constraint})$$

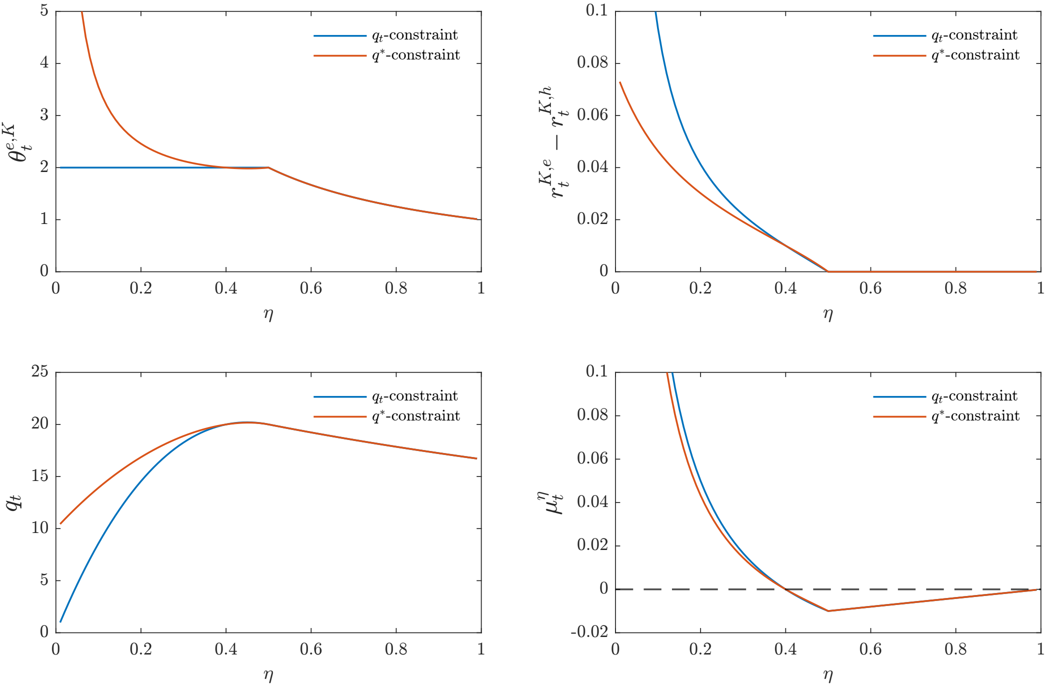
- However, debt contract was signed by old price $q^{SS} \Rightarrow \eta$ drops further
- Consider the balance sheet (first round effect):

$$\frac{\eta'}{1-\ell} q' = \frac{\ell}{1-\ell}\eta' q^{SS} + \eta'' q'$$

To get the convergence result, we need to do this procedure iteratively.

Decomposing Amplification for $t > 0$ (global solution)

$\rho^e = 0.06, \rho^h = 0.04, \ell = 0.05, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$



Decomposing Amplification for $t > 0$ (log-linearized sol.)

- Price dynamics:

$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = r^{SS}\hat{r}_t - r^{SS}\hat{a}_t^h + r^{SS}\hat{q}_t$$

- State dynamics with q_t-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

- State dynamics with q^{SS}-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{1}{1-\ell}\frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

$\hat{q}_t, \hat{a}_t^h, \hat{r}_t$ are different with different constraints.

"Single Shock Critique"

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed
- In reality an adverse shock may be followed by additional adverse shocks
  - Build-up extra safety cushion for an additional shock in a crisis
- Impulse response vs. volatility dynamics

Overview: This Lecture

Real Macrofinance Models with Heterogeneous Agents

A Two-Sector Macrofinance Model

- Complete Markets Benchmark
  - No risk and no frictions
- Basak-Cuoco Model
  - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
  - No risk, two production sectors and leverage constraints

- Physical Investment

Adding Investments/Physical Capital Formation

- Instead of exogenous capital stock,
  convert goods into physical capital (not buying it from others)
- Capital conversion function $\Phi(\iota)$ (increasing and concave)

$$dk_t^i = (\Phi(\iota_t^i)k_t^i - \delta k_t^i)dt + \Delta_t^{k,i} k_t^i$$

  - ι_t^i is investment rate of **new** physical capital (real investment is $\iota_t^i k_t^i$)
    - occurs within the period (no "time-to-build") $\Rightarrow$ static problem
  - $\Delta_t^{k,i} k_t^i$ is the purchase/sale of physical capital at q_t from/to others
    - Hint: $\Delta_t^{k,i} k_t^i$ doesn't impact return on capital $r_t^{K,i}$
  - δ is the depreciation rate of capital
- Optimal investment rate ι_t^i depends on price of physical capital q_t.
  - Tobin's Q:

$$q_t = 1/\Phi'(\iota_t)$$

  - attractive functional form with adjustment cost ϕ:
    $\Phi(\iota) = \frac{1}{\phi} \log(\phi\iota + 1)$
- K_t is a second state variable, which can be solved ex-post

- Homework: Redo continuous time KM analysis with ι-investment.

Bernanke, Gertler, Gilchrist 1999

- Fully fledged DSGE Model with price stickiness, idiosyncratic firm risk, ...
- Aggregate shocks are unanticipated zero-probability shocks (MIT shocks)
- No fire-sale to less productive household sector (unlike in KM97)
- Divestment: Convert physical capital back to consumption good at a cost (captured by $\Phi(\cdot)$-adjustment cost function)
- Financial Frictions:
  - No equity issuance
  - Debt issues with costly state verification (instead of collateral constraint)
    - If firm defaults (after negative idiosyncratic shock), creditor has to pay cost to verify true (remaining) cash flow
    - Optimal contract is a debt contract (debt payoff is hockey stick function of cash flow)
    - De-facto borrowing firms pay verification costs in expectations (in form of higher interest rate/funding costs)
  - A negative aggregate shock, lowers firms' net worth $\Rightarrow$ firm's default prob. rises $\Rightarrow$ expected verification cost rise $\Rightarrow$ Firms funding costs rise

Conclusion & Takeaways

- **Equity Issuance Friction**

1. Incomplete Markets (Basak-Cuoco)
   - No fire sales (since $a^h \to -\infty$)
   - Asset Price constant (or appreciates after adverse shock $\sigma^q \leqslant 0$) $\Rightarrow$ Fluctuating risk-free rate
   - No net worth trap, fat tails, spillovers

- **Debt Issuance Friction** (explicit)

2. Collateral Constraint (Kiyotaki-Moore)
   - Fire sales (since $a^h(\cdot) > \infty$)
   - Dynamic amplification
   - Single zero-probability shock
   - No risk dynamics, no net worth trap
   - No Volatility Paradox, Minsky Hypothesis

- **Physical Investment**/ Tobin's Q is within-period/static Decision

Extra Slides for KM97: Understanding Asset Prices

- Price dynamics (with some proper initial conditions):

$$\frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} + \frac{a^h(\kappa_t^h)}{q_t} = r_t,$$

$$q_t = \int_t^{\infty} e^{-\int_t^s r_u du} a^h(\kappa_s^h) ds$$

- Discrete time analogy:

$$\frac{q_{t+1} - q_t}{q_t} + \frac{a^h(\kappa_t^h)}{q_t} = r_t$$

$$q_t = \sum_{s=0}^{\infty} \left[\prod_{u=0}^{s} \frac{1}{(1 + r_{t+u})} \right] a^h(\kappa_{t+s}^h)$$

- Asset price = sum of discounted dividend flows.
- Asset prices are **solved backward**