Lectures on Macro, Money, and Finance

A Heterogeneous-Agent Continuous-Time Approach[1]

Markus K. Brunnermeier[2]
Princeton University

Sebastian Merkel[3]
Princeton University

Yuliy Sannikov[4]
Stanford University

May 23, 2025

– Preliminary and Incomplete –

Latest Version: [Click Here]

[1]We are grateful to Andrey Alexandrov, Max Alston, Weiqi Chen, Francisco Cocco, Zhouzhou Gu, Ziang Li, Sebastian Lopez Almirante, Fernando Mendo, Luca Neri, and Daojing Zhai, Goutham Gopalakrishna, and Jonathan Payne for great help with the project.

[2]Email: markus@princeton.edu.
[3]Email: smerkel@princeton.edu.
[4]Email: sannikov@gmail.com.

Contents

List of Symbols

i	Sector type i (experts, households...)
$\tilde{i}$	Agents of sector i with idiosyncratic risk, $\tilde{i} \in [0,1]$
c_t	Consumption
y_t	Output goods
ρ	Discount rate
a	Productivity
$\mathcal{A}(\cdot,\cdot)$	Aggregate productivity
κ	Capital share
χ	Equity share
η	Wealth share
ψ	Share of capital wealth
ϑ	Share of wealth that agents hold in money (fraction of nominal wealth)
ι	Investment rate
ω	Investment opportunities
ϕ	Adjustment costs factor in investment function
$\Phi(\cdot)$	Investment function
P	Money price of goods
q^K	Price of capital
q^M	Rescaled price of money, $q^M = \frac{M}{P \cdot K}$
M	positive net supply of money
ζ	Consumption-wealth ratio
ξ	Stochastic discount factor
V	Value function
μ_t	Drift
σ_t	Volatility
ς_t	Price of risk
$\mathrm{d}Z_t$	Brownian motions

Part I

Introduction and Solution Methods

Chapter 1

Introduction

1.1 The Three Watershed Moments in Macroeconomics

Why financial crises are so important? Figure 1.1 depicts the striking resilience of the economy of the United States. The US GDP bounced back after most recessions and returned to the previous trend growth, i.e., the economy made up previous output losses. There are two exceptions: the Great Depression in the 1930s for which the recovery took almost a decade and the global financial crisis (GFC) after 2008. In other words, Figure 1.1 suggests that regular business cycles come and go, but the economy is less resilient after financial crises.

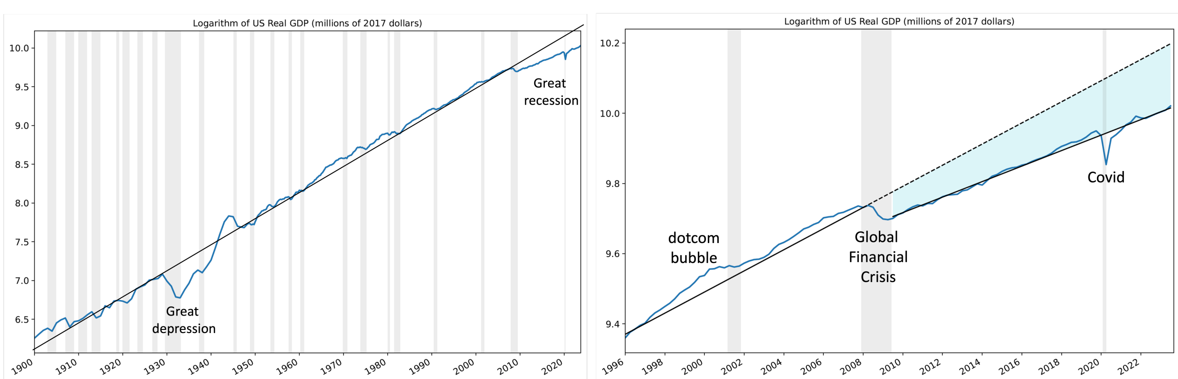


Figure 1.1: Panel A depicts the log-level of US GDP, while Panel B zooms in log GDP-level from 1996 onwards. The shaded areas show recession periods.

After the great depression in the 1930s, the economy did not bounce back for a long time, and arguably it required the fiscal spending associated with WWII to return to the previous GDP trendline. The Global Financial Crisis in 2008 led to the Great Recession, also because - as some observer argued - the fiscal spending was not aggressive enough. Panel B zooms in to stress that after 2008 both the level of GDP and the subsequent growth rate are depressed. So far, there has been no bounce back. Remarkably, the US economy was resilient to the Covid19 pandemic shock: US GDP returned to its post-GFC trend.

1.2 What is Macrofinance?

Macrofinance studies examine how finance impacts the macroeconomy. This field reached new prominence after the global financial crisis, but its roots go far back in history of economic thought. In fact, arguably all eminent economists throughout history have been concerned with the relationship between the macroeconomy and finance.

Macrofinance deals with big issues in macroeconomics, growth and efficiency, as well as with stability of the financial sector as well as the whole macroeconomy. It encompasses many strands of models and empirical analysis. All models involve a dynamic general equilibrium analysis. In most macrofinance models with financial frictions and heterogeneous agents the distribution of wealth matters. Indeed, wealth shares are important state variables. Hence, inequality is also an important policy concern for macrofinance besides growth and stability.

Macrofinance is a "broad church" that touches on most subfields of economics and finance. In finance, it is tightly connected to asset pricing, intermediary finance, corporate, household, and behavioral finance. In economics, the overlap with monetary economics and public finance is arguably the largest.

1.3 Continuous-Time Modeling

Continuous-Time macro-finance models will be the main workhorse of this class. Continuous-time modeling has several attractive features. First, there is a sharp distinction between stocks and flows. Rate changes only affect the stock over a time interval of strictly positive length. Importantly, there is no distinction between beginning-of-period and end-of-period stocks. For example, wealth is equal in the beginning and end of the period, so that consumption is the time-preference rate times the end of period wealth (for log utility). Second, when solving optimal stopping problems, one does not face integer issues.

Third, it helps overcoming time aggregation of data. Different data come in different frequencies. For example, GDP is quarterly, whereas financial data is of much higher frequency. Continuous time models provide an easy framework to deal with these discrepancies.

Fourth, continuous time modeling sometimes squares better with reality. We live in a continuous-time world and do not consume only at the end of the quarter, even though the aggregate macroeconomic data commonly come in quarterly. In discrete time modeling, consumption within periods is simply summed up. This creates an artificial distinction between consumption substitution within and across quarters. Implicitly, the elasticity of intertemporal substitution in a quarterly discrete time model is infinity within quarters, while across quarters it is given by the curvature of the per-period utility function. In contrast, in continuous-time models the elasticity of intertemporal substitution is the same within and across periods.

Besides, continuous-time modeling is often more tractable and allows for a tighter characterization of economic models. For example, in the models discussed in these notes, we can fully characterize the whole dynamical system including the volatility dynamics instead of simply studying a log-linearized representation around the steady state. The well-known Kolmogorov Forward Equation (Fokker Planck Equation) reveals state variables' distribution evolution path starting from any initial distribution, while the impulse response functions in discrete time capture only the expected path after a shock that starts at the steady state.[1] Also, the stationary distribution can be bimodal and exhibit large swings, unlike stable normal distributions that log-linearized models imply.

[1] We will introduce its continuous-time version: "distributional impulse response" in Chapter 3.

Moreover, continuous time allows one to derive more analytical steps and "more" closed form characterizations of the equilibrium before resorting to a numerical analysis. For example, the evolutions of capital, price, net worth are captured by the closed form stochastic differential equations. One can also derive explicit closed-form expressions for amplification terms, as only the slope of the price function is necessary to characterize amplification. In contrast, in discrete-time settings the whole price function is needed, as the jump size may vary. On the other hand, the numerical procedure is more straightforward and faster[2] than discrete time as one no longer needs to search the grid when looking for policy functions.

In terms of tractability, one of the largest benefits of continuous-time models for macrofinance arises in the context of portfolio choice problems. To appreciate this, let us briefly describe a fundamental difficulty of analytically handling portfolio choice in a discrete-time environment. Suppose there are J assets, $j = 1, ..., J$, with (gross) returns R_t^j. In the cross section, returns aggregate additively to portfolio returns. The portfolio assigning weight θ_t^j on asset j has a return of $\sum_{j=1}^{J} \theta_t^j R_t^j$. To retain tractability in the cross-section, we typically assume that R_t^j is normally distributed, so that the portfolio return payoff also follows a normal distribution. However, in the time dimension, returns aggregate multiplicatively. The return of asset j over time is $R_t^j \times R_{t+1}^j \times \cdots$. The returns can only be aggregated over time within the same family of distribution if they are log-normally distributed. To tackle this problem in discrete-time models, it is common place to log-linearize the first-order conditions around the steady state. However, a first-order approximation ignores all volatility terms and makes all assets equivalent. A second-order approximation around steady state is only a partial resolution: it eliminates time-varying volatility, making it impossible to study volatility dynamics. Often one resorts to a log-linearization approximation beyond the steady state à la Campbell and Shiller, (1988a, 1988b), that mimics the continuous time portfolio choice problem and is precise in the continuous time limit.

Admittedly, some of these features are not due to continuous time per se but due to the continuous nature of a particular class of stochastic processes that is typically

[2] Numerical implementations will be studied in Chapter 3. The equilibrium is characterized by partial different equations and solved numerically.

assumed in continuous-time modeling, so-called Itô Processes. An Itô process is a process whose changes over infinitesimally small time intervals are normally distributed. If X_t is the value of a stochastic process at time t, we denote by $\mathrm{d}X_t$ its *time differential*, which is to be interpreted as "$X_{t+dt} - X_t$" for a small (infinitesimal) time increment dt. An Itô process X_t satisfies

$$\mathrm{d}X_t = \mu_{X,t}\mathrm{d}t + \sigma_{X,t}\mathrm{d}Z_t.$$

Here, $\mathrm{d}Z_t$ is the time differential of a *Brownian motion*. Intuitively, one can think of $\mathrm{d}Z_t$ as the continuous-time analog of normal white noise, i.e., i.i.d. standard normal shocks, in discrete time time series models. We discuss Brownian motion in the next subsection. The coefficients $\mu_{X,t}$ and $\sigma_{X,t}$ are called the (arithmetic) *drift* and *volatility* of the Itô process at time t. The volatility dynamics are fully loaded on the Brownian shocks $\mathrm{d}Z_t$, instead of some probabilistic states as is commonly the case in discrete-time models (e.g., a Lucas tree). Importantly, even if the time differential $\mathrm{d}X_t$ of the Itô process is normal at any given time, the time increments $X_t - X_s$ of the process over any positive-length time interval $[s,t]$ can nevertheless be non-normal and exhibit, for example, skewness when the drift and volatility are time-varying.

An important property of Itô processes is that their paths are continuous. In economic models, this means information arrives smoothly in a continuous manner. Implicitly, it also assumes that agents can react continuously to a continuous information flow, so there are no jumps of any variables. On the one hand, continuous paths can greatly simplify analysis and numerical computations. For example, the discrete-time collateral constraint $b_t R_{t,t+1} \leq \min\{q_{t+1}\}k_t$ becomes $b_t \leq (q_t + \mathrm{d}q_t)k_t$ in continuous time. Since continuous paths rule out jumps, $\mathrm{d}q_t$ is infinitesimal compared to q_t, hence the condition simplifies to $b_t \leq q_t k_t$. On the other hand, continuous paths can be restrictive. For example, in an environment with only Itô processes, investors can delever continuously to avoid default, which emboldens investors ex ante and makes it impossible to study default in equilibrium. For more general purposes, one might use Lévy processes which allow for jumps.

Bibliography

Campbell, John Y. and Robert J. Shiller, "The dividend-price ratio and expectations of future dividends and discount factors," *The Review of Financial Studies*, 1988, *1* (3), 195–228.

__ **and** __ , "Stock prices, earnings, and expected dividends," *The Journal of Finance*, 1988, *43* (3), 661–676.

Chapter 2

Portfolio and Consumption Choice in Partial Equilibrium

To start with, we introduce the basic terminology and results we will follow in this course and apply them to solve Merton's portfolio choice problem. The main objective of this lecture is to introduce the basic techniques to solve continuous-time macrofinance models.

2.1 Itô Calculus

2.1.1 Brownian Motion

As described, a Brownian motion is a process Z_t whose time differentials $\mathrm{d}Z_t$ play the role of normally distributed white noise. This can be motivated from a discrete-time approximation as a binomial tree over shrinking time periods Δt, with shrinking steps $h_n = \sigma\sqrt{\Delta t/n}$. Figure 2.1 shows processes for different n. Even though the individual time steps in the discrete tree are not normally distributed (they have a Bernoulli distribution), the change in the tree over a fixed number of time units sums over more and more time steps as we refine the tree (increase n) and, by the central limit theorem, are normally distributed in the limit $n \to \infty$. In Section 2.1, we introduce a formal definition of a Brownian motion on a filtered probability space.

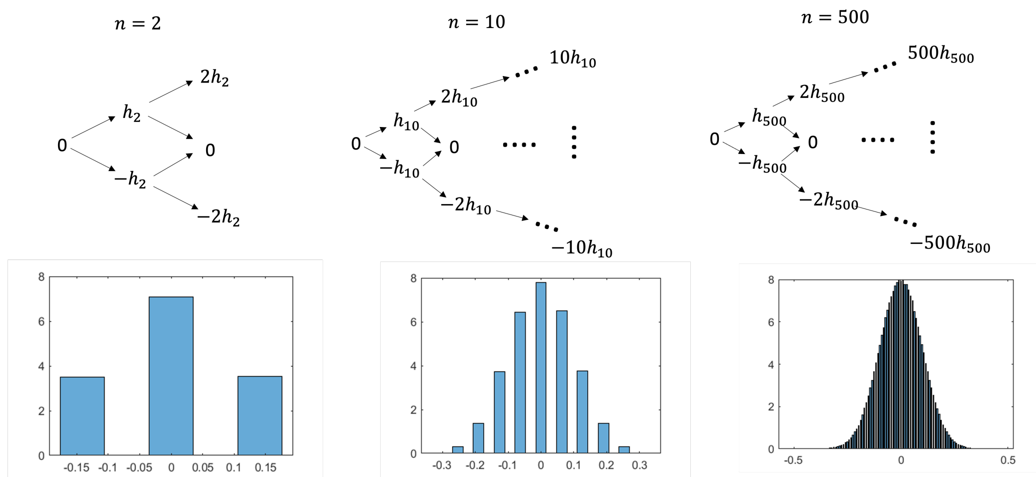


Figure 2.1: Binomial trees for different n.

This section will more formally introduce the basics of Itô calculus, which will be extensively used during the course.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and let X_t be a $\mathcal{F}_t$-measurable process, and let us start by defining Brownian motions and Itô processes.

Definition 2.1. *A Brownian motion Z_t on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies*

1. $Z_0 = 0$
2. Z_t *is almost surely continuous*
3. Z_t *has independent increments*
4. $Z_t - Z_s \sim \mathcal{N}(0, t-s)$ *for* $s \in [0, t]$
5. Z_t *is* $\mathcal{F}_t$*-measurable*

Definition 2.2. *An Itô process X_t is defined as*

$$X_t = X_0 + \int_0^t \mu_{X,s} ds + \int_0^t \sigma_{X,s} dZ_s$$

with $(\mu_{X,t})_{t\geq 0}$ a predictable Lebesgue integrable process and $(\sigma_{X,t})_{t\geq 0}$ a predictable Z_t-integrable process. That is, $\int_0^t (\sigma_{X,s}^2 + |\mu_{X,s}|) ds < \infty$ for all t. Differentiation with respect to time yields the representation

$$dX_t = \mu_{X,t} dt + \sigma_{X,t} dZ_t,$$

where $\mu_{X,t}$ and $\sigma_{X,t}$ are the arithmetic drift and volatility of X_t.

The previous definition introduced an arithmetic Itô process. However, in this course, we will work mainly with a geometric representation of Itô processes,

$$\frac{\mathrm{d}X_t}{X_t} = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t,$$

where the geometric drift and volatility are defined as $\mu_t^X \equiv \mu_{X,t}/X_t$ and $\sigma_t^X \equiv \sigma_{X,t}/X_t$. This representation is well-defined if the process is always positive or always negative. Throughout, we use the notation convention that subscripts refer to arithmetic and superscripts refer to geometric drift and volatility.

2.1.2 Itô's Lemma

Let us now introduce three important formulas: Itô's lemma, Itô's product rule, and Itô's quotient rule. The latter two are corollaries of Itô's lemma in two dimensions.

Lemma 2.1. *Itô's lemma. For any twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$, $f(X_t)$ is also an Itô process with*

$$\mathrm{d}f(X_t) = \left[f'(X_t)(\mu_t^X X_t) + \frac{1}{2}f''(X_t)(\sigma_t^X X_t)^2\right]\mathrm{d}t + f'(X_t)(\sigma_t^X X_t)\mathrm{d}Z_t$$

.

Moreover, if the function is such that $f(t, X_t)$ depends explicitly on time, then the drift of $f(t, X_t)$ includes $\partial_t f(t, X_t)$.

Corollary 2.1. *Itô's product and quotient rule. For any geometric Itô processes X_t, Y_t, $X_t Y_t$ and X_t/Y_t are Itô processes with*

$$\frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)\mathrm{d}t + (\sigma_t^X + \sigma_t^Y)\mathrm{d}Z_t.$$

$$\frac{\mathrm{d}(X_t/Y_t)}{X_t/Y_t} = \left[\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right]\mathrm{d}t + (\sigma_t^X - \sigma_t^Y)\mathrm{d}Z_t.$$

2.2 A Simple Portfolio Choice Problem

In this section we describe a simple version of the Merton portfolio choice problem with a single risky asset and a single Brownian shock process. Both assumptions are for ease of exposition only and can be generalized without additional conceptual or practical difficulties.

Preferences. Consider an agent who chooses a lifetime stream of consumption $\{c_t\}_{t=0}^{\infty}$ and portfolio weights $\{\theta_t\}_{t=0}^{\infty}$ to maximize

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) \mathrm{d}t.$$

where $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$.

Net worth evolution. The agent's net worth at $t = 0$ is given by n_0 and thereafter it evolves as follows

$$\mathrm{d}n_t = -c_t \mathrm{d}t + n_t \left[\theta_t r_t \mathrm{d}t + (1-\theta_t)\mathrm{d}r_t^a\right]$$

for all $t > 0$ subject to a solvency constraint $n_t \geq 0$. r_t is the risk-free rate and $\mathrm{d}r_t^a = (r_t + \delta_t^a)\mathrm{d}t + \sigma_t^a \mathrm{d}Z_t$ is the risky asset return process with risk premium δ_t^a.

State space. Suppose that returns are a function of an external state variable η_t so that $r_t = r(\eta_t)$, $\delta_t^a = \delta^a(\eta_t)$ and $\sigma_t^a = \sigma^a(\eta_t)$ where η evolves according to a diffusion process

$$\mathrm{d}\eta_t = \mu_t^{\eta}(\eta_t)\eta_t \mathrm{d}t + \sigma_t^{\eta}(\eta_t)\eta_t \mathrm{d}Z_t$$

with η_0 given.

Hence, the decision problem has two states variables (n_t, η_t) where n_t is a controlled state and η_t is an external state.

2.3 Solving Stochastic Control Problems: Hamilton-Jacobi-Bellman Equation, Pontryagin Stochastic Maximum Principle, and Martingale Method

We will now present three methods to solve this optimization in continuous time: the Hamilton-Jacobi-Bellman (HJB) Equation, Pontryagin's Stochastic Maximum Principle, and the Martingale Method.

2.3.1 Hamilton-Jacobi-Bellman (HJB) Equation

The HJB equation is the continuous-time analogue to the Bellman equation. To derive the HJB equation, let $\mathcal{A}(n,\eta)$ be the set of admissible choices $\{c_t,\theta_t\}_{t=0}^{\infty}$ given the initial condition $n_0 = n, \eta_0 = \eta$ and $\mathcal{A}_T(n,\eta)$ be the set of policies $\{c_t,\theta_t\}_{t=0}^{T}$ over $[0,T]$ that have admissible extensions to $[0,\infty)$, $\{c_t,\theta_t\}_{t=0}^{\infty} \subset \mathcal{A}(n,\eta)$. The value function $V(n,\eta)$ of the decision problem is defined to be

$$V(n,\eta) := \max_{\{\theta_t,c_t\}_{t=0}^{\infty}\in\mathcal{A}(n,\eta)} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho t}u(c_t)\mathrm{d}t\right].$$

Notice that V satisfies the Bellman principle of optimality: for all $T > 0$ so that

$$V(n,\eta) = \max_{\{\theta_t,c_t\}_{t=0}^{T}\subset\mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^{T} e^{-\rho t}u(c_t)\mathrm{d}t + e^{-\rho T}V(n_T,\eta_T)\right],$$

where n_T depends on the choice $\{\theta_t,c_t\}_{t=0}^{T}$ over $[0,T]$.

With $V_t := V(n_t,\eta_t)$, we can write the principle of optimality as:

$$0 = \max_{\{\theta_t,c_t\}_{t=0}^{T}\subset\mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^{T} e^{-\rho t}u(c_t)\mathrm{d}t + e^{-\rho T}V_T - V_0\right].$$

Integrating $e^{-\rho T}V_T - V_0$ by parts yields

$$e^{-\rho T}V_T - V_0 = -\rho \int_0^T e^{-\rho t} V_t \mathrm{d}t + \int_0^T e^{-\rho t} \mathrm{d}V_t.$$

Thus,

$$0 = \max_{\{\theta_t, c_t\}_{t=0}^T \subset \mathcal{A}_T(n,\eta)} \mathbb{E}_t \left[\int_0^T e^{-\rho t}(u(c_t) - \rho V_t)\mathrm{d}t + \int_0^T e^{-\rho t} \mathrm{d}V_t \right].$$

We can now divide by T and take the limit as $T \downarrow 0$ to obtain the stochastic version of the HJB

$$\boxed{\rho V_t \mathrm{d}t = \max_{c_t, \theta_t}\{u(c_t)\mathrm{d}t + \mathbb{E}[\mathrm{d}V_t]\}} \tag{2.1}$$

Now we will transform the previous equation into a non-stochastic differential equation. To do so, recall that $V_t = V(n_t, \eta_t)$ and by Itô's lemma

$$\mathbb{E}[\mathrm{d}V_t] = \left[\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn}V(n_t,\eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta}V(n_t,\eta_t)\sigma_{\eta,t}^2\right) + \partial_{n\eta}V(n_t,\eta_t)\sigma_{n,t}\sigma_{\eta,t}\right]\mathrm{d}t,$$

where $\mu_{n,t} = -c_t + n_t\left[r(\eta_t) + (1-\theta_t)\delta^a(\eta_t)\right]$ and $\sigma_{n,t} = n_t(1-\theta_t)\sigma^a(\eta_t)$.

Hence, Equation 2.1 can be written as

$$\begin{aligned}
\rho V(n,\eta) = &\max_c \left\{u(c) - \partial_n V(n,\eta)c\right\} + \max_\theta \Big\{\partial_n V(n,\eta)n\left[r(\eta) + (1-\theta)\delta^a(\eta)\right] \\
&+ \left(\frac{1}{2}\partial_{nn}V(n,\eta)n(1-\theta)\sigma^a(\eta) + \partial_{n\eta}V(n,\eta)\sigma_\eta(\eta)\right)n(1-\theta)\sigma^a(\eta)\Big\} \\
&+ \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta}V(n,\eta)\left(\sigma_\eta(\eta)\right)^2
\end{aligned}$$

which is a nonlinear partial differential equation in $V(n,\eta)$.

Special case: constant returns

Assume that returns are constant so that $r_t = r, \delta_t^a = \delta^a, \sigma_t^a = \sigma^a$. Then, we can drop η from the problem and write the HJB as

$$\rho V(n) = \max_c \left\{ u(c) - V'(n)c \right\} + \max_\theta \left\{ V'(n)n \left[r + (1-\theta)\delta^a \right] + \frac{1}{2} V''(n) n^2 \left((1-\theta)\sigma^a \right)^2 \right\}$$

The first-order conditions yield the optimal consumption and portfolio choices

$$\begin{aligned} u'(c) &= V'(n) \\ 1 - \theta &= \left(-\frac{V''(n)n}{V'(n)} \right)^{-1} \frac{\delta^a}{(\sigma^a)^2} \end{aligned}$$

where $-V''(n)n/V'(n)$ is the relative risk aversion coefficient.

To solve this problem, we will guess a functional form for the value function $V(n) = u(\omega n)/\rho$ for some constant $\omega > 0$. Plugging this guess into the HJB equation yields

$$\begin{cases} \log \omega + \log n = \log \rho + \log n - 1 + \frac{1}{\rho} \left(r + \frac{1}{2\gamma} \left(\frac{\delta^a}{\sigma^a} \right)^2 \right) & \text{if } \gamma = 1 (\text{log utility}) \\ \rho \frac{(\omega n)^{1-\gamma}}{\rho} = \gamma \rho^{1/\gamma} \omega^{1-1/\gamma} \frac{(\omega n)^{1-\gamma}}{\rho} + (1-\gamma) \left(r + \frac{1}{2\gamma} \left(\frac{\delta^a}{\sigma^a} \right)^2 \right) \frac{(\omega n)^{1-\gamma}}{\rho} & \text{if } \gamma \neq 1 \end{cases}$$

Notice that in both cases, n cancels out, thus verifying our guess, and we can then solve for ω. The full solution is given by

$$\begin{aligned} V(n) &= \frac{u(\omega n)}{\rho} \\ c(n) &= \rho^{1/\gamma} \omega^{1-1/\gamma} n \\ 1 - \theta(n) &= \frac{1}{\gamma} \frac{\delta^a}{(\sigma^a)^2} \\ \omega &= \rho \left(1 + \frac{\gamma - 1}{\gamma} \frac{1}{\rho} \left(r - \rho + \frac{1}{2\gamma} \left(\frac{\delta^a}{\sigma^a} \right)^2 \right) \right)^{\frac{\gamma}{\gamma - 1}} \end{aligned}$$

Let us analyze the optimal consumption choice. We can denote the consumption net worth ratio by $\check{\rho}$ and write it as

$$\check{\rho} := c_t/n_t = \rho^{1/\gamma}\omega^{1-1/\gamma}$$

where ω can be referred to as "investment opportunities". The reaction of c/n to investment opportunities depends on the elasticity of intertemporal substitution $\psi := 1/\gamma$. In particular

i. if $\psi < 1$, then better investment opportunities lead to an increase in consumption and a decrease in savings

ii. if $\psi > 1$, then better investment opportunities lead to a decrease in consumption and an increase in savings

iii. if $\psi = 1$, then the consumption to wealth ratio is independent of investment opportunities.

Behind these results are income and substitution effects. On the one hand, better investment opportunities make the agent wealthier, and she responds by increasing consumption in all periods. On the other hand, these better investment opportunities make saving more attractive, and to benefit from it, the agent needs to postpone consumption today to get more consumption in the future. Whenever $\psi < 1$, the substitution effect, a desire to smooth consumption, is weak and the income effect dominates. However, if $\psi > 1$, the investor is less averse to variation in consumption and the substitution effect dominates.

Special case: time-varying returns

When returns are time-varying, we can use the same approach as before where now we guess $V(n,\eta) = u(\omega(\eta)n)/\rho$. This yields the following optimal consumption and portfolio choices

$$c(n,\eta) = \underbrace{\rho^{1/\gamma}(\omega(\eta))^{1-1/\gamma}}_{\check{\rho}:=} n$$

$$1-\theta(n,\eta) = \underbrace{\frac{1}{\gamma}\frac{\delta^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\frac{\omega'(\eta)}{\omega(\eta)}\sigma^\omega(\eta)\sigma^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{hedging demand}}$$

where now investment opportunities $\omega(\eta)$ are state-dependent. Notice that there is an additional hedging demand term that depends on the covariance $\sigma^\omega\sigma^a$ of investment opportunities with asset returns. To obtain a solution for $\omega(\eta)$, we need to plug in the optimal choices into the HJB equation which will yield an ODE for $\omega(\eta)$ that can be solved numerically.

What is the role of a hedging demand? The variation in future investment opportunities is relevant for portfolio choice for two opposing motives. First, if investment opportunities are good, it is valuable to have available resources. Then, it is reasonable to invest in assets that pay off in states in which investment opportunities are good. However, if investment opportunities are bad, that is a bad time for the investor and additional wealth is valuable. Then it makes sense to invest in assets that pay off in states in which investment opportunities are bad. Which of the two dominates depends on γ. If $\gamma < 1$, the investor is not very risk averse and prefers to have resources available when it is profitable to invest. If $\gamma > 1$, the investor is sufficiently risk averse to want to hedge against bad times. When $\gamma = 1$, the two forces cancel out and the investor acts myopically. Notice that a very conservative investor ($\gamma \to \infty$) cares only about the hedging component.

2.3.2 Pontryagin's Stochastic Maximum Principle

Pontryagin's maximum principle is a method to solve optimal control problems, which is complementary to dynamic programming.

Stochastic maximum principle. Consider a control problem

$$\mathrm{d}X_t = \mu(X_t, A_t)\mathrm{d}t + \sigma(X_t, A_t)\mathrm{d}Z_t,$$

where A_t are the controls and X_t are the states. The stochastic maximum principle is formulated for finite-horizon problems with the objective function of the form

$$\mathbb{E}_0 \left[\int_0^T g\left(t, X_t, A_t\right) \mathrm{d}t + G\left(X_T\right) \right],$$

where the payoff flow $g\left(t, X_t, A_t\right)$ depends on t and hence can accommodate discounting.

To solve the optimization problem, one can work with the special adjoint process p_t, which is the dynamic Lagrange multiplier on the state variable X_t. We label p_t and its volatility q_t as *costates* of the system, and then optimize the Hamiltonian

$$H = g(t, X, A) + \langle p, \mu(X, A) \rangle + \operatorname{tr}\left[q^T \sigma(X, A) \right]. \tag{2.2}$$

Under necessary convexity conditions[a], the *stochastic maximum principle* says that p_t must satisfy the BSDE

$$\mathrm{d}p_t = -H_X\left(t, X_t, A_t, p_t, q_t\right) dt + q_t \mathrm{d}Z_t \tag{2.3}$$

with terminal condition $p_T = G'(X_T)$.

[a]See the convexity conditions in Yuliy's "Overview of Stochastic Calculus."

To solve Merton's portfolio choice problem, let ξ_t be the costate and $-\varsigma_t \xi_t$ be its volatility. Then, the Hamiltonian is given by

$$\begin{aligned} H_t &= e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \xi_t n_t \mu_t^n - \varsigma_t \xi_t n_t \sigma_t^n \\ &= e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \xi_t \left[-c_t + n_t(1-\theta_t)(r_t + \delta_t^a) + n_t \theta_t r_t - \varsigma_t n_t (1-\theta_t) \sigma_t^a \right] \end{aligned}$$

The first-order conditions with respect to $\{c_t, \theta_t\}$ yield

$$\begin{aligned} e^{-\rho t} c_t^{-\gamma} &= \xi_t \\ \delta_t^a &= \varsigma_t \sigma_t^a \end{aligned}$$

The costate equation is given by

$$\begin{aligned}
\mathrm{d}\xi_t &= -\partial_n H \mathrm{d}t - \varsigma_t \xi_t \mathrm{d}Z_t \\
&= -\xi_t \left[r_t + (1-\theta_t)(\delta_t^a - \varsigma_t \sigma_t^a) \right] \mathrm{d}t - \varsigma_t \xi_t \mathrm{d}Z_t \\
&= -r_t \xi_t \mathrm{d}t - \varsigma_t \xi_t \mathrm{d}Z_t
\end{aligned}$$

where the last line uses the first-order condition with respect to portfolio holdings.

Under the assumption $dc_t = \mu_t^c c_t \, dt + \sigma_t^c c_t \, dZ_t$, the first-order condition with respect to consumption implies by Itô's lemma

$$\mathrm{d}\xi_t = -\left[\rho + \gamma \mu_t^c - \frac{1}{2}\gamma(1+\gamma)(\sigma_t^c)^2 \right] \xi_t \mathrm{d}t - \gamma \sigma_t^c \xi_t \mathrm{d}Z_t$$

Consider the special case of constant returns and log-utility and assume $c_t = a n_t$ for some constant $a > 0$. Then $\mu_t^c = \mu_t^n = -a + r + (1-\theta_t)\delta^a$ and $\sigma_t^c = \sigma_t^n = (1-\theta_t)\sigma^a$.

The costate equation then implies

$$\begin{aligned}
r &= \rho - a + r + (1-\theta)\delta^a - (1-\theta)^2(\sigma^a)^2 \\
\varsigma &= (1-\theta)\sigma^a
\end{aligned}$$

Combining the last equation with the first order condition with respect to portfolio holdings yields $1 - \theta = \frac{\delta^a}{(\sigma^a)^2}$. Putting this result back into the drift of ξ_t implies that $a = \rho$, which confirms our guess. Therefore, we obtain the same solution as with the HJB equation.

2.3.3 Martingale Method

Now we introduce the martingale approach, a powerful tool for many macro-finance models.

Martingale approach in discrete time.

Consider a standard dynamic portfolio choice problem in discrete time:

$$\begin{aligned} \max_{\{\boldsymbol{\theta}_\tau, c_\tau\}_{\tau=t}^{\infty}} \quad & \mathbb{E}_t\left[\sum_{\tau=t}^{\infty} \frac{1}{(1+\rho)^{\tau-t}} u(c_\tau)\right] \\ \text{s.t.} \quad & \boldsymbol{\theta}_t \boldsymbol{p}_t = \boldsymbol{\theta}_{t-1}(\boldsymbol{p}_t + \boldsymbol{d}_t) - c_t \qquad \forall t, \end{aligned}$$

where $\{\boldsymbol{\theta}_t, \boldsymbol{p}_t, \boldsymbol{d}_t\}$ are the vectors of holdings, prices and dividends of different assets. WLOG, we focus on an environment with one asset. The FOC w.r.t. θ_t is

$$\xi_t p_t = \mathbb{E}_t\left[\xi_{t+1}(p_{t+1} + d_{t+1})\right],$$

where $\xi_t = \frac{1}{(1+\rho)^t}\frac{u'(c_t)}{u'(c_0)}$ is the (multi-period) SDF. Consider a self-financing trading strategy A where one reinvests dividend d_t in every period. The price of the strategy p_t^A satisfies

$$\xi_t p_t^A = \mathbb{E}_t\left[\xi_{t+1} p_{t+1}^A\right],$$

i.e., the process $\xi_t p_t^A$ is a martingale.

Martingale approach in continuous time.

Consider a similar portfolio choice problem in continuous time:

$$\begin{aligned} \max_{\{\iota_t^e, \boldsymbol{\theta}_t^e, c_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t} u(c_t)\mathrm{d}t\right] \\ \text{s.t.} \quad & \frac{\mathrm{d}n_t}{n_t} = -\frac{c_t}{n_t}\mathrm{d}t + \sum_j \theta_t^j \mathrm{d}r_t^j + \text{ labor income/endowment/taxes} \\ & n_0 \text{ given.} \end{aligned}$$

Here n_t is the net worth of the agent. r_t^j denotes the return of asset j. Let x_t^A be the value of a self-financing trading strategy A where one reinvests all dividends. Again, define the SDF as $\xi_t^i = e^{-\rho t}u'(c_t^i)$. Then it must be that $\xi_t x_t^A$ follows a martingale. (For proof, see Brunnermeier and Sannikov (2016, pg. 19) or separate notes prepared by Sebastian.) Let

$$\frac{\mathrm{d}x_t^A}{x_t^A} = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t.$$

Assume that the SDF follows

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -r_t \mathrm{d}t - \varsigma_t \mathrm{d}Z_t.$$

Using Itô's product rule,

$$\frac{\mathrm{d}(\xi_t x_t^A)}{\xi_t x_t^A} = \left[-r_t + \mu_t^A - \varsigma_t \sigma_t^A\right] \mathrm{d}t + \text{volatility terms}.$$

Since $\xi_t x_t^A$ follows a martingale, its drift equals zero, i.e.,

$$\mu_t^A = r_t + \varsigma_t \sigma_t^A.$$

Example 1. For risk-free asset, $\sigma_t^A = 0$. Hence, $r_t^F = r_t$.

Example 2. For any two assets A, B, we have $\mu_t^A - \mu_t^B = \varsigma_t(\sigma_t^A - \sigma_t^B)$.

Risky Asset as Asset A and Bonds as Asset B. The bond follows the return process

$$\mathrm{d}r_t^B = r_t \mathrm{d}t,$$

and the risky asset follows the return process

$$\mathrm{d}r_t^A = (r_t + \delta_t^a)\mathrm{d}t + \sigma_t^a \mathrm{d}Z_t.$$

Then, the martingale asset pricing condition is

$$\begin{aligned} &\mathbb{E}_t \left[\mathrm{d}r_t^A - \mathrm{d}r_t^B\right] / \mathrm{d}t = \varsigma_t(\sigma_t^a - 0) \\ \iff &(r_t + \delta_t^a) - r_t = \varsigma_t \sigma_t^a \end{aligned}$$

which yields

$$\delta_t^a = \varsigma_t \sigma_t^a. \tag{2.4}$$

We can then recover ς_t by Itô's lemma. Indeed, ξ_t is $e^{-\rho t}u'(c_t) = e^{-\rho t}c_t^{-\gamma}$. [Note: $dc_t = \mu_t^c c_t \mathrm{d}t + \sigma_t^c c_t \mathrm{d}Z_t$]. And, note $u'' = -\gamma c^{-\gamma-1}, u''' = \gamma(\gamma+1)c^{-\gamma-2}$,

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -\underbrace{\left(\rho + \gamma\mu_t^c - \frac{1}{2}\gamma(\gamma+1)(\sigma_t^c)^2\right)}_{r_t^f}\mathrm{d}t - \underbrace{\gamma\sigma_t^c}_{\varsigma_t}\mathrm{d}Z_t.$$

Consider the special case of constant returns and log-utility and assume $c_t = an_t$ for some constant $a > 0$. Then $\mu_t^c = \mu_t^n = -a + r + (1-\theta_t)\delta^a$ and $\sigma_t^c = \sigma_t^n = (1-\theta_t)\sigma^a$. Then, by (2.4) and the previous equation,

$$\frac{\delta^a}{\sigma^a} = \varsigma_t = \sigma_t^c = (1-\theta_t)\sigma^a$$

which reduces to the same solution as before

$$1 - \theta = \frac{\delta^a}{(\sigma^a)^2}.$$

Net Worth as Asset A and Bonds as Asset B. We can use a self-financing strategy that reinvests consisting of an agent's net worth with consumption reinvested. The return on this strategy is

$$\begin{aligned}
\mathrm{d}r_t^n = \frac{\mathrm{d}n_t + c_t\mathrm{d}t}{n_t} &= \theta_t r_t \mathrm{d}t + (1-\theta_t)\mathrm{d}r_t^a \\
&= \theta_t r_t \mathrm{d}t + (1-\theta_t)(r_t + \delta_t^a)\mathrm{d}t + (1-\theta_t)\sigma_t^a \mathrm{d}Z_t \\
&= (r_t + (1-\theta_t)\delta_t^a)\mathrm{d}t + (1-\theta_t)\sigma_t^a\mathrm{d}Z_t,
\end{aligned}$$

The martingale asset pricing condition is therefore

$$\begin{aligned}
\mathbb{E}_t\left[dr_t^n - dr_t\right]/\mathrm{d}t &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff (r_t + (1-\theta_t)\delta_t^a) - r_t &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff (1-\theta_t)\delta_t^a &= \varsigma_t(1-\theta_t)\sigma_t^a \\
\iff \delta_t^a &= \varsigma_t\sigma_t^a
\end{aligned}$$

which is the same as (2.4).

2.4 Exercises

2.4.1 Solving Differential Equations

1. Read Section 1 of Sebastian's notes on differential equations.

2. Solve the following ODEs

$$y' = y^{-19} \tag{2.5}$$
$$y' = x\cos(x^2)y^2 \tag{2.6}$$
$$y'' = -y \tag{2.7}$$

on the interval $[0, 10]$ with the initial condition $y(0) = 1$ for all three equations and an additional initial condition $y(0) = 0$ for equation 2.7 using the following three methods:

(a) explicit Euler method (Section 1.2.1);

(b) implicit Euler method (Section 1.2.2), using a built-in root-finder of your numerical software;

(c) a built-in ODE solver of your numerical software.

Compare the accuracy of explicit and implicit methods across different grid sizes $(N = 11, 51, 501, 10001)$. For each of the three equations, find the grid size that you like the most and plot the results from the three approximation methods together with the respective true solution. These are given by:

$$y(x) = (20x+1)^{1/20} \qquad y(x) = \frac{1}{1-\sin x^2/2} \qquad y(x) = \cos x$$

3. Now consider a variation of the implicit method: instead of using a built-in root-finder, perform <u>one</u> step of Newton's method. It is an iterative root-finding algorithm that solves $F(y) = 0$ starting from an initial guess y^0 and updating via

$$y^{n+1} = y^n - (J^n)^{-1}F(y^n)$$

where J^n is the Jacobian of $F(y^n)$, so that $J^n_{ij} = \partial F_i(y^n)/\partial y^n_j$ for the multivariate case. The idea is to compute the tangent of $F(\cdot)$ at y^n and find the point y^{n+1} where this tangent intersects zero. Instead of iterating the algorithm until convergence, we can make a single step and hopefully save some time without losing a lot of accuracy. For our purposes, define $F_i(\cdot)$ at every grid point as follows:

$$F_i(y) \equiv \frac{y - y_{i-1}}{x_i - x_{i-1}} - g(x_i, y)$$

where $g(x_i, y)$ is the RHS of the explicitly written ODE (see equation (2) in Sebastian's notes). In each step, use y_{i-1} as the initial guess and compute y_i via one step of the Newton's method. Compare the results with the "fully-fledged" implicit method. When does the variation work well and when does it fail?

Bibliography

Brunnermeier, Markus K. and Yuliy Sannikov, "Macro, money, and finance: A continuous-time approach," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 1497–1545.

Part II

Real Models with Financial Frictions

Chapter 3

Simple Macrofinance Models

In the last chapter, we studied the Merton portfolio choice problem as an introduction to continuous-time modeling. The main objective of this lecture is to illustrate some main building blocks of a large body of macro-finance models that employs continuous-time methods. First, we develop a general model with a leverage constraint and risk. Then we cover three specific examples of this model: a complete markets benchmark with no risk, the Basak-Cuoco model (Basak and Cuoco, 1998) which features risk but no leverage constraint, and the Kiyotaki-Moore (Kiyotaki and Moore, 1997) model which features a leverage constraint but no risk.

3.1 Setup with Two-Type of Agents

3.1.1 Model Setup

Environment. Time is continuous. There is no labor and hence capital is the only factor used in production. The economy consists of two types of agents – experts and households. We denote the two types by $i = \{e, h\}$.[1] There is a continuum (with mass one) of both types, $\tilde{i} \in [0, 1]$.[2] Aggregate capital stock evolves exogenously according

[1] In general, i can denote different types/sectors, or different subgroups within the same sector.
[2] Individual-specific analysis will only be needed in environments with idiosyncratic risk.

to

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t,$$

with $g > 0$ and $\sigma > 0$ model parameters. Notice there is no investment in capital.

Notation. We denote the net worth of each sector as N_t^i while N_t is the net worth of the economy. The single endogenous state variable is the wealth share of experts, denoted by $\eta_t \equiv N_t^e / N_t$. The share of the aggregate capital stock held by the expert sector is denoted as κ_t. The portfolio share on capital held by each sector is $\theta_t^{K,i}$. We denote r_t as the risk free rate and $r_t^{K,i}$ as the rate of return on capital for each sector. In general, the sector capital stock is obtained by

$$K_t^i \equiv \int_0^1 k_t^{i,\tilde{i}} \mathrm{d}\tilde{i}.$$

Since in this model there is only aggregate risk, $K_t^i = k_t^{i,\tilde{i}}$ for all $\tilde{i}$ in each sector. The same is true for consumption

$$C_t^i \equiv \int_0^1 c_t^{i,\tilde{i}} \mathrm{d}\tilde{i},$$

where $C_t^i = c_t^{i,\tilde{i}}$ for all $\tilde{i}$ in each sector. The price of capital is denoted by q_t and sector capital holdings are denoted by $K_t^e = \kappa_t K_t$ and $K_t^h = (1 - \kappa_t) K_t$. The net worth of the economy is given by the value of the aggregate capital stock so that $N_t = q_t K_t$.

Financial Frictions. No equity issuance is allowed. There is debt issuance D_t from experts to households, but with the collateral constraint $D_t^e \leq \ell \kappa_t^e q_t K_t$. This can be rearranged to $\frac{D_t^e}{N_t^e} \leq \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1 - \theta_t^{K,e}) \leq \ell \theta_t^{K,e} \Leftrightarrow (1 - \ell)\theta_t^{K,e} \leq 1$. ℓ is an exogenous parameter measuring the degree of the collateral constraint. Finally, each sector is subject to a non-negativity constraint on capital holdings.

Experts' Problem. Experts have a CRS production function $y_t^e = a^e k_t^e$. Denote experts' consumption by c_t^e.[3] Since experts invest $\theta_t^{K,e}$ of their net worth into capital and

[3] Again, we will suppress the superscript $\tilde{i}$ throughout this chapter.

consume at rate c_t^e their net worth process follows

$$\frac{\mathrm{d}n_t^e}{\mathrm{d}t} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right].$$

Experts have a log-utility function and solve the following optimization problem

$$\begin{aligned}
\max_{c_t^e,\theta_t^{K,e}} & \int_s^\infty e^{-\rho^e t} u(c_t^e)\mathrm{d}t, \quad \text{s.t.} \\
& (1-\ell)\theta_t^{K,e} \leq 1, \\
& \theta_t^{K,e} \geq 0, \\
\frac{\mathrm{d}n_t^e}{\mathrm{d}t} &= \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right].
\end{aligned}$$

Households' Problem. Households' production function $a^h(1-\kappa_t)k_t^h$ is a function of (aggregate) κ_t^h. Productivity $a^h(1-\kappa_t) \leq a^e$, with equality for $\kappa = 1$ and is strictly decreasing in $1-\kappa$. This means output is given by $y_t^h = a^h(1-\kappa_t)k_t^h = a^h(\cdot)(1-\kappa_t)K_t$. Households consume c_t^h. Since households invest $\theta_t^{K,h}$ of their net worth into capital, their net worth process follows

$$\frac{\mathrm{d}n_t^h}{\mathrm{d}t} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right].$$

Households have a log-utility function and solve the following optimization problem

$$\begin{aligned}
\max_{c_t^h,\theta_t^{K,h}} & \int_s^\infty h^{-\rho^h t} u(c_t^h)\mathrm{d}t, \quad \text{s.t.} \\
& \theta_t^{K,h} \geq 0, \\
\frac{\mathrm{d}n_t^h}{\mathrm{d}t} &= \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right].
\end{aligned}$$

Market Clearing. We have that the final goods market clears, following

$$C_t^e + C_t^h = a^e K_t^e + a^h(1-\kappa_t)K_t^h.$$

The capital market clears, such that

$$K_t^e + K_t^h = K_t.$$

3.2 A Complete Markets Benchmark

In this subsection, we study a frictionless case in which there is no risk, i.e. $\sigma = 0$, and the leverage constraint is not active, i.e. $\ell \to \infty$. These assumptions imply that markets are complete (risk-free assets span all possible payoffs without risk and, without a constraint, they can be traded frictionlessly).

3.2.1 Solution Method

Step 1 Goods Market Clearing

The final goods market must clear at each t, giving the following condition

$$C_t^e + C_t^h = a^e K_t^e + a^h(1-\kappa_t)K_t^h$$

For the case of log-utility, we know from Chapter 2 that the consumption-net worth ratio is equal to the discount rate so that $C_t^e = \rho^e N_t^e = \rho^e \eta_t q_t K_t$. Analogously, $C_t^h = \rho^h(1-\eta_t)q_t K_t$. Plugging this in and manipulating yields

$$q_t = \frac{a^e \kappa_t + a^h(1-\kappa_t)(1-\kappa_t)}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

Step 2 Portfolio choice

Let us first obtain the return processes for capital for each sector. Each return process is composed of a dividend yield as well as a capital gain term so that

$$r_t^{K,e} = \frac{a^e}{q_t} + g + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t}$$

$$r_t^{K,h} = \frac{a^h(1-\kappa_t)}{q_t} + g + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t}$$

By virtue of $a^h(1-\kappa_t) \leq a^e$, we have that $r_t^{K,h} \leq r_t^{K,e}$ so that experts always have a higher rate of return on capital relative to households. This implies that the non-negativity constraint on capital holdings for the household sector must bind. Since the capital market must clear in equilibrium, we have that experts hold all the capital, i.e. $\kappa_t = 1$. Hence the price of capital is given by

$$q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

which is a function of the net worth share of experts.

Since there is no leverage constraint, the experts instantly take out loans to finance the purchase of all available capital in the economy. Because of the capital market clearing condition

$$\theta_t^{K,e}\eta_t q_t K_t + \theta_t^{K,h}(1-\eta_t)q_t K_t = q_t K_t,$$

this implies that $\theta_t^{K,e} = 1/\eta_t$ instantaneously. Then, in the transition, experts repay their loans to households which in turn finances their consumption.

The non-negativity constraint on capital holdings for the household sector binding also implies that the risk-free rate must equal the return on capital if held by experts. This is since only experts will have an interior solution.

Step 3 Evolution of the net worth share

To find the evolution of the net worth share, recall that $\eta_t = N_t^e/N_t$ so that by Itô's lemma $\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$ since $\mu_t^N = \frac{1}{N_t}\frac{\mathrm{d}N_t}{\mathrm{d}t} = \eta_t\mu_t^{N^e} + (1-\eta_t)\mu_t^{N^h}$. The evolution of the net worth for experts is given by

$$\mu_t^{N^e} = -\rho^e + \theta^{K,e}(r_t^{K,e} - r_t) = -\rho^e,$$

since $r_t^{K,e} = r_t$. The evolution of the net worth for households is given by

$$\mu_t^{N^h} = -\rho^h + \theta^{K,h}(r_t^{K,h} - r_t) = -\rho^h,$$

since $\theta^{K,h} = 0$. Putting these together yields the evolution of the net worth share

$$\mu_t^\eta = -(1 - \eta_t)(\rho^e - \rho^h).$$

Step 4 Solving the ODE

Notice that the evolution of the net worth share can be written as the following ODE

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t} = -\eta_t(1 - \eta_t)(\rho^e - \rho^h)$$

Solving this with an initial condition for η_0 yields the following closed-form expression for η_t

$$\eta_t = \frac{e^{-(\rho^e - \rho^h)t}}{\frac{1-\eta_0}{\eta_0} + e^{-(\rho^e - \rho^h)t}}$$

Hence, for $\rho^e > \rho^h$ we have $\eta_t \to 0$ as $t \to \infty$ while $\eta_t \to 1$ for $\rho^e < \rho^h$. On the other hand, if $\rho^e = \rho^h$, then $\eta_t = \eta_0$ is constant over time.

3.2.2 Benchmark Model Conclusions

This frictionless model shows that

i) capital is always held by the most efficient sector which in this case are the experts,

ii) the consumption allocation is determined by the initial wealth distribution and wealth only moves due to differences in preferences for the timing of consumption, i.e. $\rho^e - \rho^h$. For $\rho^e = \rho^h$, every initial condition leads to a steady state, while for different time preferences, the model converges in the long run to a boundary

$\eta = 0$ for $\rho^e > \rho^h$ or $\eta = 1$ for $\rho^e < \rho^h$. However, these dynamics do not affect production,

iii) the price of capital is constant for when there are no differences in discount rates but otherwise, it rises over time because the agents with the lower marginal propensity to consume become richer as time goes by.

3.3 Basak-Cuoco Model

In this Section we study the heterogeneous agents model of Basak and Cuoco (1998), a simple yet classic model. This model is a special case of the general model in which there is no leverage constraint, i.e. $\ell \to \infty$, and households cannot produce, i.e. $a^h \to -\infty$.

3.3.1 Solution Method

Step 1 Postulate aggregates, price processes and obtain return processes

In general, the aggregate capital stock is obtained by

$$K_t \equiv \int_0^1 k_t^{e,\tilde{i}} \mathrm{d}\tilde{i}.$$

With only aggregate risk, all experts (households) are identical (i.e., $k_t^{e,\tilde{i}} = k_t^e, \forall \tilde{i}$), so total capital stock, expert net worth and household net worth can be simply obtained by $K_t = k_t^e, N_t^e = n_t^e, N_t^h = n_t^h$.

Denote the price of capital by q_t. The total wealth of the economy is $q_t K_t$. The wealth share of experts is $\eta_t = N_t^e / (N_t^e + N_t^h) = N_t^e / q_t K_t$. We then *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Importantly, volatility of price loads on the same Brownian motion as capital stock. Given the price process and the consumption decision of experts, we can calculate the return rate to capital, $r_t^{K,e}$.[4] Using Itô's product rule,

$$
\begin{aligned}
\mathrm{d}r_t^{K,e} &= \overbrace{\frac{a^e}{q_t}\mathrm{d}t}^{\text{Dividend Yield}} + \overbrace{\frac{\mathrm{d}(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\
&= \left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right]\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t.
\end{aligned} \tag{3.1}
$$

We then postulate that the stochastic discount factor (SDF, e.g., $\xi_t^i = e^{-\rho^i t}u'(c_t^i)$) is a diffusion process:

$$
\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \mu_t^{\xi^i}\mathrm{d}t + \sigma_t^{\xi^i}\mathrm{d}Z_t, \qquad i \in \{e, h\},
$$

As we showed in Chapter 2, it turns out that its drift $\mu_t^{\xi^i}$ is the negative of the risk-free rate r_t, and its volatility loading $\sigma_t^{\xi^i}$ is the negative of the price of risk ς_t^i. As a result, the SDF follows

$$
\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t\mathrm{d}t - \varsigma_t^i\mathrm{d}Z_t, \qquad i \in \{e, h\}, \tag{3.2}
$$

Step 2 For given SDF processes, derive individual equilibrium conditions

Since all experts are identical, aggregate capital stock K_t also follows

$$
\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma\mathrm{d}Z_t.
$$

We use the Stochastic Maximum Principle to solve the optimization problems. Following the exposition in Chapter 2, the Hamiltonian for the experts is given by

$$
\begin{aligned}
\mathcal{H}_t^e &= e^{-\rho^e t}\log c_t^e + \xi_t^e n_t^e \mu_t^{n^e} - \varsigma_t^e \xi_t^e n_t^e \sigma_t^{n^e} \\
&= e^{-\rho^e t}\log c_t^e + \xi_t^e\left[-c_t^e + n_t^e r_t + n_t^e\theta_t^{K,e}\left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right)\right]\mathrm{d}t - \varsigma_t^e\xi_t^e n_t^e\theta_t^{K,e}(\sigma + \sigma_t^q)
\end{aligned}
$$

[4]For superscripts, we use lowercase letters for different *types* and capital letters for different *assets*.

The first-order conditions for the expert's choices of consumption c_t^e and capital shares $\theta_t^{K,e}$ are given by

$$
\begin{aligned}
e^{-\rho^e t}\left(c_t^e\right)^{-1} &= \xi_t^e \\
\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t &= \varsigma_t^e(\sigma + \sigma_t^q)
\end{aligned}
$$

The costate equation reads by virtue of the

$$
\begin{aligned}
\mathrm{d}\xi_t^e &= -\frac{\partial H_t^e}{\partial n_t^e}\mathrm{d}t - \varsigma_t^e \xi_t^e \mathrm{d}Z_t \\
&= -r_t \xi_t^e - \varsigma_t^e \xi_t^e \mathrm{d}Z_t
\end{aligned}
$$

where the equality follows from the first-order condition for $\theta_t^{K,e}$.

Using the first-order condition for consumption, we also find by Itô's lemma that

$$
\mathrm{d}\xi_t^e = \left[-\rho^e - \mu_t^{c^e} + (\sigma_t^{c^e})^2\right]\mathrm{d}t - \sigma_t^{c^e}\mathrm{d}Z_t
$$

We already know that for the case of log-utility $c_t^e = \rho^e n_t^e$, so that $\mu_t^{c^e} = \mu_t^{n^e}$ and $\sigma_t^{c^e} = \sigma_t^{n^e}$. Then the price of risk is given by $\varsigma_t^e = \sigma_t^{c^e} = \sigma_t^{n^e} = \theta_t^{K,e}(\sigma + \sigma_t^q)$.

Similarly, the household sector will consume according to $c_t^h = \rho^h n_t^h$.

Finally, we combine two sectors' problems with market clearing conditions and solve the model. We will see the price of capital q_t is actually a constant in section Step 4.

Step 3 Evolution of state variable η_t

In this model, agents start with some initial endowments of capital. Over time, they allocate their wealth between the assets available to them by solving their respective utility maximization problems, subject to budget constraints and taking prices as given. Given prices, markets for capital and consumption goods have to clear.

Recall that

$$\eta_t = \frac{N_t^e}{q_t K_t} \in [0,1].$$

The total wealth of experts N_t follows

$$\begin{aligned}
\frac{dN_t^e}{N_t^e} = \frac{dn_t^e}{n_t^e} &= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left[dr_t^K - r_t dt\right] \\
&= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right]dt + (\sigma + \sigma_t^q)dZ_t\right\} \\
&= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\varsigma_t^e(\sigma + \sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{d(q_t K_t)}{q_t K_t} &= \left[\mu_t^q + g + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t \\
&= \left[r_t - \frac{a^e}{q_t} + \varsigma_t^e(\sigma + \sigma_t^q)\right]dt + (\sigma + \sigma_t^q)dZ_t.
\end{aligned}$$

Apply Itô's quotient rule to $\eta_t = N_t^e / q_t K_t$:

$$\frac{d\eta_t}{\eta_t} = \left[-\frac{c_t^e}{n_t^e} + \frac{a^e}{q_t} - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)\left(\varsigma_t^e - (\sigma + \sigma_t^q)\right)\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t. \tag{3.3}$$

Using what we found in the previous step for $c_t^e = \rho^e n_t^e$ and $\varsigma_t^e = \theta_t^{K,e}(\sigma + \sigma_t^q)$, we have

$$\frac{d\eta_t}{\eta_t} = \left[-\rho^e + \frac{a^e}{q_t} + (1 - \theta_t^{K,e})^2(\sigma + \sigma_t^q)^2\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t. \tag{3.4}$$

Step 4 Market clearing

Finally, we can close the model using market clearing conditions. Consumption good market clearing yields

$$C_t = \rho^e N_t^e + \rho^h N_t^h = \rho^e \eta_t q_t K_t + \rho^h (1-\eta_t) q_t K_t = a^e K_t \implies q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1-\eta_t)}$$

Capital market clearing yields

$$\theta_t^{K,e} = \frac{q_t K_t}{N_t^e} = \frac{1}{\eta_t}. \tag{3.5}$$

Then, the law of motion of η_t is

$$\frac{\mathrm{d}\eta_t}{\eta_t} = (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{1-\eta_t}{\eta_t^2}(\sigma + \sigma_t^q)^2\right]\mathrm{d}t + \frac{1-\eta_t}{\eta_t}(\sigma + \sigma_t^q)\mathrm{d}Z_t. \tag{3.6}$$

where by Itô's lemma

$$\sigma_t^q = -\frac{\rho^e - \rho^h}{\rho^e \eta_t + \rho^h (1-\eta_t)} \sigma_t^\eta \eta_t$$

Hence

$$\sigma_t^\eta = \frac{1-\eta_t}{\eta_t} \frac{\rho^e \eta_t + \rho^h (1-\eta_t)}{\rho^e} \sigma$$

and

$$\begin{aligned}\frac{\mathrm{d}\eta_t}{\eta_t} =& (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{1-\eta_t}{\eta_t^2}\frac{\left[\rho^e \eta_t + \rho^h (1-\eta_t)\right]^2}{(\rho^e)^2}\sigma^2\right]\mathrm{d}t \\ &+ \frac{1-\eta_t}{\eta_t}\frac{\rho^e \eta_t + \rho^h (1-\eta_t)}{\rho^e}\sigma \mathrm{d}Z_t\end{aligned}$$

a simple one-dimensional stochastic differential equation (SDE).

Numerical example. The figure below shows the price of capital, its volatility, the arithmetic drift and volatility of net worth as a function of net worth η. The parameter values are ρ^e = 0.06, ρ^h = 0.04, a^e = 0.11, σ = 0.10 and g=0.1.

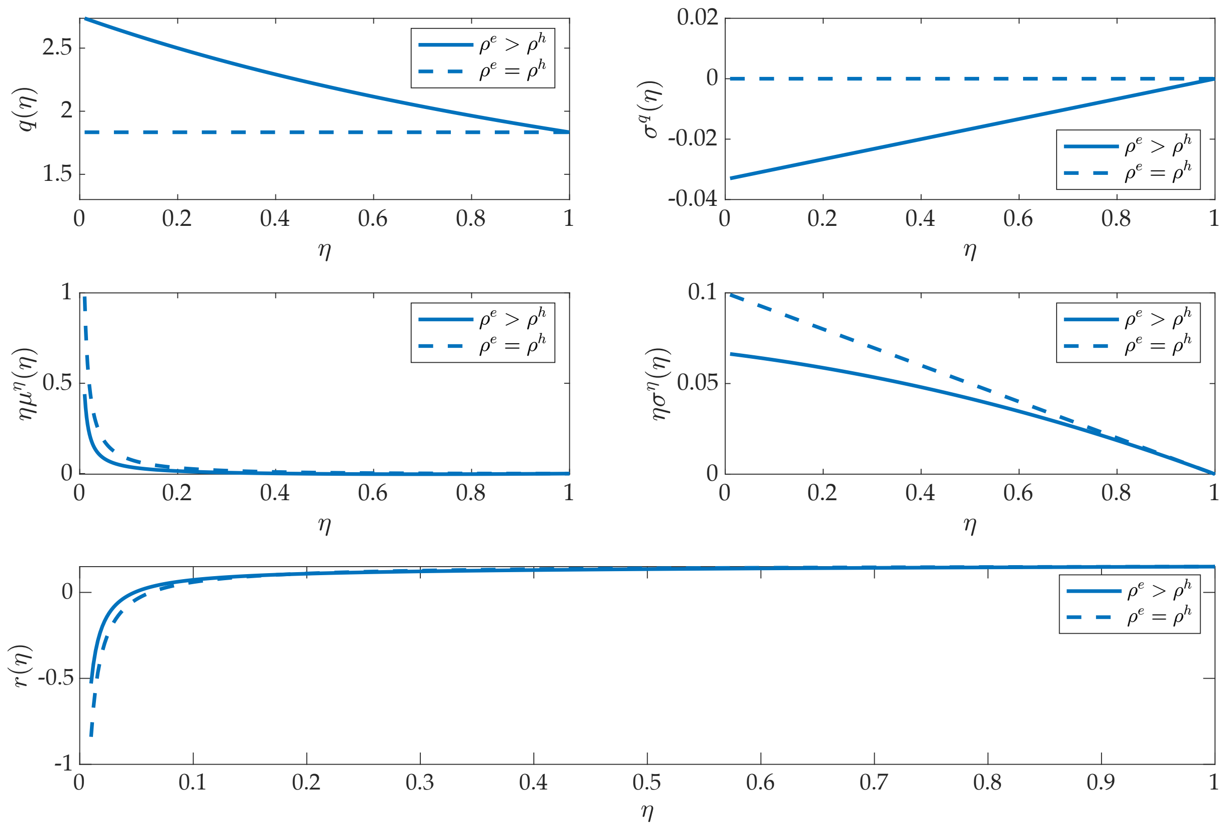


Figure 3.1: Basak-Cuoco numerical example

3.4 Kiyotaki-Moore Model

The third specific example of the general model is the Kiyotaki-Moore model. It is a deterministic model with a single zero probability shock rather than a model that features risk, i.e. $\sigma = 0$. Here we present a modified version of the model in Kiyotaki and Moore (1997). In particular, we convert the discrete-time model into a continuous-time setting.

3.4.1 Solution Method

Step 1 Postulate aggregates, price processes and obtain return processes

We *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t.$$

Importantly, there is no stochasticity as $\sigma = 0$. Given the price process and the consumption decision of experts, we can calculate their return rate to capital, $r_t^{K,e}$.[5] Using Itô's product rule,

$$\begin{aligned}
\mathrm{d}r_t^{K,e} &= \overbrace{\left(\frac{a^e}{q_t} + g\right)\mathrm{d}t}^{\text{Dividend Yield}} + \overbrace{\frac{\mathrm{d}(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\
&= \left[\frac{a^e}{q_t} + g + \mu_t^q\right]\mathrm{d}t.
\end{aligned} \tag{3.7}$$

Similarly,

$$\mathrm{d}r_t^{K,h} = \left[\frac{a^h(1-\kappa)}{q_t} + g + \mu_t^q\right]\mathrm{d}t. \tag{3.8}$$

We then postulate that the stochastic discount factor (SDF, e.g., $\xi_t^i = e^{-\rho^i t}u'(c_t^i)$) is the process:

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \mu_t^{\xi^i}\mathrm{d}t, \qquad i \in \{e, h\},$$

As we showed in Chapter 2, it turns out that its drift $\mu_t^{\xi^i}$ is the negative of the risk-free rate r_t. As a result, the SDF follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t, \qquad i \in \{e, h\}, \tag{3.9}$$

[5] For superscripts, we use lowercase letters for different *types* and capital letters for different *assets*.

Step 2 For given SDF processes, derive individual equilibrium conditions

The Hamiltonians can be constructed as

$$
\begin{aligned}
\mathcal{H}_t^e =& e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left(1 - (1-\ell)\theta_t^{K,e}\right),\\
\mathcal{H}_t^h =& e^{-\rho^h t}u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h\left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right].
\end{aligned}
$$

where the ξ_t^i is the multiplier on the budget constraint and the $\xi_t^e n_t^e \lambda_t^\ell$ is the multiplier on the leverage constraint. Later we show that co-state variable ξ_t^i equals SDF, which for log-utility $= e^{-\rho^i t}\frac{1}{\rho^i n_t^i}$. Note that the Fisher Separation Theorem between consumption and portfolio choice applies. That is the first order conditions with respect to consumption, c_t^i, and portfolio choice, $\theta_t^{K,i}$ are independent. The first order conditions with respect to c_t^i and $\theta_t^{K,i}$ are given by

$$
\left.\begin{aligned}
e^{-\rho^e t}u'(c_t^e) =& \xi_t^e\\
e^{-\rho^h t}u'(c_t^h) =& \xi_t^h
\end{aligned}\right. \Rightarrow c_t^i = \rho^i n_t^i, \text{under log utility}
$$

and,

$$
\begin{aligned}
r_t^{K,e} - r_t &= (1-\ell)\lambda_t^\ell\\
r_t^{K,h} - r_t &= 0.
\end{aligned}
$$

The additional first order condition is given by the costate equation,

$$
\mathrm{d}\xi_t^i = -\frac{\mathrm{d}H_t^i}{\mathrm{d}n_t^i},
$$

which yields

$$
\begin{aligned}
\mathrm{d}\xi_t^e &= \xi_t^e\left(\left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right) + \lambda_t^\ell\left(1 - (1-\ell)\theta_t^{K,e}\right)\right),\\
\mathrm{d}\xi_t^h &= \xi_t^h\left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right).
\end{aligned}
$$

Plugging in the first order conditions for portfolio choices yields, as expected,

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t, \qquad i \in \{e, h\}. \tag{3.10}$$

Step 3 Evolution of state variable η_t

In this model, agents start with some initial endowments of capital. Over time, they allocate their wealth between the assets available to them by solving their respective utility maximization problems, subject to budget constraints and taking prices as given. Given prices, markets for capital and consumption goods have to clear. To think about dynamics first note that equilibrium objects are functions of the single state, the net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t K}$. The state dynamics are solved by noting

$$\mu_t^N dt = \frac{dN_t}{N_t} = \underbrace{\frac{N_t^e}{N_t}}_{\eta_t} \mu_t^{N^e} dt + \underbrace{\frac{N_t^h}{N_t}}_{(1-\eta_t)} \mu_t^{N^h} dt$$

such that,

$$\begin{aligned}
\mu_t^\eta &= \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h}) \\
&= (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}(r_t^{K,e} - r_t) - \theta_t^{K,h} \overbrace{(r_t^{K,h} - r_t)}^{=0 \text{ from above}}\right] \\
&= (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}\left(\underbrace{\frac{a^e}{q_t} - \frac{a^h(1-\kappa_t)}{q_t}}_{=r_t^{K,e} - r_t^{K,h}}\right)\right].
\end{aligned}$$

Step 4 Market clearing

Finally, we can close the model using market clearing conditions. Consumption good market clearing yields

$$q_t K_t[\rho^e \eta_t + \rho^h(1-\eta_t)] = [a^e \kappa_t^e + a^h(1-\kappa_t)(1-\kappa_t)]K_t.$$

Capital market clearing yields

$$\underbrace{\theta_t^{K,e}\eta_t}_{=\kappa_t} q_t K_t + \underbrace{\theta_t^{K,h}(1-\eta_t)}_{=1-\kappa_t} q_t K_t = q_t K_t \tag{3.11}$$

Equilibrium Conditions Summary. The equilibrium objects (κ, q, r) are functions of the state η_t. Following some algebra, they are pinned down by the following conditions,

$$\begin{aligned}
q_t[(\rho^e-\rho^h)\eta_t+\rho^h] &= \kappa_t a^e + (1-\kappa_t)a^h(1-\kappa_t) \\
\kappa_t &\leq \frac{\eta_t}{1-\ell} \\
\mu_t^\eta &= (1-\eta_t)\left[-(\rho^e-\rho^h) + \frac{\kappa_t}{\eta_t}\frac{a^e-a^h(1-\kappa_t)}{q_t}\right].
\end{aligned}$$

3.4.2 Solution and Comments

Global Non-Linear Solution. Figure 3.2 plots the global solution to this model for the parameter values specified.

When experts have a low share of the total wealth their collateral constraint is binding. They put twice their wealth in capital ($\theta_t^{e,K} = 2$). They would like to lever up more but cannot and hence we observe an excess return over the households, which is increasing as the experts net worth share decreases. For larger wealth shares the portfolio choice of the experts becomes an interior solution. The excess return is then 0 as the constraint is slack.

Furthermore, the price of capital, q_t, is decreasing as wealth share decreases, in the binding collateral constraint region. The reason for this is that as the experts' wealth

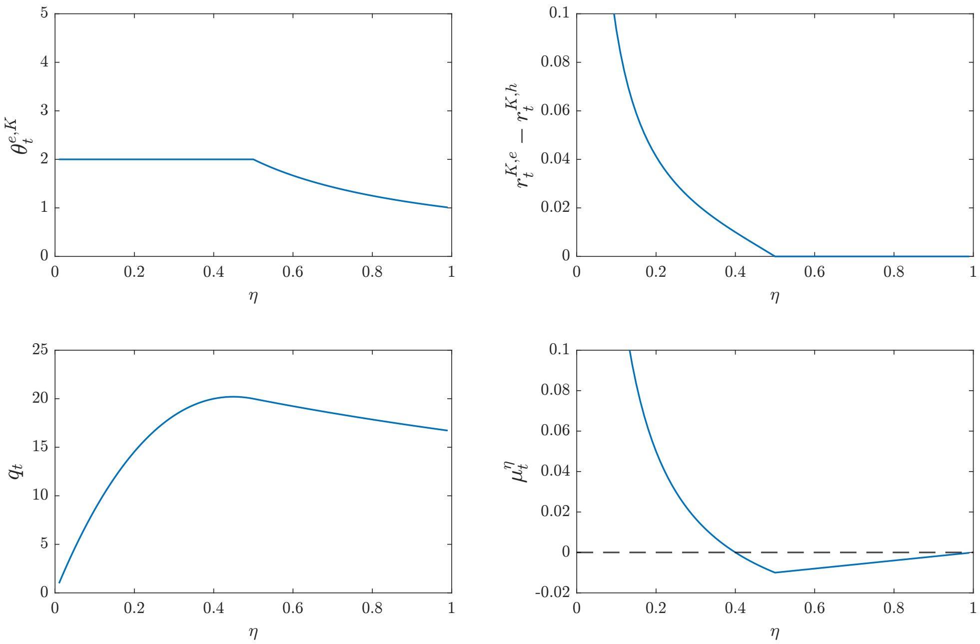


Figure 3.2: Global non-linear solution for parameters: $\rho^e = 0.06$, $\rho^h = 0.04$, $\ell = 0.05$, $a^e = 1.0$, $a^h(1-\kappa) = a^e\kappa$

share decreases they can hold less and less capital due to the collateral constraint. As capital is in fixed supply, households hold more and more, meaning each marginal value unit of capital becomes less productive, depressing the price. For high η values the decline in the price is from the fact that experts are less patient than households. The steady state is 0.4. For any value of η above this η drifts down, and for any value below η drifts up.

Impulse Responses. One can do analysis of an unanticipated shock to the steady state. Figure 3.3 plots impulse response functions for the same system above, for a 30% (of η) negative redistribution shock. We can observe the wealth share and price drifting back up to the steady state.

Aside: Understanding Asset Prices. With some proper initial conditions, the price dynamics are given by the solution of the following differential equation,

$$\frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} + \frac{a^h(1-\kappa_t)}{q_t} = r_t,$$

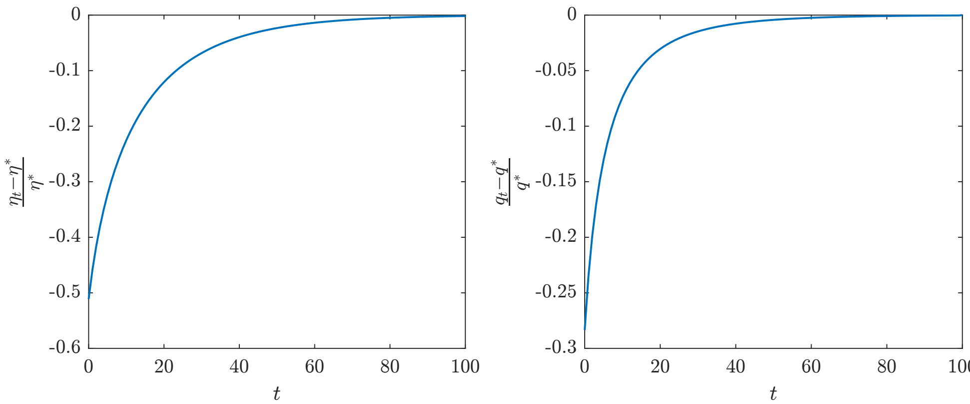


Figure 3.3: Impulse response function with 30% (of η) negative redistribution shock. Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

which is

$$q_t = \int_t^\infty e^{-\int_t^s r_u du} a^h(1 - appa_s^h)ds.$$

The discrete time analogy for this is the difference equation and it's solution,

$$\frac{q_{t+1} - q_t}{q_t} + \frac{a^h(1-\kappa_t)}{q_t} = r_t,$$

$$q_t = \sum_{s=0}^{\infty} \left[\prod_{u=0}^{s} \frac{1}{(1 + r_{t+u})} \right] a^h(1 - \kappa_{t+s}).$$

That is, the asset price is the sum of discounted dividend flows, and is solved backwards.

3.4.3 Log-linearized Dynamics around Steady State

The way the Kiyotaki-Moore model was solved originally, also in discrete time, was through a log-linearization around the steady state. We will now cover this and compare. First we derive the steady state with $\mu^\eta = 0$ and its' properties. Then we will log-linearize and characterize the dynamical system locally around the steady state.

The Steady State. First note that the collateral constraint always binds in the steady state. If the collateral constraint were not binding $\lambda_t^\ell = 0$ and hence $r^{K,e} = r^{K,h}$, i.e. $a^e = a^h(\cdot)$. The constraint does not need to bind *only if* $\kappa_t = 1$. Then $\mu_t^\eta = (1-\eta_t)(\rho^h - \rho^e)$

and since $\rho^e > \rho^h$, we have $\mu_t^\eta < 0$, i.e. η declines. Now, since the collateral constraint binds, the steady state capital share is given by

$$\kappa^{SS} = \frac{\eta^{SS}}{1-\ell}.$$

The expert sector's net worth share is $\eta_t := \frac{N_t^e}{q_t \bar{K}}$, is constant, i.e. $\mu_t^\eta := \frac{\mathrm{d}\eta_t}{\mathrm{d}t} = 0$. From the good's market clearing condition and the last equation of the equilibrium conditions,

$$\begin{aligned} q^{SS}[(\rho^e - \rho^h)\eta^{SS} + \rho^h] &= \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS}) \\ (\rho^e - \rho^h) &= \frac{\kappa^{SS}}{\eta^{SS}} \frac{a^e - a^h(1-\kappa^{SS})}{q^{SS}} \quad \text{for } \mu^\eta = 0. \end{aligned}$$

Combining the above,

$$\begin{aligned} \kappa^{SS} a^e - \kappa^{SS} a^h (1-\kappa^{SS}) + q^{SS}\rho^h &= \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS}) \\ \Rightarrow \quad q^{SS} &= a^h(1-\kappa^{SS})/\rho^h, \end{aligned}$$

where the steady state κ^{SS} is implicitly given by:

$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell} \frac{a^e - a^h(1-\kappa^{SS})}{a^h(1-\kappa^{SS})}.$$

Finally, for specific functional form $a^h(1-\kappa_t) = a^e \kappa_t$,

$$\kappa^{SS} = \frac{1}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1} \quad \Rightarrow \eta^{SS} = \frac{1-\ell}{(1-\ell)(\rho^e-\rho^h)/\rho^h + 1}.$$

One can do comparative statics with this derivation. For example for higher leverage, ℓ, (i.e. less tight collateral constraint)

- κ^{SS}, SS-capital share, is higher.
- η^{SS}, SS-net worth share, is lower.
- $q^{SS} = \frac{a^h}{\rho^h}$, price of capital, is higher. $q^{SS}\bar{K}$, total wealth in the economy, is higher too.

- $N^{e,SS}$ SS-experts' net worth, is higher (Check?)

Comparative statics analyzes a permanent (long-run) shift to new steady state, but not the dynamics between them. It compares two separate steady states. To analyze dynamics we can log-linearize.

Log-linearized Dynamics. Analytical solutions to η_t, q_t dynamics are hard to obtain. Expanding around the steady state,

$$
\begin{aligned}
\log(\eta_t/\eta^{SS}) &= \hat{\eta}_t \\
\log(q_t/q^{SS}) &= \hat{q}_t \\
\log(r_t/r^{SS}) &= \hat{r}_t \\
\log(a_t^h/a^{h,SS}) &= \hat{a}_t^h
\end{aligned}
$$

As an exercise one can derive an expression for $\hat{a}_t^h, \hat{q}_t^h$ as a function of $\hat{\eta}_t$ with a first order Taylor approximation. The state dynamics and price dynamics become,

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} &= \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right) \\
\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} &= r^{SS}(\hat{r}_t + \hat{q}_t - \hat{a}_t^h).
\end{aligned}
$$

This works well for small shocks around the steady state, where the drift of η is close to linear. However, further away from the steady state the log-linearization can lead to misleading conclusions about dynamics of the system.

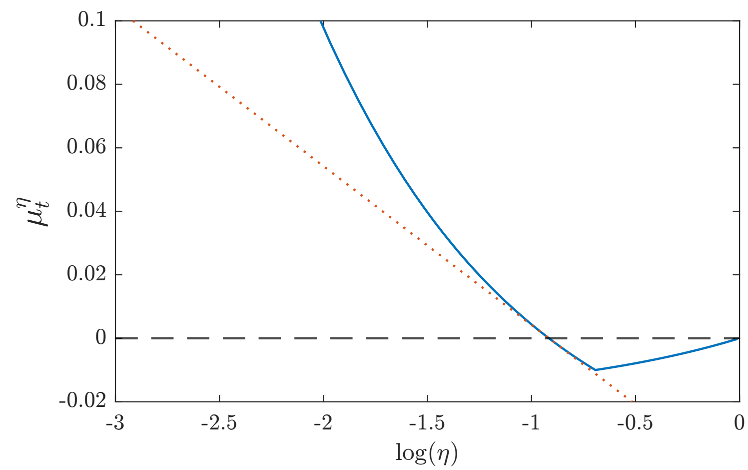


Figure 3.4: Global vs. Log-linearized Solution for η-drift. Note: x-axis is $\log(\eta)$, since we log-linearized.

3.4.4 Decomposing Amplification Effects

To decompose amplification effects, we first start at the steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$. We then shock the system by redistributing a fraction of experts' net worth share to households. In the original Kiyoataki-Moore model, the productivity shock lasts for one period (not for an instant), which causes initial redistribution. With determenistic recovery the immediate impacts, at $t = 0$, are as follows. First there is a direct redistributive effect/shock. Second, there is a price-net worth effect. The decline in q_t reduces experts' net worth share as they are levered, which feeds back into the price-net worth effect. Third, there is a price-collateral effect. The decline in q_t tightens collateral constraints, which also feeds back into price-net worth effect. Subsequent impact $t > 0$ feeds back into immediate impacts.

To carry out a decomposition, we switch off the price-collateral effect by assuming that the collateral constraint is determined by SS-price q^{SS} instead of the equilibrium price q_t. Formally, the collateral constraint, $\kappa_t \leq \frac{\eta_t}{1-\ell}$, becomes $\kappa_t \leq \frac{\eta_t}{1-\ell q^{SS}/q_t}$. Figure 3.5 plots the impulse response function of this decomposition, for a 30% (of η) negative redistribution shock. Because of this amplification, when all channels are on this translates to just over a 50% negative redistribution. If we switch off the the price-collateral effect the amplification translates to just under a 40% negative redistribution. The difference in amplification effects at $t = 0$ is also clear for the difference in price.

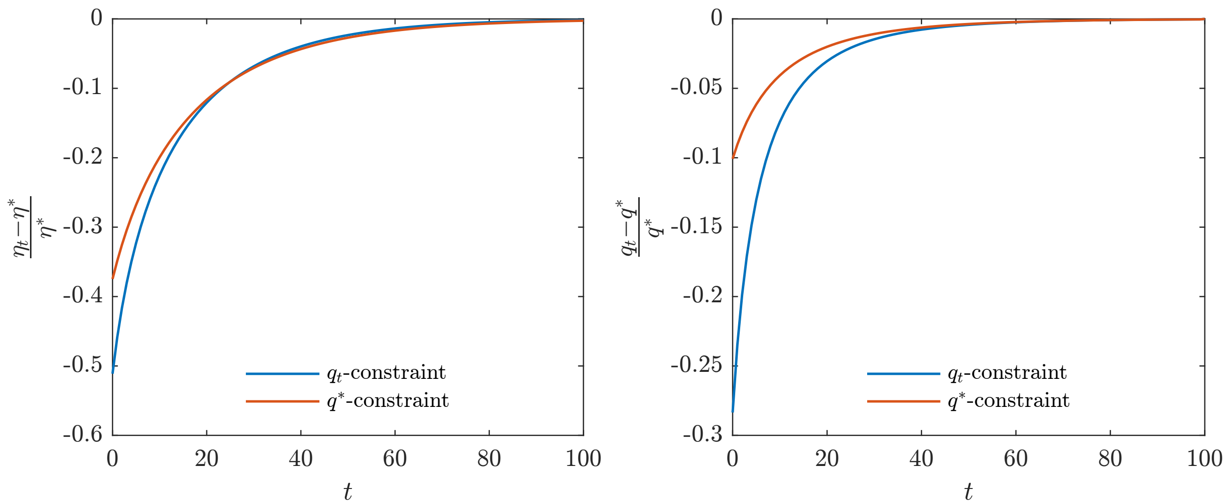


Figure 3.5: Impulse response function with 30% (of η) negative redistribution shock. Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = \kappa$

To show more clearly how this works, consider that economy is at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$. There is then a negative initial/direct redistributive shock $\eta' = (1 - \epsilon)\eta^{SS}$. The new price q', and capital holding κ' solve,

$$
\begin{aligned}
q' &= \frac{\kappa' a^e + (1-\kappa')a^h(1-\kappa')}{(\rho^e - \rho^h)\eta' + \rho^h}, && \text{(Goods market)} \\
\kappa' &= \frac{\eta^{SS}(1-\epsilon)}{1-\ell}, && \text{(q_t-constraint)} \\
\kappa' &= \frac{\eta^{SS}(1-\epsilon)}{1-\ell q^{SS}/q'}. && \text{(q^{SS}-constraint)}
\end{aligned}
$$

However, the debt contract was signed by the old price q^{SS} so η drops further. The first round effect is done by considering the balance sheet,

$$
\frac{\eta'}{1-\ell} q' = \frac{\ell}{1-\ell} \eta' q^{SS} + \eta'' q'.
$$

To then get the full full convergence result, we need to do this procedure iteratively, both with the q_t-constraint and the q^{SS}-constraint.

The global solution for the $t > 0$ decomposition of amplification is plotted in Figure 3.6. The blue line (q_t-constraint) is the same as the global solution from before, in Figure 3.2. With the q^{SS}-constraint, the constraint is less binding as it takes into account the higher, steady-state, price of capital. This means that for lower η the excess return will be lower than the q_t-constraint since experts will hold more capital (as they are allowed to lever up more). It also means that the capital price is not so depressed as the experts hold a higher share of capital (and they are more productive). This portrays the additional amplification from keeping the price-collateral effect channel open. One interesting point is that with the q_t-constraint the system recovers faster. The drift of η is higher since the excess return is higher.

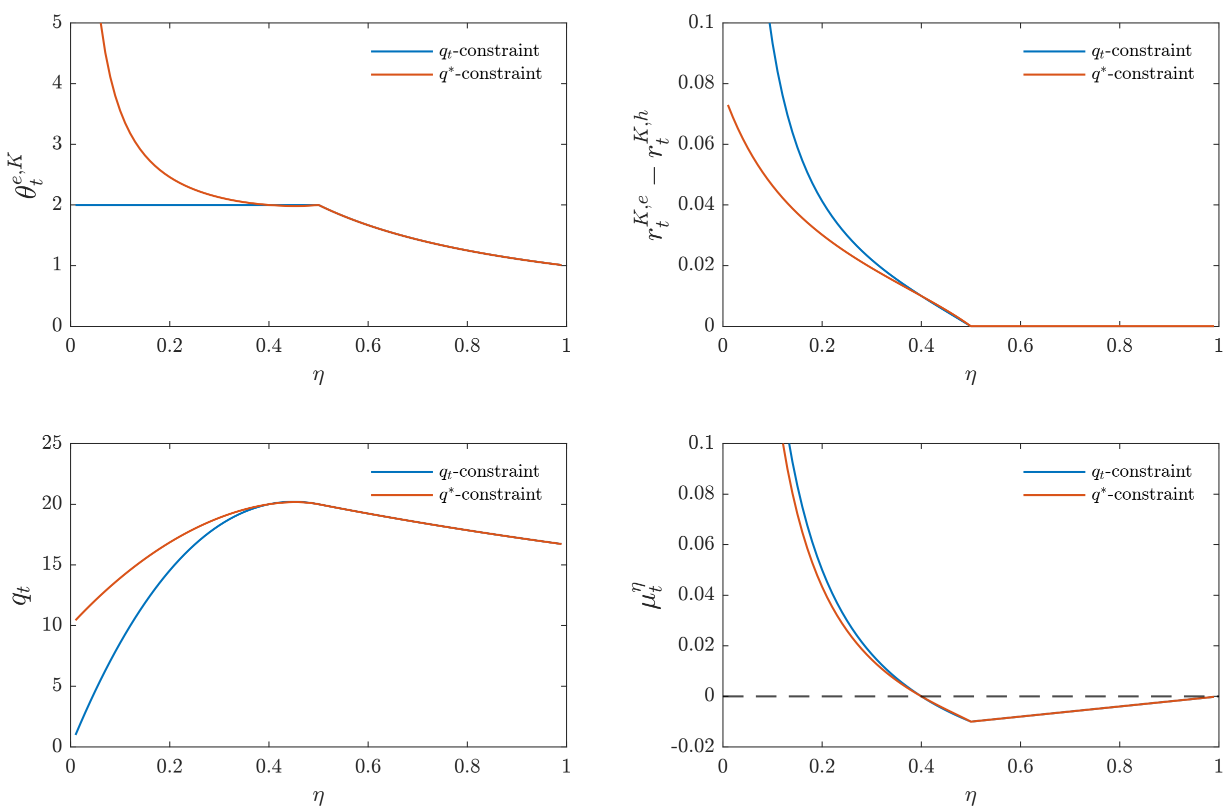


Figure 3.6: Decomposing Amplification for $t > 0$ (global solution). Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

For the log-linearized solution, we can also decompose the amplification for $t > 0$. The dynamics are given by

- Price dynamics:

$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = r^{SS}\hat{r}_t - r^{SS}\hat{a}_t^h + r^{SS}\hat{q}_t$$

- State dynamics with q_t-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

- State dynamics with q^{SS}-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{1}{1-\ell}\frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

Note that $\hat{q}_t, \hat{a}_t^h, \hat{r}_t$ are different with the different constraints. For large changes in η there will be a significant difference between these dynamics and the global solution.

3.5 Introducing Physical Investment

So far, we have considered a two sector model in which aggregate capital K_t evolves exogenously. We now add physical investment to it. Consumption goods can be converted in new capital. The conversion is not necessarily one-for-one, but concave in the investment rate ι, an agent's new physical investment divided by his capital.[6] This is captured by the concave capital conversion function $\Phi(\iota)$ with the assumption $\Phi'(\cdot) > 0, \Phi''(\cdot) < 0$. In addition, capital depreciates at a rate of δ. For example, consider an agent with capital k_t at an investment rate ι_t at time t. His real investment is $\iota_t k_t$, and the capital accumulation is $(\Phi(\iota_t) - \delta)k_t$. This is equivalent to a convex adjustment cost assumption. More generally, capital accumulation follows

$$\frac{\mathrm{d}k_t}{k_t} = \big(\Phi(\iota_t) - \delta\big)\,\mathrm{d}t + \sigma \mathrm{d}Z_t$$

[6] Assuming that $\Phi(\cdot)$ is concave in investment rate I_t/K_t rather than the investment I_t ensures tractable aggregate across agents.

where dZ_t is a Brownian motion capturing shocks to the capital accumulation process.

Note that investment is an intraperiod/static decision in models, in which capital does not take "time to build". Hence, maximizing agents' Hamiltonian with respect to ι_t only enters via individual agent's capital return r_t^k. The capital return consists of two terms: a dividend yield term as well as a capital gain term. We can write the return on capital as a function of the investment rate ι_t as

$$dr_t^k(\iota_t) = \underbrace{\frac{a-\iota_t}{q_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gain}} = \left[\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t.$$

The agent will now choose the investment rate ι_t to maximize their expected return on capital accumulation. That is,

$$\max_{\iota_t} \mathbb{E} dr_t^k(\iota_t)/dt$$

This yields the following Tobin Q optimality condition

$$\frac{1}{q_t} = \Phi'(\iota_t).$$

For the investment conversion function of the form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, this implies the following relationship between the investment rate and the price of capital

$$\phi\iota_t = q_t - 1.$$

This investment conversion function has a parameter ϕ that measures the degree of investment adjustment costs. As $\phi \to 0$, $\Phi(\iota) \to \iota$ so that there are no adjustment costs in capital investment. However, as $\phi \to \infty$ these costs become infinitely large and all new investment is lost, that is, $\Phi(\iota) \to 0$.

3.6 Exercises

3.6.1 Coding Continuous Time Kiyotaki-Moore Model

Please replicate the global solution figures (of Figure 3.2) for the case with ι-investments.

3.6.2 Capital (Quality) and Technology Shocks

Consider the simple model from chapter 3. There, we assumed that expert capital follows

$$\frac{\mathrm{d}k_t^e}{k_t^e} = \left(\Phi(\iota_t^e) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t \tag{3.12}$$

and produces an output flow[7]

$$y_t^e = \bar{a}k_t^e. \tag{3.13}$$

In this problem set you are asked to consider a different specification with consumption-specific technology shocks instead of capital shocks. Specifically, suppose instead of equations (3.12) and (3.13) that capital evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = \left(\Phi\left(\frac{\bar{a}}{a_t}\iota_t^e\right) - \delta\right)\mathrm{d}t \tag{3.14}$$

and produces an output flow

$$y_t = a_t k_t \tag{3.15}$$

where a_t is now a stochastic process given by[8]

$$\frac{\mathrm{d}a_t}{a_t} = \psi\left(\log\bar{a} - \log a_t\right)\mathrm{d}t + \sigma \mathrm{d}Z_t. \tag{3.16}$$

[7] I use here $\bar{a}$ instead of a from the lecture to distinguish this more clearly from the process a defined below.

[8] One gets to this equation by imposing that $\log a_t$ follows an Ornstein-Uhlenbeck process, the continuous-time equivalent of a discrete-time AR(1) process, and correcting by a deterministic time drift, such that the long-run mean of A_t is not growing/shrinking over time. The equivalent in discrete time is often taken as a productivity process in standard macro models.

For $\psi = 0$ this specification implies a geometric Brownian motion for productivity, for $\psi > 0$, a_t mean-reverts to the level $\bar{a}$ in the long run. The additional term $\bar{a}/a_t$ in the Φ function implies that only consumption production is impacted by changes of a_t.

(a) Show that without productivity mean reversion ($\psi = 0$), the model with capital shocks (evolution (3.12) and output (3.13)) and the model with consumption-specific technology shocks (capital evolution (3.14), output (3.15) and productivity process (3.16)) are isomorphic in the sense that they imply the same dynamics for output, consumption, net worth, the expert wealth share η^e and the risk-free rate.

(b) How are the two models related, if $\psi > 0$?

(c) Explain economically, why the two shock types are not equivalent for neutral technology shocks (i.e. if the investment technology is $\Phi\left(\iota_t^e\right)$) and an inconstant ψ function (1-2 sentences are sufficient).

3.6.3 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the model from chapter 3 (now again with capital shocks), but unlike there assume that households are more patient than experts, i.e. they have a discount rate $\rho^h < \rho^e$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

(a) Derive closed-form expressions for ι^e, q, σ^q, μ^{η^e} and σ^{η^e} as a function of η and model parameters.[9] You do not actually have to follow the order of steps in the lecture. In this simple model it pays off to start with goods market clearing.

(b) Assume $\phi > 0$. Show that in this model asset price movements mitigate exogenous risk (i.e. $\sigma^q + \sigma < \sigma$). Explain economically why this happens and why the effects disappears if $\phi = 0$.

[9] As in the lecture, assume the specific functional form $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$ for $\Phi(\cdot)$.

(c) Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a fully spelled-out formal proof). Compute the stationary density of η^e by numerically solving the ODE stated on page 16 of Yuliy's stochastic calculus notes using the same parameters as in part 2. What is the stationary density of q?

Bibliography

Basak, Suleyman and Domenico Cuoco, "An equilibrium model with restricted stock market participation," *The Review of Financial Studies*, 1998, *11* (2), 309–341.

Kiyotaki, Nobuhiro and John Moore, "Credit Cycles," *Journal of Political Economy*, 1997, *105* (2), 211–248.

Chapter 4

A Macro-Model with Endogenous Risk Dynamics: Amplification, Fire-sales, and Speculation

In last chapter, we studied simple models to illustrate the basic structure of continuous-time macro-finance models.

In this chapter, we present a more complex model in which apart from risk-free debt, experts can also issue outside equity. In terms of economic insights, we enrich the model to obtain the following properties:

- The risk as well as the price of risk is endogenous and hence time-varying depending on the wealth distribution across the heterogeneous agents in the economy.
- Equilibrium dynamics contains two regimes – a normal regime around the steady state and a crisis regime. The economy should be relatively stable near the steady state, where experts are adequately capitalized and able to absorb most shocks. However, an unexpected large shock or a series of negative shocks can significantly damage the experts and bring the economy to the crisis regime. In a crisis, experts are undercapitalized and financially constrained. As a result, market liquidity can suddenly dry up and shocks do affect demand for and prices of assets.

This generates *endogenous risk and volatility* through feedback effects of fire-sales and financial constraints can become occasionally binding.

- Volatility is high in the crisis regime, which might push experts' net worth towards zero. In this case, the economy needs a long time to recover. Ex ante, the system will spend a large amount of time away from the steady state and the stationary distribution can be bimodal.
- Assets are more correlated during crises due to endogenous risk.
- Endogenous risk-taking gives rise to a *volatility paradox*, which means that the economy does not become more stable when the fundamental risk σ is lower. This is because when risk is lower, experts take on greater leverage, making the economy more prone to crises.
- Financial innovations (e.g., securitization) that improve risk-sharing among experts might destabilize the economy in equilibrium. The logic is similar. Being able to diversify (idiosyncratic) risk emboldens the experts, leading to higher leverage and amplifying systemic risk.

In terms of modeling, we highlight the essential techniques for solving large-scale macro-finance models in continuous time:

- We introduce an *occasionally binding constraint* in this setting. The "skin in the game constraint" is not binding in the normal regime, while it binds in the crisis regime in which fire sales occur and volatility spikes.
- We rely on the "Fisher separation theorem" in order to solve the model from the viewpoint of a "price-taking" social planner.
- We introduce a change from a consumption numeraire to a total wealth numeraire, which simplifies many algebraic steps.

This chapter builds on Brunnermeier and Sannikov (2016), which expands on Brunnermeier and Sannikov (2014).

4.1 Model Setup

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions. Upon death of an expert/household, a new agent takes their place, inherits their wealth, and becomes an expert with probability $\zeta^e \in (0, 1)$.

Experts. Experts have a CRS technology $y_t^e = a^e k_t^e$. Denote their consumption and investment rate by c_t^e, ι_t^e. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = (\Phi(\iota_t^e) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Still, we have only aggregate risk in the environment. Experts have a log utility function and they each maximize

$$\mathbb{E}_0 \left[\int_0^T e^{-\rho_0^e t} \log c_t^e \mathrm{d}t \right]$$

where T is exponentially distributed with parameter ρ_d^e. Define $\rho^e := \rho_0^e + \rho_d^e$. The objective is equivalent to infinite lifetime with higher discount rate ρ^e

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log c_t^e \mathrm{d}t \right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

We let households hold capital to capture fire-sales. Households are more patient than the experts, i.e., $\rho^h \leq \rho^e$. As we have discussed in section 3, assuming that households are more patient than the experts, i.e., $\rho^h \leq \rho^e$, is a modeling trick to ensure that the experts do not hold all the capital in the long run. However, here we achieve the same outcome by introducing death. The households maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^h t}\log c_t^h \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^h. Similarly, the objective is equivalent to infinite lifetime with higher discount rate $\rho^h := \rho_0^h + \rho_d^h$

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log c_t^h \mathrm{d}t\right].$$

Financial Friction. The financial friction in this chapter is due to incomplete markets (see, e.g., Dumas and Luciano, 2017). Although experts are allowed to issue equity, they must hold at least α fraction of their risk. The balance sheets of the two sectors are as following:

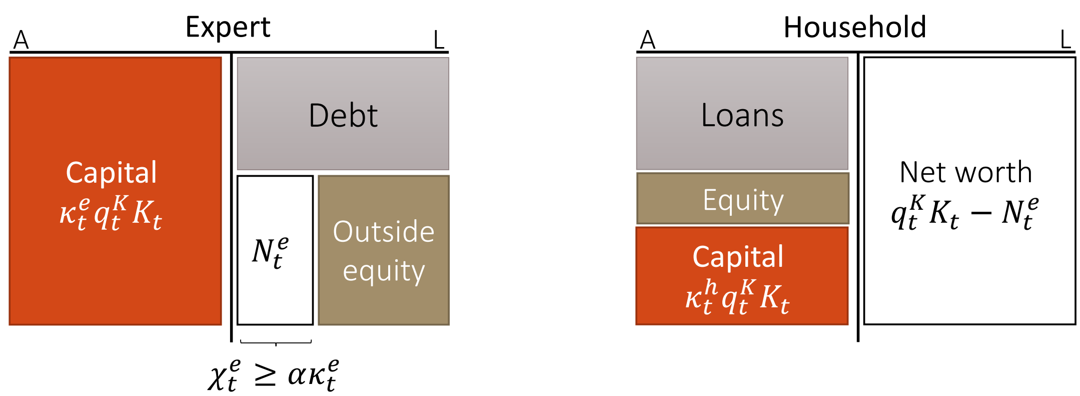


Figure 4.1: Balance sheets of experts and households

The skin-in-the-game constraint can be expressed as $\chi_t^e \geq \alpha\kappa_t^e$, where χ_t^e is the fraction of risk held by experts and κ_t^e is the fraction of capital held by experts. We will discuss carefully this relationship in later sections.

4.2 Solution Method

4.2.0 Postulate aggregates, price processes and obtain return processes

Again, with only aggregate risk, all experts (households) are identical, so total capital stock and net worth in each sector are obtained by $K_t^i = k_t^i$, $N_t^i = n_t^i, i = \{e, h\}$. Denote the price of capital by q_t. The total wealth of the economy is $q_t \sum_i K_t^i = q_t K_t = N_t = \sum_i N_t^i$. Define the capital shares as

$$\kappa_t^i = \frac{K_t^i}{\sum_{i'} K_t^{i'}},$$

and the net worth share as

$$\eta_t^i = \frac{N_t^i}{\sum_{i'} N_t^{i'}} = \frac{N_t^i}{q_t K_t}.$$

We then *postulate* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Given the price process and the consumption-investment decision of the expert, we can calculate the return rate to capital for both sectors, $r_t^{i,K}(\iota_t^i)$. Same as (3.7), we have

$$\mathrm{d}r_t^{i,K}(\iota_t^i) = \left[\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right]\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t. \tag{4.1}$$

We then postulate that SDF ($\xi_t^i = e^{-\rho t} u'(c_t^i)$) follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t, \tag{4.2}$$

where r_t is the risk-free rate.

4.2.1 For given SDF processes, derive individual equilibrium conditions

With an additional asset available, sector i's problem becomes

$$\begin{aligned}
\max_{\{\iota_t^i, \theta_t^i, c_t^i\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho^i t} \log c_t^i \mathrm{d}t \right] \\
\text{s.t.} \quad & \frac{\mathrm{d}n_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}\mathrm{d}t + \theta_t^{i,K}\mathrm{d}r_t^{i,K}(\iota_t^i) + \theta_t^{i,OE}\mathrm{d}r_t^{OE} + \theta_t^{i,D} r_t \mathrm{d}t \\
& n_0^i \text{ given,}
\end{aligned} \tag{4.3}$$

where r_t^{OE} is the return to outside equity. Note that the outside equity has the same risk (volatility) as capital but possibly different expected returns (drifts) due to the skin-in-the-game constraint. The experts' allocation satisfies

$$\theta_t^{e,K} \geq 0, \quad -(1-\alpha)\theta_t^{e,K} \leq \theta_t^{e,OE} \leq 0, \quad \theta_t^{e,D} \leq 0, \quad \text{and} \quad \theta_t^{e,K} + \theta_t^{e,OE} + \theta_t^{e,D} = 1. \tag{4.4}$$

The households' allocation satisfies

$$\theta_t^{h,K} \geq 0, \quad \theta_t^{h,OE} \geq 0, \quad \theta_t^{h,D} \geq 0, \quad \text{and} \quad \theta_t^{h,K} + \theta_t^{h,OE} + \theta_t^{h,D} = 1. \tag{4.5}$$

Optimal investment ι. The choice of investment rate is still a static and time-separable problem. An agent chooses ι_t^i to maximize her return $r_t^K(\iota_t^i)$. The first-order condition yields the Tobin's q equation

$$\frac{1}{q_t} = \Phi'(\iota_t^i). \tag{4.6}$$

With the special functional form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, $\phi\iota_t^i = q_t - 1$.

Asset and risk allocation. We solve the portfolio choice problem via the "price-taking planner's problem", which is widely applicable to environments with multiple assets. Intuitively, the **price-taking planner's Theorem** means that a social planner that takes prices as given chooses a real asset (capital) allocation κ_t and risk allocation χ_t that coincides with the choices implied by all individuals' portfolio choices. The planner's problem is often of the form

$$\max_{\{\kappa,\chi\}} \ \mathbb{E}\left[\text{Capital Return}\right] - (\text{weighted ave. price of risk}) \times (\text{incremental capital risk}),$$
$$\text{s.t.} \qquad \text{Financial Friction(s).}$$

Let's see how this seemingly magical result works in our environment. The price-taking planner's problem is

$$\max_{\{\kappa_t,\chi_t\}} \left\{ \underbrace{\mathbb{E}_t\left[\mathrm{d}r_t^K(\kappa_t)\right]/\mathrm{d}t}_{\text{Expected Return}} - \overbrace{\underbrace{\left(\sum_{i=\{e,h\}} \varsigma_t^i \chi_t^i\right)}_{\text{(Weighted) Price of Risk}} \times \underbrace{\sigma^{r^K}}_{\text{Risk}}}^{\text{(Weighted) Risk Premium}} \right\} \quad \text{s.t.} \quad \underbrace{\chi_t^e \geq \alpha\kappa_t^e}_{\text{Financial Friction}}.$$

Here $r_t^K(\kappa_t)$ is the overall return to capital: $(\iota_t = \iota_t^e = \iota_t^h = \frac{1}{\phi}(q_t - 1))$

$$\mathrm{d}r_t^K(\kappa_t) = \sum_i \kappa_t^i \mathrm{d}r_t^{i,K}(\iota_t^i) = \left[\frac{\sum_i \kappa_t^i a^i - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right]\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t.$$

Hence, the planner's problem can be written as[1]

$$\max_{\{\kappa_t,\chi_t\}} \left\{ \frac{\sum_i \kappa_t^i a^i - \iota_t}{q_t} - \left(\sum_i \varsigma_t^i \chi_t^i\right)(\sigma + \sigma_t^q) \right\} \quad \text{s.t.} \quad \chi_t^e \geq \alpha\kappa_t^e. \tag{4.7}$$

Theorem 4.1. *The equilibrium allocation of physical capital, κ_t^e, as well as the allocation of risk, χ_t^e, that arises from agents' portfolio decisions can be more directly obtained by solving the "price-taking social planner problem" (4.7).*

Proof. The proof takes three steps

1. By **Fisher's Separation Theorem**[2], each individual's portfolio maximization is equivalent to the following maximization problem of a "firm". In our model,

[1]Note that $\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q$ does not depend on κ_t, χ_t.
[2]We postpone the proof of this result till chapter 7. See also Kelsey and Milne (2006) for more details.

individuals in sector i solve[3]

$$\max_{\{\theta_t^{i,K},\,\theta_t^{i,OE},\,\theta_t^{i,D}\}} \quad \theta_t^{i,K}\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t + \theta_t^{i,OE}\mathbb{E}_t\left[\mathrm{d}r_t^{OE}\right]/\mathrm{d}t + \theta_t^{i,D}r_t - \varsigma_t^i(\theta_t^{i,K}+\theta_t^{i,OE})\sigma^{r^{i,K}}$$

$$\text{s.t.} \quad \begin{cases} (4.4) & \text{if } i=e \\ (4.5) & \text{if } i=h \end{cases}.$$

2. Aggregate $\{\boldsymbol{\eta}_t\}$-weighted sum of the two sectors' problems:[4]

$$\begin{aligned}\max_{\{\boldsymbol{\theta}_t^i\}_{i=\{e,h\}}} \quad & \sum_i \eta_t^i\theta_t^{i,K}\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t + \sum_i \eta_t^i\theta_t^{i,OE}\mathbb{E}_t\left[\mathrm{d}r_t^{OE}\right]/\mathrm{d}t + \sum_i \eta_t^i\theta_t^{i,D}r_t \\ & -\sum_i \varsigma_t^i\eta^i(\theta_t^{i,K}+\theta_t^{i,OE})\sigma_t^{r^K}, \qquad \text{s.t.} \quad (4.4) \text{ and } (4.5).\end{aligned}$$

3. Market clearing conditions are

$$\begin{aligned} \text{Capital:} \quad & \eta_t^i\theta_t^{i,K} = \kappa_t^i, \quad \eta_t^i(\theta_t^{i,K}+\theta_t^{i,OE}) = \chi_t^i, \\ \text{Outside Equity:} \quad & \sum_i \eta_t^i\theta_t^{i,OE} = 0, \\ \text{Debt:} \quad & \sum_i \eta_t^i\theta_t^{i,D} = 0. \end{aligned}$$

Note that (4.4) togerther with the capital market clearing condition implies

$$\chi_t^e = \eta_t^e[\theta_t^{e,K}+\theta_t^{e,OE}] \geq \eta_t^e[\theta_t^{e,K}-(1-\alpha)\theta_t^{e,K}] = \alpha\kappa_t^e.$$

Therefore, the aggregated problem can be simplified to

$$\max_{\{\kappa_t,\chi_t\}} \quad \sum_i \kappa_t^i\mathbb{E}_t\left[\mathrm{d}r_t^{i,K}(\iota_t^i)\right]/\mathrm{d}t - \left(\sum_i \varsigma_t^i\chi_t^i\right)\sigma_t^{r^K}, \qquad \text{s.t.} \quad \chi_t^e \geq \alpha\kappa_t^e,$$

which is equivalent to the planner's problem (4.7).

□

[3] Recall that outside equity and capital have the same risk (volatility).
[4] Note that $\sigma^{r^{i,K}} = \sigma^{r^K}$ as the two sectors face the same aggregate risk.

Although we proved the theorem for this specific model, the three-step argument is generally valid for more complicated models. Now we can solve the planner's problem (4.7) to obtain the risk/capital allocations. The KKT conditions are

$$\chi_t: \quad \min\left\{\varsigma_t^e - \varsigma_t^h, \chi_t^e - \alpha\kappa_t^e\right\} = 0, \tag{4.8}$$

$$\kappa_t: \quad \min\left\{\frac{a^e - a^h}{q_t} - \alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q), 1 - \kappa_t^e\right\} = 0. \tag{4.9}$$

The derivation of the KKT conditions and how these conditions overlap may not be obvious, first notice that apart from the financial friction constraint stated in 4.7, we have another constraint on the capital holding by experts, which is $\kappa_t^e \leq 1$. Hence the Lagrangian reads

$$\mathcal{L} = \frac{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t}{q_t} - \left(\chi_t^e\varsigma_t^e + (1-\chi_t^e)\varsigma_t^h\right)(\sigma + \sigma_t^q) + l_1(\chi_t^e - \alpha\kappa_t^e) + l_2(1-\kappa_t^e)$$

where l_1, l_2 are the Lagrangian multipliers.

Then we focus on χ, taking FOC and the complementary slackness gives

$$(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \geq 0, \chi_t^e - \alpha\kappa_t^e \geq 0, (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)(\chi_t^e - \alpha\kappa_t^e) = 0$$

Hence we have two cases here,

$$\text{Case 1}: \varsigma_t^e(\sigma + \sigma_t^q) > \varsigma_t^h(\sigma + \sigma_t^q), \chi_t^e = \alpha\kappa_t^e,$$
$$\text{Case 2}: \varsigma_t^e(\sigma + \sigma_t^q) = \varsigma_t^h(\sigma + \sigma_t^q), \chi_t^e > \alpha\kappa_t^e.$$

Now under Case 1, we plug $\chi_t^e = \alpha\kappa_t^e$ into the Lagrangian and take FOC and complementary slackness regarding to κ_t^e, we get cases 1a and 1b,

$$\text{Case 1a: } \frac{a^e - a^h}{q_t} > \alpha(\varsigma_t^e - \varsigma_t^h), \kappa_t^e = 1,$$
$$\text{Case 1b: } \frac{a^e - a^h}{q_t} = \alpha(\varsigma_t^e - \varsigma_t^h), \kappa_t^e < 1.$$

Under Case 2, there is no such relationship between χ_t^e and κ_t^e, so the conditions are

$$\text{Case 2a: } \frac{a^e - a^h}{q_t} > 0, \kappa_t^e = 1,$$
$$\text{Case 2b: } \frac{a^e - a^h}{q_t} = 0, \kappa_t^e < 1.$$

where 2b is impossible because $a^e > a^h$.

Intuitively, $\frac{a^e - a^h}{q_t}$ is the benefit of shifting a unit of capital from households to experts and $\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$ is the associated cost, which is the difference in the required risk premium. When the "skin in the game" constraint binds, there might be an interior solution where the cost and benefit equalizes, and experts do not hold all the capital, which is case 1a, there might also be a corner solution where experts already hold all the capital but the benefit still exceeds the cost, which is case 1b. If the "skin in the game" constraint does not bind, then the cost of shifting capital is zero as they require the same risk premium, hence there is only a corner solution given by case 2a.

The FOCs can be visualized as following:

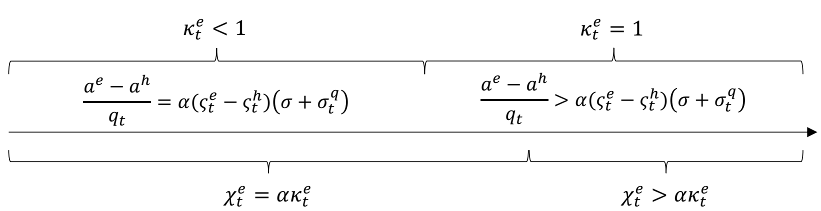


Figure 4.2: First-order conditions of the planner's problem

4.2.2 Evolution of state variable η_t

The definition of equilibrium is similar to the previous chapter – a map from histories of shocks to equilibrium prices.

Definition 4.1. *Given any initial allocation of capital among the agents, an* ***equilibrium*** *is a map from histories* $\{Z_s, s \in [0,t]\}$ *to price* q_t*, investment rate* ι_t^i*, price of risk* ς_t^i *and capital/risk allocation* κ_t^i, χ_t^i*, such that*

(1) all agents choose portfolios and consumption rates to maximize utility,

(2) all markets, for capital, equity and consumption goods, clear.

Drift of η_t^i. One might solve for the process of the state variable $\eta_t^i = N^i/(q_t K_t^i)$ by brute-force, i.e., combining process (6.1) and the market clearing conditions by Itô's quotient rule. However, this method involves a formidable amount of algebra and is thus error-prone. As an alternative, we introduce a new method – "change of numeraire" – to compute the *drift* of wealth shares, $\mu_t^{\eta^i}$.

Note that as a ratio of two equilibrium quantities, the wealth share η_t^i (and its volatility) remain unchanged under a different numeraire, so the change-of-numeraire method does not provide any new information on $\sigma_t^{\eta^i}$.

Change of Numeraire.

Consider two different numeraires – call them dollars (\$) and euros (€). Let x_t^A be the value of a self-financing strategy in \$. Denote the exchange rate by Y_t:

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t.$$

Then, x_t^A/Y_t is the value of the self-financing strategy in €. Applying the martingale approach, $\xi_t^{\$} x_t^A$ and $\xi_t^{€}(x_t^A/Y)$ are both martingales. Recall that for any two assets A, B, the martingale approach implies

$$\begin{aligned} \mu_t^A - \mu_t^B &= \varsigma_t^{\$}(\sigma_t^A - \sigma_t^B), \\ \mu_t^{A/Y} - \mu_t^{B/Y} &= \varsigma_t^{€}(\sigma_t^{A/Y} - \sigma_t^{B/Y}). \end{aligned}$$

By Itô's quotient rule,

$$\begin{aligned} \mu_t^{A/Y} - \mu_t^{B/Y} &= (\mu_t^A - \mu_t^B) - \sigma_t^Y(\sigma_t^A - \sigma_t^B), \\ \sigma_t^{A/Y} - \sigma_t^{B/Y} &= (\sigma_t^A - \sigma_t^Y) - (\sigma_t^B - \sigma_t^Y) = \sigma_t^A - \sigma_t^B. \end{aligned}$$

Hence,

$$(\mu_t^A - \mu_t^B) - \sigma_t^Y(\sigma_t^A - \sigma_t^B) = \varsigma_t^{€}(\sigma_t^A - \sigma_t^B)$$

$$\iff (\varsigma_t^{\$} - \sigma_t^Y)(\sigma_t^A - \sigma_t^B) = \varsigma_t^{€}(\sigma_t^A - \sigma_t^B)$$
$$\iff \boxed{\varsigma_t^{€} = \varsigma_t^{\$} - \sigma_t^Y}. \tag{4.10}$$

We change the numeraire from consumption goods to the total wealth in the economy $N_t = \sum_i N_t^i$. Consider two assets:

- Asset A: sector i's portfolio return in terms of total wealth, that is $N_t^i/N_t = \eta_t^i$. Extra drift terms are included due to reshuffling (death). The return to this asset is

$$\frac{\mathrm{d}\eta_t^i + (C_t^i/N_t)\mathrm{d}t}{\eta_t^i} = \left(\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} + \rho_d^i \zeta^{\neg i} - \rho_d^{\neg i}\zeta^i \frac{N_t^{\neg i}}{N_t^i}\right)\mathrm{d}t + \sigma_t^{\eta^i}\mathrm{d}Z_t.$$

- Asset B: a benchmark asset that everyone can hold (e.g., risk-free asset or money in terms of total wealth). In this chapter, asset B is the risk-free loan from the households to the experts, which has return $r_t\mathrm{d}t$.

The martingale asset pricing formula implies

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} + \rho_d^i \zeta^{\neg i} - \rho_d^{\neg i}\zeta^i \frac{N_t^{\neg i}}{N_t^i} - r_t = \underbrace{(\varsigma_t^i - \sigma_t^N)}_{\substack{\text{price of risk} \\ \text{under } N_t \text{ numeraire}}} \sigma_t^{\eta^i}. \tag{4.11}$$

Aggregate $\{\boldsymbol{\eta}_t\}$-weighted sum of the two sectors

$$\underbrace{\sum_{i'} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t = \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma_t^N)\sigma_t^{\eta^{i'}}, \tag{4.12}$$

where the first item equals zero because it is the drift of $\sum_{i'} \eta^{i'} = 1$. Subtracting (4.12) from (4.11), the drift of η_t^i is

$$\begin{aligned}
\mu_t^{\eta^i} &= (\varsigma_t^i - \sigma_t^N)\sigma_t^{\eta^i} - \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma_t^N)\sigma_t^{\eta^{i'}} - \left(\frac{C_t^i}{N_t^i} - \frac{C_t}{N_t}\right) - \rho_d^i \zeta^{\neg i} + \rho_d^{\neg i}\zeta^i \frac{N_t^{\neg i}}{N_t^i} \\
&= (\varsigma_t^i - \sigma - \sigma_t^q)\sigma_t^{\eta^i} - \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma - \sigma_t^q)\sigma_t^{\eta^{i'}} - \left(\frac{C_t^i}{N_t^i} - \frac{C_t}{N_t}\right) - \rho_d^i \zeta^{\neg i} + \rho_d^{\neg i}\zeta^i \frac{N_t^{\neg i}}{N_t^i},
\end{aligned}$$

where the second equality holds because $N_t = q_t K_t$, and hence $\sigma_t^N = \sigma + \sigma_t^q$.

Volatility of η_t^i. Recall that the wealth share η_t^i is numeraire invariant, so we still use Itô's quotient rule to solve for its volatility. Since $\eta_t^i = N_t^i / N_t$,

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = \left[\frac{\chi_t^i}{\eta_t^i} - \sum_{i'} \eta_t^{i'} \frac{\chi_t^{i'}}{\eta_t^{i'}} \right] (\sigma + \sigma_t^q) = \frac{\chi_t^i - \eta_t^i}{\eta_t^i} (\sigma + \sigma_t^q),$$

where the third equality follows from (6.1) and market clearing conditions, as

$$\sigma_t^{N^i} = \sigma_t^{n^i} = (\theta_t^{i,K} + \theta_t^{i,OE})(\sigma + \sigma_t^q) = \frac{\chi_t^i}{\eta_t^i} (\sigma + \sigma_t^q).$$

Amplification. Applying Itô's lemma to $q(\eta_t^e)$,

$$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} (\eta_t^e \sigma_t^{\eta^e}) = \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q).$$

The total volatility is

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \dfrac{q'(\eta_t^e)}{q/\eta_t^e} \dfrac{\chi_t^e - \eta_t^e}{\eta_t^e}} > \sigma. \tag{4.13}$$

The amplification effect arises due to fire-sales of capital from the experts to the households. A negative shock increases market illiquidity, leading to expert losses on their capital stock. Since the experts are levered, when hit by a negative shock, they are forced to sell their capital stock to the households, causing a further price drop, and so on – a *loss spiral*.

4.2.3 Goods market clearing

Since both experts and households may hold capital, the goods market clearing condition takes the form:

$$\sum_i \kappa_t^i a^i - \iota_t = \sum_i \frac{C_t^i}{K_t} = q_t \sum_i \eta_t^i \frac{C_t^i}{N_t^i} \tag{4.14}$$

4.2.4 The special case of log-utility and the Inner Loop

So far we have characterized the optimal investment decision (4.6), the capital and risk allocation (4.8 and 4.9), the volatility of capital price (4.13) and the goods market clearing condition (4.14). If we know the prices of risk ς_t^i and consumption-to-wealth ratios C_t^i/N_t^i, we can already solve the model. We see that under log-utility $\varsigma_t^i = \sigma_t^{n^i} = \frac{\chi_t^i}{\eta_t^i}(\sigma + \sigma_t^q)$ and $C_t^i/N_t^i = \rho^i$. It is straightforward to show that (4.8) and (4.9) reduce to the following two conditions:

$$\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2, \text{ with equality if } \kappa_t^e < 1 \tag{4.15}$$

$$\chi_t^e = \max\left\{\alpha\kappa_t^e, \eta_t^e\right\}, \tag{4.16}$$

It follows that for the case of log-utility we only need to solve a system of non-linear equations, one of which is an ODE (4.13). We refer to the algorithm as the "inner loop", as it will also be part of solution procedure for more general types of preferences.

Inner loop. Our goal is to solve for equilibrium variables $(\chi_t^i, \kappa_t^i, \sigma_t^q, \iota_t^i, q_t)$ as functions of η_t^e using conditions (4.6), (4.13), (4.14), (4.15), (4.16), and given expressions for ς_t^i and C_t^i/N_t^i. The algorithm goes as follows:

Start at $q(0)$ (the autarky economy) and solve to the right. Use different procedures for two η^e regions:

i. If $\kappa_t^e < 1$, solve ODE for $q(\eta_t^e)$ using conditions (4.13), (4.14) and (4.15). Specifically, we have to first plug (4.6) into (4.14) and (4.16) into (4.15). Then solve the three equilibrium conditions to the right using Newton's method.

ii. If $\kappa_t^e = 1$, (4.15) is no longer informative about σ_t^q. Instead, we solve (4.6) and (4.14) for $q(\eta_t^e)$, again, using Newton's method.

Newton's Method.

The classic one-dimensional Newton's Method finds the root of a real-valued function by successively computing the intercept of the tangent line approximation of this function. Mathematically, it iteratively computes

$$z_{n+1} = z_n - [f'(z_n)]^{-1} f(z_n).$$

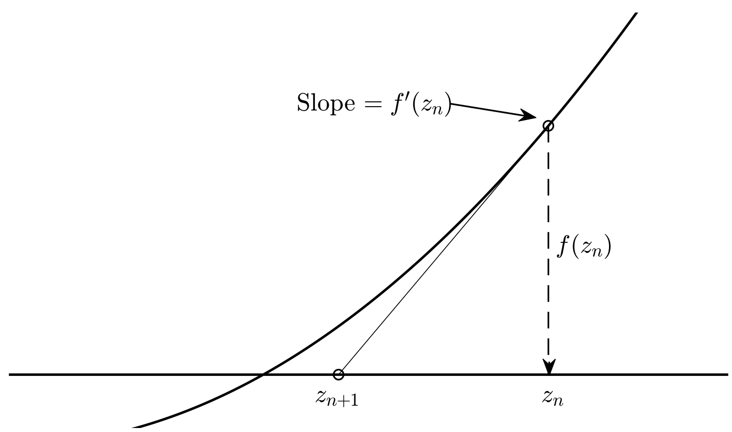


Figure 4.3: One-dimensional Newton's method

For a multi-dimensional system $F(\mathbf{z}) = \mathbf{0}$, the Newton's method proceeds as

$$\mathbf{z}_{n+1} = \mathbf{z}_n - J_n^{-1} F(\mathbf{z}_n), \tag{4.17}$$

where J_n is the Jacoban matrix, i.e., $J_{i,j} = \partial f_i(\mathbf{z}) / \partial z_j$.

In the algorithm above, we use Newton's method to approximate the solution to our ODE. Specifically, we let $F(\cdot)$ be the the three equilibrium conditions and $\mathbf{z} = \{q, \kappa^e, (\sigma + \sigma^q)\}$. Starting from the autarky solution $\mathbf{z}_0$, we then iteratively compute to the right $\mathbf{z}_n$ by (4.17) on the η^e grid. Note that for each grid point, we essentially only conduct the first step in the Newton's method. It can be shown that the error is of order $O((\mathbf{z} - \mathbf{z}^*)^2)$, where $\mathbf{z}^*$ is the true solution. Since our η^e grid will be very dense and all variables are continuous in η^e, the error should be negligible in practice.

If using only the first step is insufficient, one can easily switch to a multi-step Newton's method. For example, in the inner loop function presented in Section 4.2.5, one only needs to add an additional loop for codes between line 21 and 34. However, it is important to keep in mind that Newton's method does not guarantee global convergence. In practice, a multi-step Newton's method might diverge, and one should check the gain from each additional step to deter possible divergence.

4.2.5 Implementation in MATLAB: log-utility

Following is main code executing the algorithm under the assumption of log-utility.

```
%% Parameters and grid
a_e = 0.11; a_h = 0.03;        % production rates
rho_0 = 0.04;                  % time preference
rho_e_d = 0.01;rho_h_d = 0.01;% death rates
rho_e = rho_0 + rho_e_d;       % expert's discount rate
rho_h = rho_0 + rho_h_d;       % household's discount rate
zeta = 0.05;                   % probability of becoming an expert
delta = 0.05; sigma = 0.1;     % decay rate/volatility
phi = 10; alpha = 0.5;         % adjustment cost/equity constraint

N = 501;                         % grid size
eta = linspace(0.0001,0.999,N)'; % grid for \eta

%% Solution
% Solve for q(0)
q0 = (1 + a_h*phi)/(1 + rho_h*phi);

% Inner loop
[Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h, sigma, phi, alpha);

S = (Chi - eta).*SSQ; % \sigma_{\eta^e} -- arithmetic volatility of \eta^e
Sg_e = S./eta;        % \sigma^{\eta^e} -- geometric volatility of \eta^e
Sg_h = -S./(1-eta);   % \sigma^{\eta^h} -- geometric volatility of \eta^h

VarS_e = Chi./eta.*SSQ;          % \varsigma^e -- experts' price of risk
VarS_h = (1-Chi)./(1-eta).*SSQ;  % \varsigma^h -- households' price of risk

CN_e = rho_e; % experts' consumption-to-networth ratio
CN_h = rho_h; % households' consumption-to-networth ratio

MU = eta .* (1-eta) .* ((VarS_e - SSQ).*(Sg_e + SSQ) -  (VarS_h - SSQ).*(Sg_h + SSQ) - (CN_e - CN_h) + (rho_h_d.*zeta.*(1-eta)-rho_e_d.*(1-zeta).*eta)./(eta.*(1-eta))); % \mu_{\eta^e} -- arithmetic drift of \eta^e
```

The inner loop procedure is implemented in the function `inner_loop_log.m`:

```
function [Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h, sigma, phi, alpha)

N = length(eta);
deta = [eta(1); diff(eta)]; % imposes the correct grid step for numerical derivative at \eta^e = 0
```

```

% variables
Q = ones(N,1);     % price of capital q
SSQ = zeros(N,1);   % \sigma + \sigma^q
Kappa = zeros(N,1); % capital fraction of experts \kappa

Rho = eta*rho_e + (1-eta)*rho_h; % auxiliary variable: average consumption-to-networth ratio

% Initiate the loop
kappa = 0; q_old = q0; q = q0; ssq = sigma;

% Iterte over eta
% At each step apply Newton's method to F(z) = 0 where z = [q, kappa, ssq]'
% Use chi = alpha*kappa
for i = 1:N
    % Compute F(z_{n-1})
    F = [kappa*(a_e - a_h) + a_h - (q-1)/phi - q*Rho(i);
         ssq*(q - (q - q_old)/deta(i) * (alpha*kappa - eta(i))) - sigma*q;
         a_e - a_h -  q*alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq^2];

    % Construct Jacobian J^{n-1}
    J = zeros(3,3);
    J(1,:) = [-1/phi - Rho(i), a_e - a_h, 0];
    J(2,:) = [ssq*(1 - (alpha*kappa - eta(i))/deta(i)) - sigma, ...
          -ssq*(q-q_old)/deta(i)*alpha, q - (q-q_old)/deta(i)*(alpha*kappa - eta(i))];
    J(3,:) = [- alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq^2, ...
          -q*alpha^2/(eta(i)*(1-eta(i)))*ssq^2, -2*q*alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq];

    % Iterate, obtain z_{n}
    z = [q, kappa, ssq]' - J\F;

    % If the new kappa is larger than 1, break
    if z(2) >= 1
        break;
    end

    % Update variables
    q = z(1); kappa = z(2); ssq = z(3);

    % save results
    Q(i) = q; Kappa(i) = kappa; SSQ(i) = ssq;
    q_old = q;
end

% Set kappa = 1, use chi = max(alpha, eta) and compute the rest
n1 = i;
for i = n1:N
    q = (1 + a_e*phi)/(1 + Rho(i)*phi);
    qp = (q - q_old)/deta(i);

```

```
    Q(i) =  q; Kappa(i) = 1;
    SSQ(i) = sigma/(1 - (max(alpha, eta(i)) - eta(i))*qp/q);
    q_old = q;
end

% Compute chi, iota
Chi  = max(alpha*Kappa, eta);
Iota = (Q - 1)/phi;
```

4.3 Stationary Distribution and Fan Charts

4.3.1 Stationary distribution

Recall that in the simple model in Chapter 2, experts hold all the capital in the long run. In this chapter, we introduce death for experts and households to avoid a degenerate stationary distribution. Figure 4.4 shows the drift and volatility of η^e. There exists an η^* where the drift of η^e becomes zero. η^* can be viewed as the "steady state" of this model. In the absence of shocks, the system will converge to and stay at the steady state. In response to small shocks, drifts of η^e can still push the economy back to the steady state. Moving away from the crisis regime, risk premia decline, which boosts experts' consumption and lowers the drift of η^e. At η^*, risk premia decline sufficiently so experts' income is exactly offset by their consumption propensity, and hence their wealth share stays constant.

The region where $\eta^e \geq \alpha = 0.5$ reflects perfect risk sharing between experts and households, where the volatility of η^e is zero. Since the drift is negative, the system will never stay in this region. If we start there, the system deterministically moves to $\eta^e = \alpha$.

The stationary distribution of the system can be derived using Kolmogorov Forward Equation. Consider a n-dimensional Itô diffusion X with law of motion

$$\mathrm{d}X = \mu(X)\mathrm{d}t + \sigma(X)\mathrm{d}Z, \tag{4.18}$$

where $\mu : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and Z is a m-dimensional Brownian motion. The

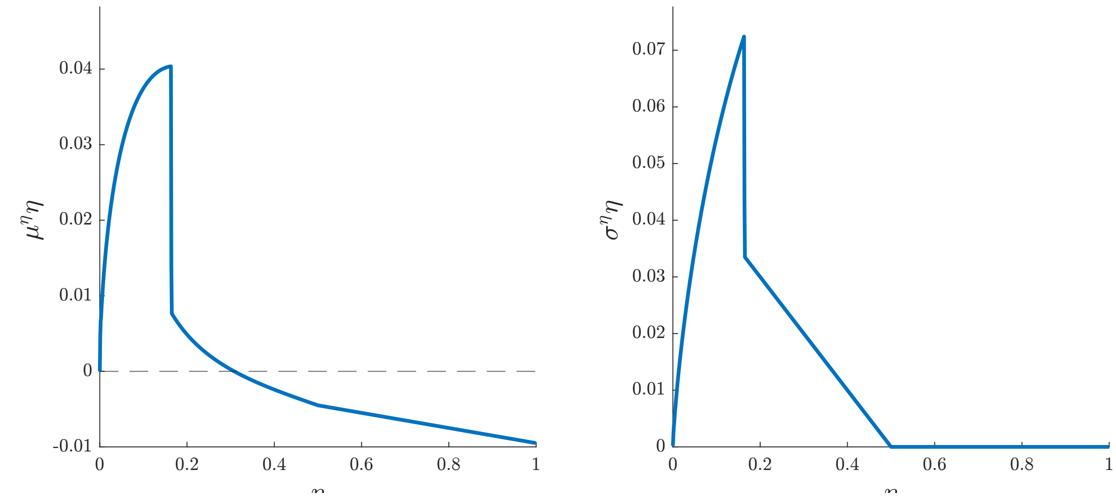


Figure 4.4: Drift and Volatility of η^e

stationary KFE for f_X is

$$0 = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\mu_i(x) f_X(x)\right) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\sigma(x)\sigma(x)^T\right)_{ij} f_X(x) \right).$$

This is a linear equation for the function f_X, namely $Tf_X = 0$ with the differential operator T defined by

$$Tf := -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\mu_i f\right) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\sigma\sigma^T\right)_{ij} f \right). \tag{4.19}$$

One can show by integration by parts that

$$\int (Tf)(x) g(x) \mathrm{d}x = \int f(x) \underbrace{\left(\sum_{i=1}^{n} \mu_i(x) \frac{\partial}{\partial x_i} g(x) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sigma\sigma^T\right)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} g(x) \right)}_{=:(Sg)(x)} \mathrm{d}x,$$

so T is the adjoint of the operator S. We then discretize the differential operator S by a matrix A. In finite dimensions, forming adjoints means taking transposes, so that A^T is a finite-dimensional approximation of T. We interpret A as the transition matrix of a continuous-time Markov chain. Then its basis $\boldsymbol{y}$ (i.e., $A^T\boldsymbol{y} = 0$) is a multiple of the invariant distribution of this Markov chain, so one can divide $\boldsymbol{y}$ by its (unweighted) sum to obtain invariant probabilities. Then we can form the cumulative sum to get the CDF and approximate the density by taking finite differences.

In next section we provide MATLAB program `KFE.m` to solve stationary and time-dependent Kolmogorov forward equation. Figure 4.5 plots the stationary distribution. Note that any monotone transformation of η^e is also a valid state variable, including the CDF of η^e.

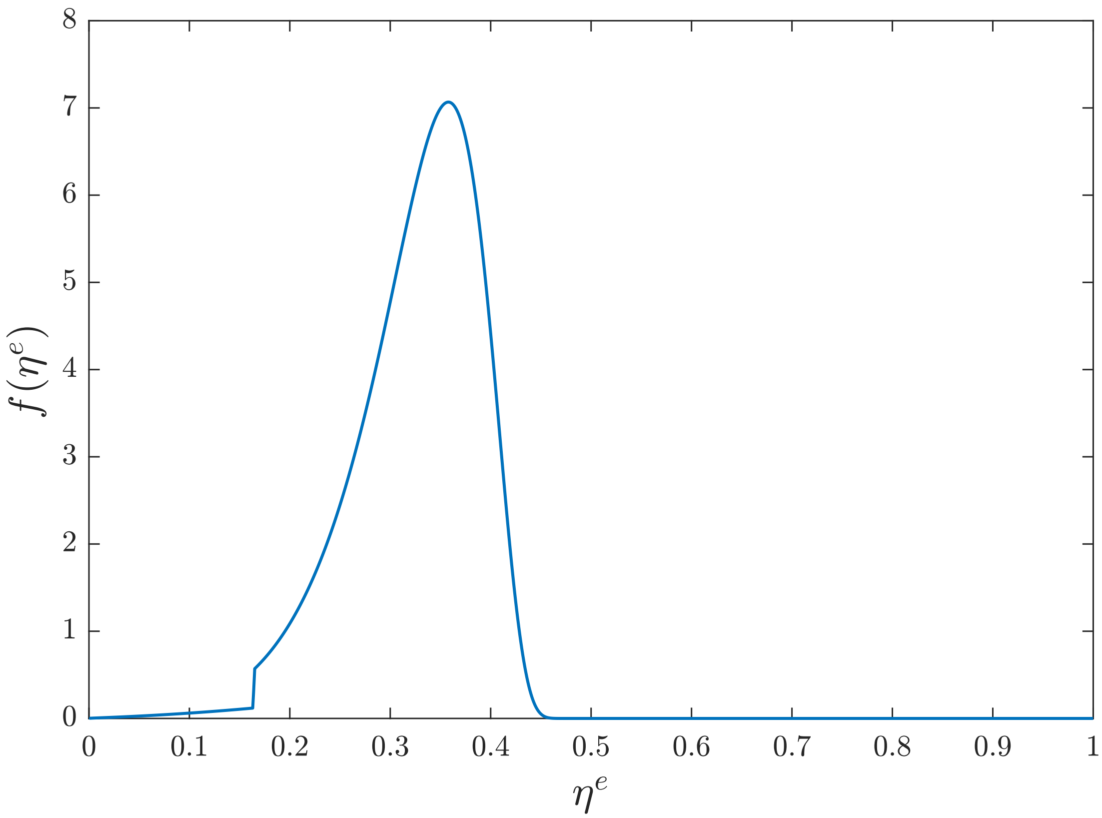


Figure 4.5: Stationary distribution

4.3.2 Fan Charts

In standard macroeconomic models with a fixed steady state, it is commonplace to start at the deterministic steady state and shock the state variable with a one standard deviation (negative) shock (bad unanticipated realization). Subsequently one can observe how the system/economy converges back to the steady state, i.e., by plotting the

the impulse response function.

The mathematical tool for studying transition paths in a continuous-time environment is given by the time-dependent Kolmogorov forward equation. For a process X with the evolution (4.18), the KFE is

$$\frac{\partial}{\partial t} f_X(x,t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu_i(x) f_X(x,t)\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(\left(\sigma(x)\sigma(x)^T\right)_{ij} f_X(x,t)\right)$$

or, shorter, $\partial f_X(x,t)/\partial t = \left[T f_X(\cdot,t)\right](x)$ with T as defined in (4.19). To solve such a linear parabolic PDE, we can borrow the A^T approximation matrix from the stationary case and iterate the right hand side of KFE forward.[5]

The following MATLAB function outlines this solving method for both stationary and time-dependent Kolmogorov forward equation:

```
function [pdf_stat,varargout] = KFE(X,MU,S,T,F0)
% KFE Solve one dimensional stationary and time-dependent KFE by finite
% difference method The process being studied is dX = MU(X)dt + S(X)dZ_t.

% REQUIRED INPUT:
% X: [X(1), X(2) ... X(N)]' is the state space (can be uneven grid)
% MU: a drift vector of length N, with MU(1) >= 0 and MU(N) <= 0
% S: a volatility vector of length N, with S(1) = S(N) = 0
% OPTIONAL INPUT FOR TIME-DEPENDENT KFE:
% T: time grid, M*1
% F0: initial distribution vector, N*1

% IMPLEMENT
% 1. For stationary distribution, [pdf_stat] = KFE(X,MU,S);
% 2. For distribution diffusion, [pdf_stat,cdf] = KFE(X,MU,S,T,F0);

% NOTE: 1. Fokker Planck operator (KFE) is the adjoint operator of Feynman Kac
% operator (KBE). We first build Feynman Kac operator and then transpose it.
% 2. We use upwind scheme and implicit scheme for monotonicity and stability.

N = length(X);
dX = X(2:N) - X(1:N-1);
%% 1. Build Fokker-Planck operator
% approximate drift terms with an upwind scheme
% upper diagonal
AU = max(MU(1:N-1),0)./dX;
% lower diagonal
AD = - min(MU(2:N),0)./dX;
% main diagonal
A0 = zeros(N,1); A0(1:N-1) = A0(1:N-1) - AU; A0(2:N) = A0(2:N) - AD;
```

[5] In the one-dimensional case can also be solved with MATLAB built-in solver pdepe.

```
% matrix A
A = sparse(1:N,1:N,A0,N,N) + sparse(1:N-1,2:N,AU,N,N) + sparse(2:N,1:N-1,AD,N,N);

% approximate volatility terms
% sigma^2/(x_{n+1} - x_{n-1})
S0 = zeros(N,1); S0(2:N-1) = S(2:N-1).^2./(dX(1:N-2) + dX(2:N-1));
% upper diagonal
BU = S0(1:N-1)./dX;
% lower diagonal
BD = S0(2:N)./dX;
% main diagonal
B0 = zeros(N,1); B0(1:N-1) = B0(1:N-1) - BU; B0(2:N) = B0(2:N) - BD;
% matrix B
B = sparse(1:N,1:N,B0,N,N) + sparse(1:N-1,2:N,BU,N,N) + sparse(2:N,1:N-1,BD,N,N);

% Fokker Planck operator
FP = (A+B)';

%% 2. Find stationary distribution.
% MATLAB doesn't have build-in kernel solver for sparse matrix, for higher
% efficiency one can use online package like spnull, etc.
F_stat = null(full(FP));
cdf_stat = cumsum(F_stat(:,1)./sum(F_stat(:,1)));
pdf_stat = [0;(cdf_stat(2:end)-cdf_stat(1:end-1))./dX];

%% 3. Solve the time-dependent KFE.
if nargin == 5
    F = F0;
    DT = [0 T(2:end)-T(1:end-1)];
    pdf_diffusion = zeros(length(T),length(F));

    for i = 1:length(T)
        F = (speye(N,N) - DT(i)*FP)\F;
        pdf_diffusion(i,:) = F;
    end
    varargout{1} = pdf_diffusion;
end

end
```

To visualize the transition, we can use fan charts originally introduced by the Bank of England. The first type of fan chart plots the evolution of (the distribution of) the state variable after a shock – the "distributional impulse response". The idea is similar to the impulse response function in the DSGE literature, but instead of imposing a one-time shock, it studies the dynamics of the whole system. Figure 4.6 plots the convergence back to the stationary distribution after a shock to an economy originally at the median of the stationary distribution. To simulate a negative shock, we set the original Brownian shock at its 1% quantile ($dZ_t = -2.32dt$) for a period of $\Delta t = 1$. The dashed lines indicate different quantiles of the distribution and the solid line is the median response. The color is monotone in density.

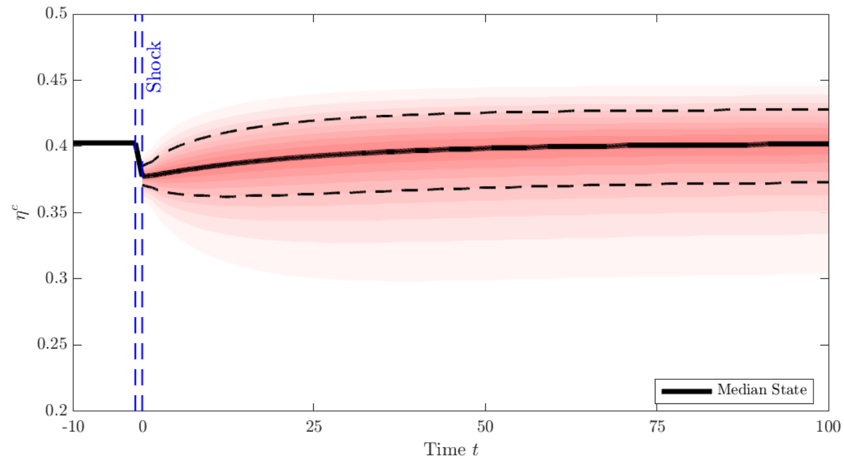


Figure 4.6: Distributional impulse response at stochastic steady state

More interestingly, the second type of fan chart plots the *difference* between distributions with and without the shock. As we find in Figure 4.7, the difference converges to zero in the long run.

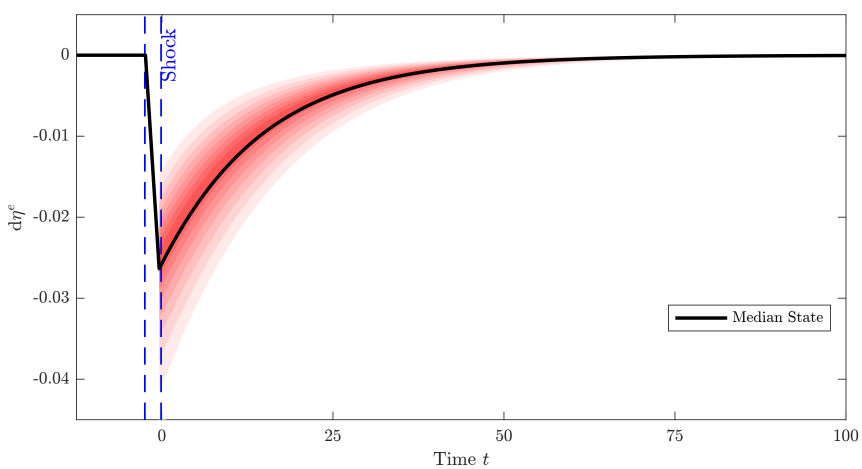


Figure 4.7: Distributional impulse response (difference to unshocked path), $\sigma = 0.1$

4.4 Discussions

In this section we study how specific parameters (a^h, σ, α) affect the equilibrium.

Figure 4.8 shows the effect of σ on the equilibrium. The steady state η^* drops as σ declines, while risk premia fall in the normal region, until η^* reaches the boundary of the crisis region (the kink in q).[6] A volatility paradox emerges: endogenous risk σ_t^q does not necessarily fall as σ declines. Note that as $\sigma \to 0$, the boundary of the crisis region does not converge zero – there is always some positive endogenous risk.

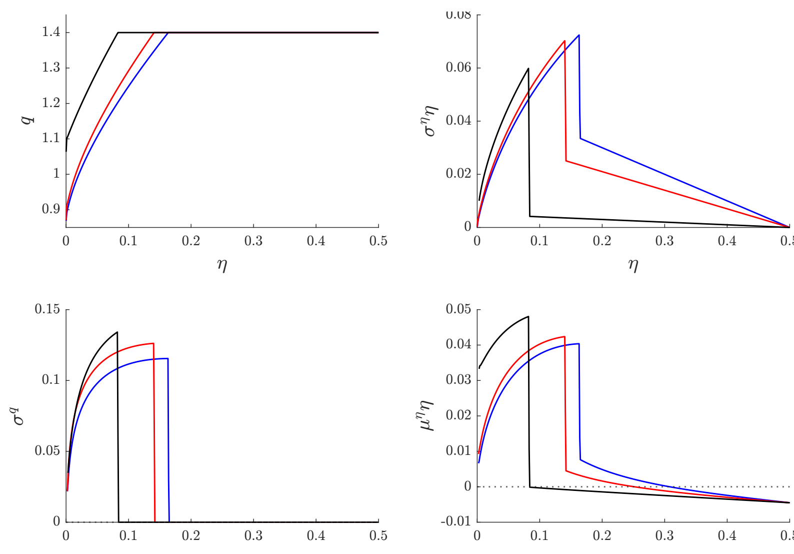


Figure 4.8: Equilibrium for σ= .1 (blue), .07 (red) and .01 (black)

The effect of relaxed financial friction is similar. As shown in Figure 4.9, endogenous risk σ_t^q increases as α falls.

[6] This happens for $\sigma = .01$ in Figure 4.8.

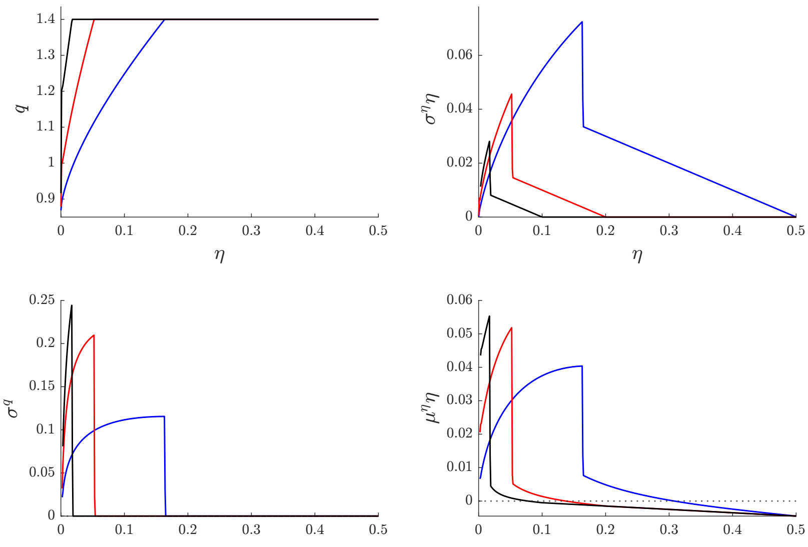


Figure 4.9: Equilibrium for α= .5 (blue), .2 (red) and .1 (black)

The household productivity a^h has a major impact on stability of the system. a^h reflects how households value capital, when they are forced to hold it. Figure 4.10 shows the equilibrium dynamics for different values of a^h. Endogenous risk significantly increases as a^h declines while the behavior in the normal regime and η^* are insensitive to a^h.[7] It is surprising that although expert leverage responds to fundamental risk σ, it does not respond to endogenous tail risk.

[7] As noted in Brunnermeier and Sannikov (2016), for log utility, it can be analytically proved that the dynamics in the normal regime is independent of a^h.

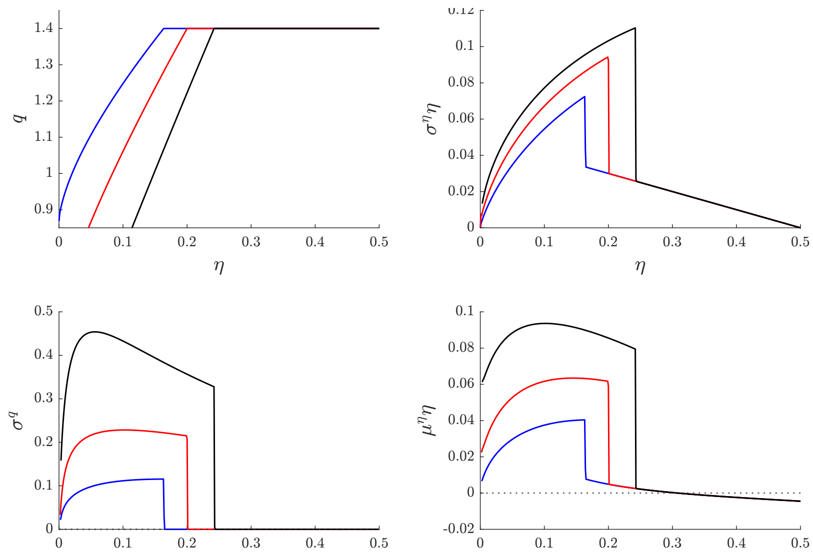


Figure 4.10: Equilibrium for a^h= .03 (blue), -.02 (red) and -.07 (black)

4.5 Exercises

4.5.1 Fire Sales

In this exercise you will solve the model from Lecture 04 numerically, under the assumption of log utility.

1. Our goal is to construct functions $q(\eta), \iota(\eta), \kappa(\eta)$ and $\sigma^q(\eta)$ on the $[0,1]$ grid. Slides 47-48 provide the parameter values, and slide 46 provides the set of equations and the algorithm.

    (a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$.

    (b) Create a uniform grid for $\eta \in [0.0001, 0.9999] = \{\eta_1 = 0.0001, \eta_2, \ldots, \eta_N = 0.9999\}$.

    (c) Using the implicit method with the one-step Newton's algorithm, solve the system of equations on slide 46 for $\eta_1, \eta_2, \ldots$ and so on.

    (d) Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$, solve for q and σ^q.

(e) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$ and comparing it with the graph on slide 50. Do your functions converge to the boundary solution for $\eta = 1$ that you obtained in (a) as $\eta \to 1$?

(f) Plot the remaining variables: $\iota(\eta), \kappa(\eta)$.

(g) We can also look at the experts' balance sheet: derive expression for the scaled version of issued debt: $\frac{D_t^e}{q_t K_t}$ and plot it against η.

2. Recall from the lecture that drift and volatility of η in the general case are given by:

$$\mu_t^\eta = (1-\eta_t)\Bigg[(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) -$$
$$(\varsigma_t^h - \sigma - \sigma_t^q)\left(-\frac{\eta_t}{1-\eta_t}\sigma_t^\eta + \sigma + \sigma_t^q\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t^h}{N_t^h}\right) + \frac{\rho_d^h \zeta(1-\eta_t) - \rho_d^e(1-\zeta)\eta_t}{\eta_t(1-\eta_t)}\Bigg]$$

$$\sigma_t^\eta = \frac{\kappa_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q)$$

(a) Which terms in the above equations can we simplify/substitute because of log utility and why? Perform these substitutions and derive the drift and volatility of η under log utility.

(b) Verify your solution by plotting $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$ and comparing them with the graph on slide 50.

4.5.2 Brunnermeier and Sannikov (2014)

In Brunnermeier and Sannikov (2014), experts and less productive households are risk neutral. However, while consumption of households can go negative, for experts it has to stay non-negative. Hence, experts become extremely risk averse when consumption approaches zero. As a result, the stationary distribution is bimodal.

Try to replicate the paper with tools studied in this chapter.

Bibliography

Brunnermeier, Markus K. and Yuliy Sannikov, "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 2014, *104* (2), 379–421.

— **and** — , "Macro, money, and finance: A continuous-time approach," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 1497–1545.

Dumas, Bernard and Elisa Luciano, *The Economics of Continuous-Time Finance*, MIT Press, 2017.

Kelsey, David and Frank Milne, "Externalities, Monopoly and the Objective Function of the Firm," *Economic Theory*, 2006, *29* (3), 565–589.

Chapter 5

Contrasting Financial Frictions

In this chapter, we extend the previous model by including a leverage contrainst on the amount of debt the expert sector can issue.

5.1 Model Setup

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions. Upon death of an expert/household, a new agent takes their place, inherits their wealth, and becomes an expert with probability $\zeta^e \in (0, 1)$.

Experts. Experts have a CRS technology $y^e_t = a^e k^e_t$. Denote their consumption and investment rate by c^e_t, ι^e_t. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k^e_t}{k^e_t} = (\Phi(\iota^e_t) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta^{k,e}_t$$

Still, we have only aggregate risk in the environment. Experts have a log utility function and they each maximize Experts have a log utility function and they each maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^e t}\log c_t^e \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^e. Define $\rho^e := \rho_0^e + \rho_d^e$. The objective is equivalent to infinite lifetime with higher discount rate ρ^e

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^e t}\log c_t^e \mathrm{d}t\right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,h}.$$

We let households hold capital to capture fire-sales. As we have discussed in section 3, assuming that households are more patient than the experts, i.e., $\rho^h \leq \rho^e$, is a modeling trick to ensure that the experts do not hold all the capital in the long run. However, here we achieve the same outcome by introducing death. The households maximize

$$\mathbb{E}_0\left[\int_0^T e^{-\rho_0^h t}\log c_t^h \mathrm{d}t\right]$$

where T is exponentially distributed with parameter ρ_d^h. Similarly, the objective is equivalent to infinite lifetime with higher discount rate $\rho^h := \rho_0^h + \rho_d^h$

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log c_t^h \mathrm{d}t\right].$$

Financial Friction. The financial friction in this chapter is due to incomplete markets (see, e.g., Dumas and Luciano, 2017). Although experts are allowed to issue equity, they must hold at least α fraction of their risk. At the same time, experts are subject to a leverage constraint by which they can issue risk-free debt up to a fraction ℓ of the value of their capital holdings. As for households, they are subject to a no-short-selling constraint on capital. The balance sheets of the two sectors are as following:

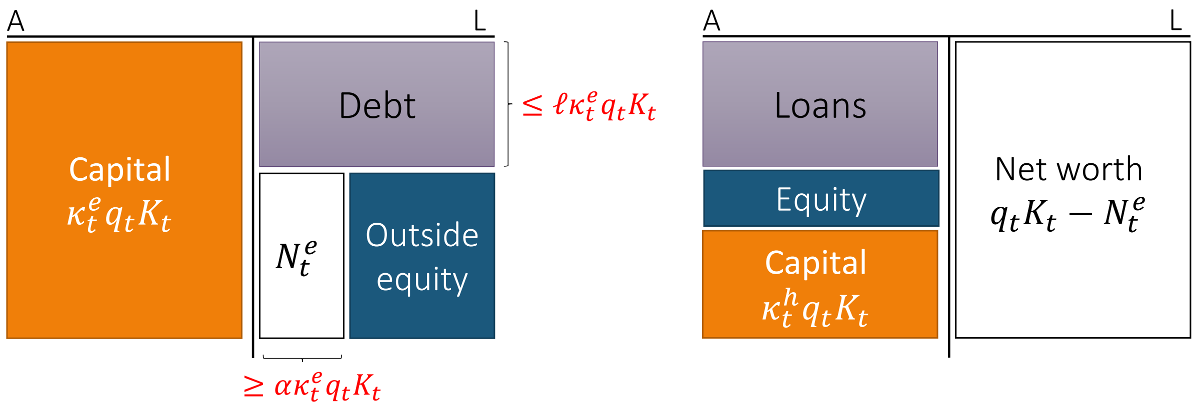


Figure 5.1: Balance sheets of experts and households

Letting $\theta^{e,K} = \kappa^e qK/N^e, \theta^{e,D} = -D^e/N^e$ and $\theta^{e,OE} = OE^e/N^e$, the skin-in-the-game constraint can be expressed as $(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geq 0$ and the leverage constraint as $(1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leq 1$. In addition, the no-short-selling constraint for the household sector is $\theta_t^{h,K} \geq 0$.

Leverage Constraints in Practice. The international regulation for leverage is imposed by the Basel Accords. In 1988 Basel I was published, setting minimal capital requirements for credit risk. In 2004 this was updated to Basel II which introduced risk dependent weights (which are not time varying) on capital requirements. Basel III, introduced in 2010/11, tightened capital requirements and set a leverage ratio requirement.

5.2 Solution Method

We solve this problem using the Stochastic Maximum Problem.

5.2.0 Agents' portfolio choice

Experts' portfolio choice. Let $r_t^{e,j} := \mathbb{E}[dr_t^{e,j}]/dt$. Then the experts solve the following optimization problem

$$\begin{aligned}
\max_{c_t^e,\iota_t^e,\theta_t^{e,K},\theta_t^{e,OE}} \; & \mathbb{E}\left[\int_s^\infty e^{-\rho^e t}u(c_t^e)\mathrm{d}t\right] \text{ subject to} \\
\mathrm{d}n_t^e = & \left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]\mathrm{d}t \\
& + n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t \\
& (1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geq 0 \text{ (skin-in-the-game constraint)} \\
& (1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leq 1 \text{ (leverage constraint)}
\end{aligned}$$

Denote by λ_t^ℓ the multiplier on the leverage constraint and by λ_t^χ the multiplier on the skin-in-the-game constraint. The Hamiltonian can be constructed as

$$\begin{aligned}
\mathcal{H}_t^e = & e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} \\
& - \varsigma_t^e \xi_t^e \overbrace{n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell\left[1 - (1-\ell)\theta_t^{e,K} - \theta_t^{e,OE}\right] \\
& + \xi_t^e n_t^e \lambda_t^\chi\left[(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE}\right]
\end{aligned}$$

Notice that the objective function is linear in θ, hence the solution is of bang-bang type: the agents are either indifferent or at a constraint. In addition to this, the Fisher Separation Theorem applies between $c_t^e, \grave{}_t^e, \iota_t^e$ since their first-order conditions are decoupled.

Households' portfolio choice. The households solve the following optimization problem

$$\max_{c_t^h,\iota_t^h,\theta_t^{h,K},\theta_t^{h,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t}u(c_t^h)\mathrm{d}t\right], \; s.t.$$

$$
\begin{aligned}
\mathrm{d}n_t^h &= \left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K} - r_t) + \theta_t^{h,OE}(r_t^{h,OE}(\iota_t^h) - r_t)\right)\right]\mathrm{d}t \\
&+ n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t \\
&\theta_t^{h,K} \geq 0 \text{ (household short-sale constraint)}
\end{aligned}
$$

Denote the multiplier on the no-short-selling constraint on capital as λ_t^h. The Hamiltonian can be constructed as

$$
\begin{aligned}
\mathcal{H}_t^h = e^{-\rho^h t}u(c_t^h) + \xi_t^h \overbrace{\left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t) + \theta_t^{h,OE}(r_t^{h,OE} - r_t)\right)\right]}^{\mu_t^{n^h} n_t^h} \\
- \varsigma_t^h \xi_t^h \overbrace{n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^h} n_t^h} \;+\; \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}
\end{aligned}
$$

which is also linear in θ_t^h and Fisher Separation Theorem also applies.

θ-choice. The first-order conditions with respect to θs are

- Experts' sector

$$
\begin{cases}
r_t^{e,K} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi \\
r_t^{OE} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi
\end{cases}
$$

- Households' sector

$$
\begin{cases}
r_t^{h,K} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) - \lambda_t^h \\
r_t^{OE} - r_t = \varsigma_t^h(\sigma + \sigma_t^q)
\end{cases}
$$

Taking the difference between them, we obtain

$$
\begin{aligned}
\frac{a^e - a^h}{q_t} &= (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^h + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi, \\
0 &= (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi,
\end{aligned}
$$

Hence, if we focus on the return gaps $r_t^{OE} - r_t^{h,K}$ and $r_t^{e,K} - r_t^{OE}$, we can write the

FOCs as

$$\begin{cases} r_t^{e,K} - r_t^{OE} = \alpha\lambda_t^{\chi} - \ell\lambda_t^{\ell} \\ r_t^{OE} - r_t^{h,K} = \lambda_t^h \end{cases}$$

Let us now consider different cases in which the constraints may or may not bind.

- Household short-selling constraint not binding ($\lambda_t^h = 0 \implies r_t^{OE} = r_t^{h,K}$)
  - $\lambda_t^{\chi} = 0, \lambda_t^{\ell} > 0$ is impossible because $r_t^{e,K} > r_t^{h,K} = r_t^{OE}$
  - $\lambda_t^{\chi} > 0, \lambda_t^{\ell} > 0$ or $\lambda_t^{\chi} > 0, \lambda_t^{\ell} = 0$ are both possible. The skin-in-the-game constraint always binds while the leverage constraint may or may not bind.
- Household short selling constraint binding ($\lambda_t^h > 0$)
  - If we define $\eta^{e,*}$ to be the smallest η_t^e such that $\lambda_t^h > 0$, then $\lambda_t^{\ell} > 0$ is impossible because $1/\eta_t^e < 1/\eta^{e,*}$. Hence, only the skin-in-the-game constraint may bind. The intuition behind this is that outside equity cannot generate a higher return than physical capital.

$(\kappa, \chi)-$ **Asset/Risk Allocation.** We now translate the constraints from the θ space to the $\kappa - \chi$ space. These become

$$\text{Skin-in-the-game constraint } \chi_t^e = \eta_t^{e,K}\theta_t^e + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\geq -(1-\alpha)\kappa_t^e} \geq \alpha\kappa_t^e$$

$$\text{Leverage constraint } \chi_t^e = \eta_t^e\theta_t^{e,K} + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\leq (1-(1-\ell)\theta_t^{e,K})} \leq \ell\kappa_t^e + \eta_t^e$$

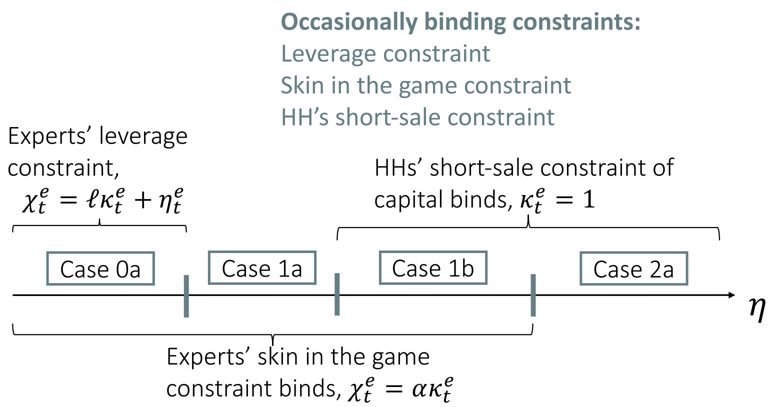
And the FOCs can be rewritten as

$$\frac{a^e - a^h}{q_t} \geq \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta-\text{risk premia}}, \qquad \text{with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

$$\varsigma_t^e \geq \varsigma^h, \qquad \text{with equality if } \chi_t^e > \alpha\kappa_t^e$$

With this in mind, we can now consider four different cases across η which are summarized in the table and figure below

Cases	0a	1a	1b	2a
Leverage	$\chi^e_t = \ell\kappa^e_t + \eta^e_t$	$\chi^e_t < \ell\kappa^e_t + \eta^e_t$	$\chi^e_t < \ell\kappa^e_t + \eta^e_t$	$\chi^e_t < \ell\kappa^e_t + \eta^e_t$
Skin-in-the-game	$\chi^e_t = \alpha\kappa^e_t$	$\chi^e_t = \alpha\kappa^e_t$	$\chi^e_t = \alpha\kappa^e_t$	$\chi^e_t > \alpha\kappa^e_t$
Short-sale	$\kappa^e_t < 1$	$\kappa^e_t < 1$	$\kappa^e_t = 1$	$\kappa^e_t = 1$
Δ-risk premia	$>$	$=$	$>$	$>$
Risk-sharing	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$



5.2.1 Market Clearing

Determination of κ, χ. The determination of κ is based on the difference in risk premia $\alpha(\varsigma^e_t - \varsigma^h_t)(\sigma + \sigma^q_t)$. Hence, κ is determined by the FOC

$$\boxed{\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi^e_t - \eta^e_t}{(1-\eta^e_t)\eta^e_t}(\sigma + \sigma^q_t), \text{ with equality if } \kappa^e_t < 1 \text{ and } \chi^e_t < \ell\kappa^e_t + \eta^e_t.}$$

The risk share of experts χ^e_t is given by

$$\boxed{\chi^e_t = \max\{\alpha\kappa^e_t, \eta^e_t\}}$$

while the determination of κ^e_t in the leverage constrained region follows from

$$\boxed{\kappa^e_t = \frac{\eta^e_t}{\alpha - \ell}}$$

Investment and capital prices. Optimal investment dictates

$$\boxed{\phi\iota = q - 1}$$

and from the previous lecture on amplification, we have that

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

Output good market clearing. This requires

$$(\kappa_t^e a^e + (1 - \kappa_t^e) a^h - \iota_t) K_t = C_t$$
$$\Rightarrow \quad \boxed{\kappa_t^e a^e + (1 - \kappa_t^e) a^h - \iota_t = q_t[\eta_t \rho^e + (1 - \eta_t)\rho^h]}$$

5.2.2 Algorithm – Static Step

We have five static conditions,

1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e: $\frac{a^e - a^h}{q_t} \geq \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Planner condition for χ_t^e: $\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$
4. $\kappa_t^e a_t^e + (1 - \kappa_t^e) a^h - \iota(q_t) - q_t[\eta_t \rho^e + (1 - \eta_t)\rho^h] = 0$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$
   $\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

The algorithm strategy is to start at $q(0)$, solve to the right, and use a different procedure for the two η regions depending on κ:

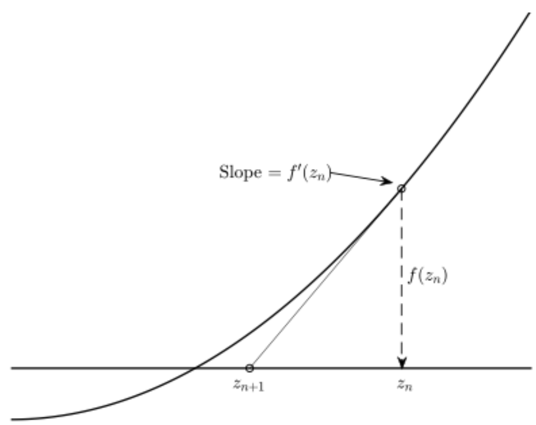
1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (4)

- Plug in the Planner's condition of χ_t
- Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method
- if $\chi_t^e \geq \ell\kappa_t^e + \eta_t^e$, replace κ_t^e by $\frac{\eta_t^e}{\alpha-\ell}$, solve (3) (4) (5) for $\chi(\eta^e)$,$q(\eta^e)$,$\sigma^q(\eta^e)$

2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (4) for $q(\eta^e)$
   (HINT: When constraint binds, we directly substitute in κ^e)

Aside: Newton's Method The system of equations to use for Newton's method in this case is as follows,

$$\mathbf{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, F(\mathbf{z}_n) = \begin{bmatrix} \kappa_t^e a_t^e + (1-\kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t\rho^e + (1-\eta_t)\rho^h] \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}\left(\sigma + \sigma_t^q\right)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{Planner.} \end{bmatrix}$$



5.3 Numerical Solution and Model Properties

In this Section we cover the numerical solution, which was computed with the algorithm covered above.

5.3.1 Capital Price and Volatility

We first analyze the price of capital and amplification. Figure 5.2 plots these, for the displayed parameters. Note that $\ell = 0.55$, which corresponds to one being able to borrow up to 55% of their collateral. We observe four regions split by the dashed lines. In the left most region both the leverage constraint ad outside equity constraint are binding. In the second left most region the leverage constraint no longer binds. These two regions make up the crisis regimes, where fire-sales are occurring and capital price is increasing as a function of η. The third region is where 100% of the capital is held by the experts, and households are short-sale constrained. In the final, right most, region there is perfect risk sharing, where $\chi = \kappa$.

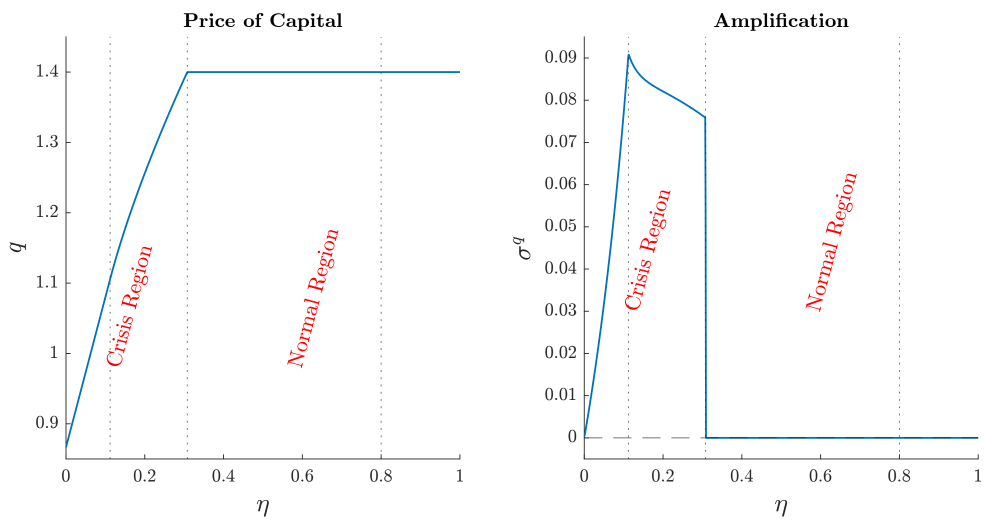


Figure 5.2: $\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

It is interesting to compare the volatility of the price of capital in the model over different cases. Figure 5.2 plots this for the full model, with outside equity and leverage. Figure 5.3 plots σ^q for a benchmark model (dashed black line) with no outside equity and no leverage, as well as a model with outside equity only (red line). As outside equity only is added to the model, the volatility goes up in the fire sale region. As the leverage constraint is added on top of this, the fire-selling region in which it does not bind (the second left most region) experiences more price volatility.

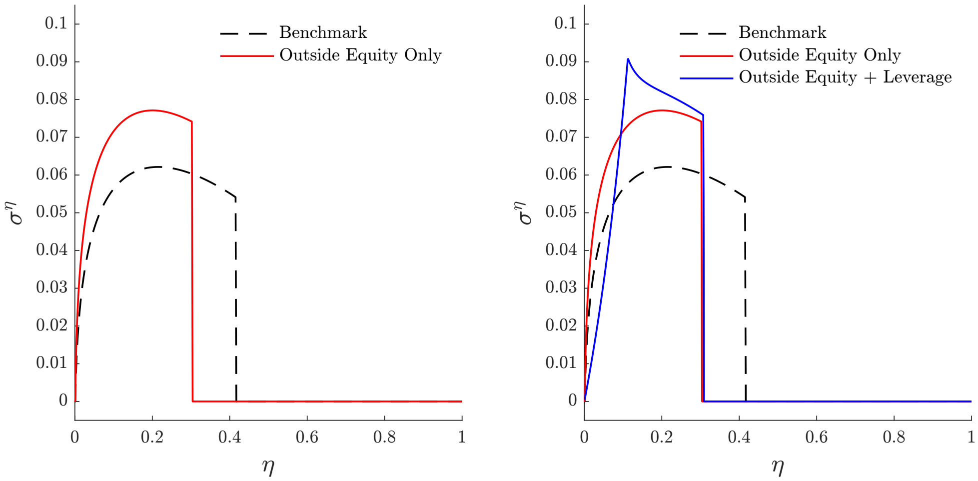


Figure 5.3: Same parameters as Figure 5.2, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

5.3.2 Net Worth Evolution: Drift & Volatility

Figure 5.4 plots the drift and volatility of η in arithmetic terms. When the drift of the state variable is 0 we are at the steady state. Below the steady state the system drifts up and above it it drifts down. The volatility is highest in the region where the leverage constraint is not binding but fire-sales are still happening.

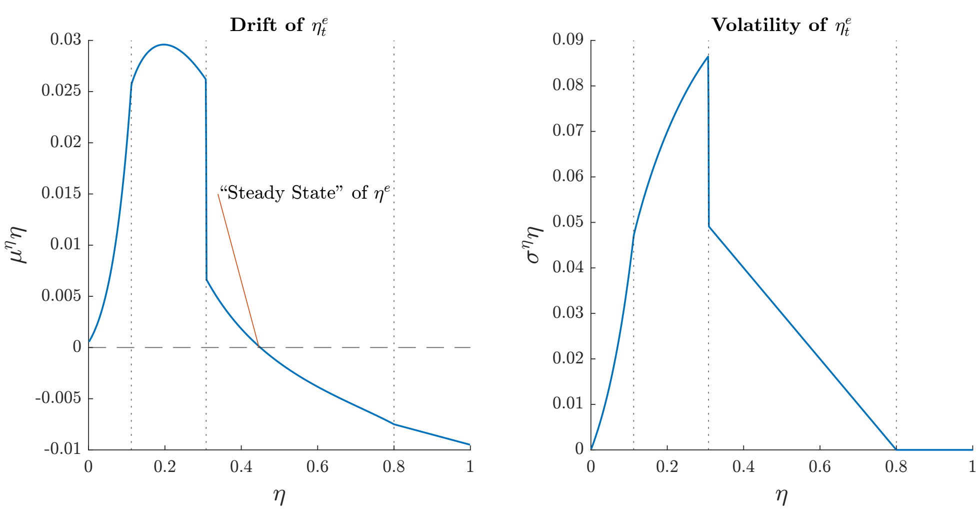


Figure 5.4: $\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, δ =0.05, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

5.3.3 Risk Allocation & Leverage

Figure 5.5 plots the risk allocation and leverage for the full model, with both outside equity and the leverage constraint. The risk allocation plot shows what proportion of risk is held by the experts. It is increasing in η, and we observe perfect risk sharing in the right most region. When the experts hold, for example, 90% of the wealth, they also hold 90% of the risk. Therefore from $\eta = 0.8$ onwards the risk holding plot is the identity line. In the second right most region, the experts already hold all of the capital but only 80% of the risk. This is because the outside equity constrain of $\alpha = 0.8$ is binding. In the capital net worth ratio plot we observe the leverage constraint kicking in in the left most region. Experts would like to lever up more but cannot. Note that the lever up less as they capture more of the wealth share.

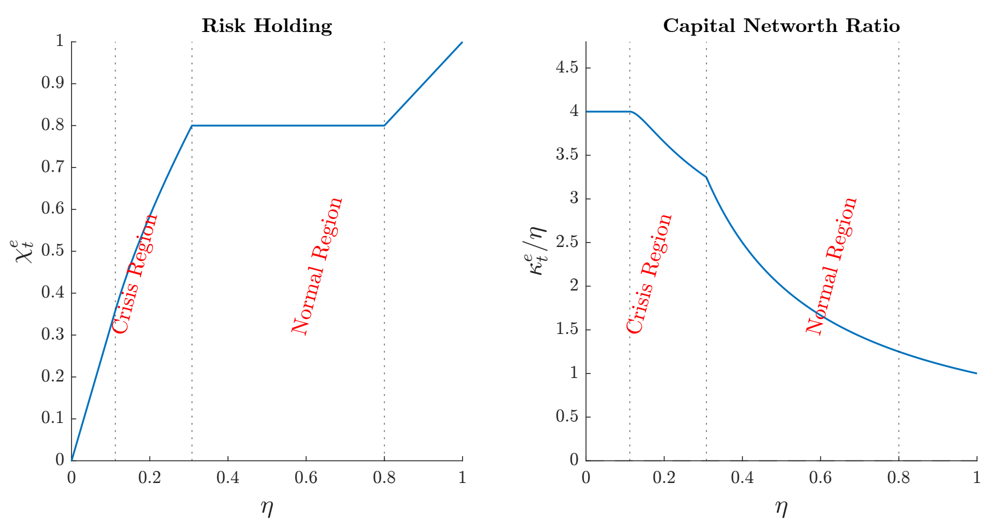


Figure 5.5: $\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

Risk Allocation Comparison. Figure 5.6 plots risk allocation for the benchmark model (dashed black line) with no outside equity and no leverage, as well as the model with outside equity only (red line). The full model is again plotted in blue. The addition of outside equity allows experts to offload a proportion of their risk, which is why the red line is lower.

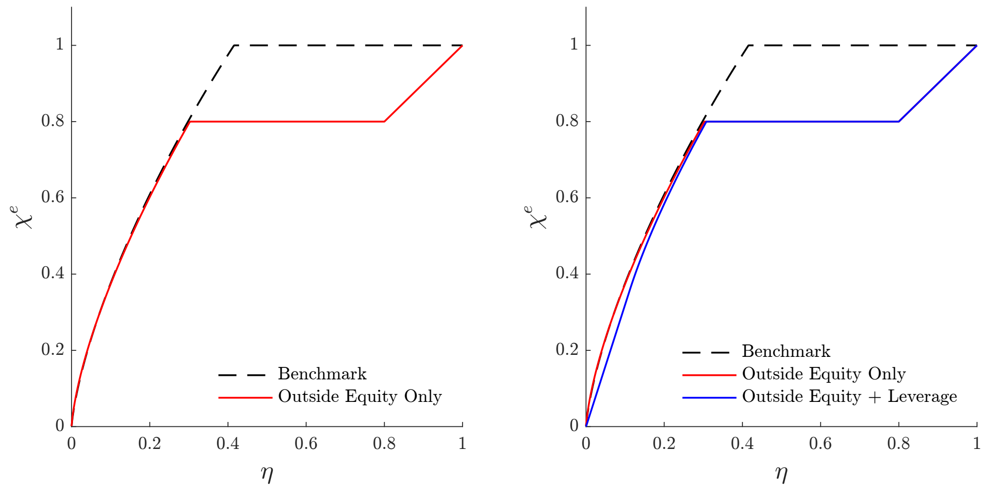


Figure 5.6: Same parameters as Figure 5.5, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

Leverage Comparison. Figure 5.6 plots leverage for the benchmark model (dashed black line) with no outside equity and no leverage, as well as the model with outside equity only (red line). The full model is again plotted in blue. Allowing for outside equity lets experts hold even more capital in relation to net worth, but if there is a leverage constraint on top this mechanically tightened again.

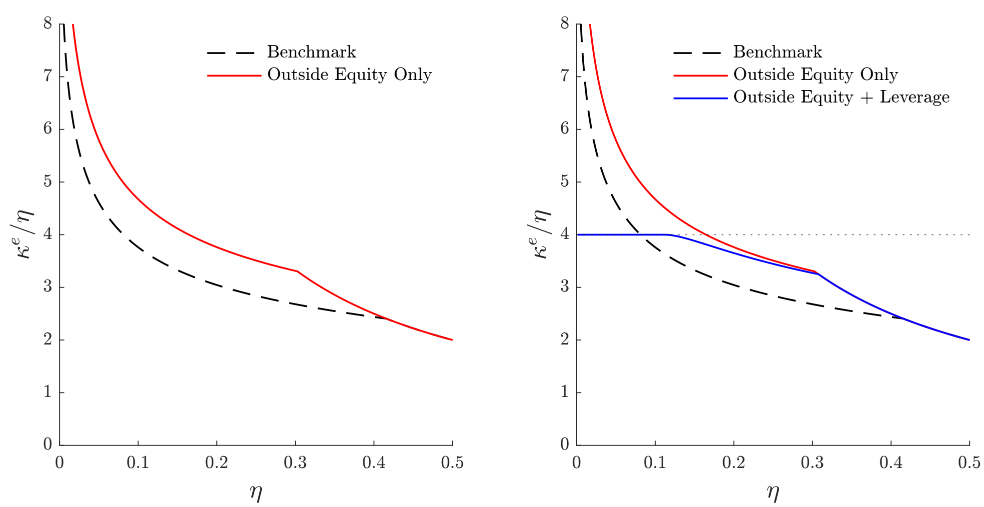


Figure 5.7: Same parameters as Figure 5.5, but black line has $\alpha = 1$, $\ell = 1$ and red line has $\alpha = .8$, $\ell = 1$.

5.3.4 Volatility Paradox

We now consider the model with outside equity $\alpha = 0.8$, but no leverage constraint. Then the *volatility paradox* is a phenomenon which shows that σ^η (as well as $\sigma + \sigma^q$) stays roughly constant as σ varies (even when $\sigma \to 0$). Figure 5.8 plots various objects for different σ. We observe that the regions shift slightly, since the fire-selling regions start for lower η if there is lower σ. However, we observe that the total volatility, at the point where fire sales start in each parameterization, is roughly the same. This is because as you lower the fundamental volatility, σ, the price volatility, σ^q, increases.

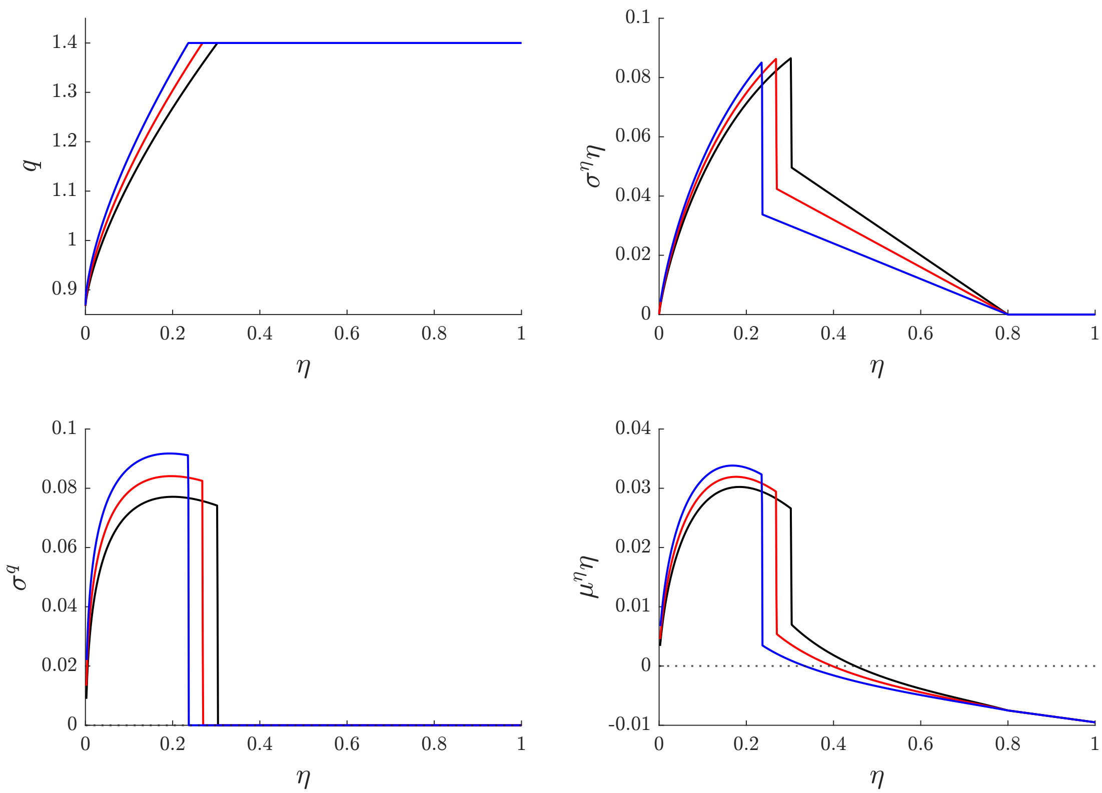


Figure 5.8: Volatility Paradox $\alpha = 0.8$, $\sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$

If we add a leverage constraint on top of the model $\ell = 0.55$ we get even more interesting results. Figure 5.9 plots the same objects as before for different σ. Now σ^q is really increasing in σ for the second left most region, where fire-sales are happening but the leverage constraint does not bind. In the left most region of the binding leverage constraint and fire-sales, the $\sigma^\eta \eta$ and σ^q are decreasing in σ. The leverage constraint binds earlier on for lower σ and makes the volatility go down. This was not the case in the model without the leverage constraint. Interestingly, the drift also collapses compared to the case of no leverage constraint. this is because experts cannot lever up as they wish to take advantage of investment opportunities. The upwards drift in the left most region is lower for lower σ. The lower the σ the longer it takes to get out of the leverage constrained region.

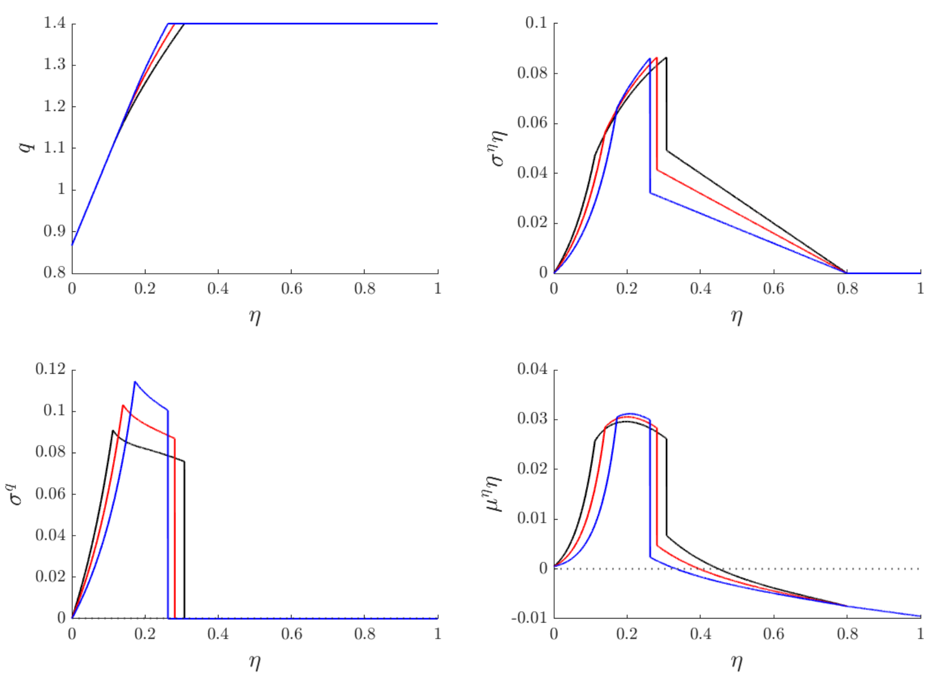


Figure 5.9: Volatility Paradox $\alpha = 0.8, \ell = 0.55, \sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$

5.4 Stationary Distribution & Net Worth Trap

Recall from Chapter 4 that one can find the stationary distribution from the "Kolmogorov Forward Equation". Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows

$$\frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial [f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2 [f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}.$$

A corollary is that if stationary distribution $f(\eta)$ exists, it satisfies ODE

$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2},$$

which has the closed form solution,

$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x\right).$$

Aside: KFE Analytical Example.

- Reflected Geometric Brownian Motion (Reflecting barrier at $x = d$):

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}Z_t - \mathrm{d}U_t, X_t \in (0, d]$$

- KFE:

$$\frac{\partial f}{\partial t} = -\frac{\partial(\mu x f)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2 x^2 f)}{\partial x^2}$$

- Stationary distribution

$$f(x) = \frac{\text{Const}}{\sigma^2 x^2} \exp\left(\int_0^x \frac{2\mu y}{\sigma^2 y^2}\mathrm{d}y\right) = \frac{\frac{2\mu}{\sigma^2} - 1}{d^{\frac{2\mu}{\sigma^2}-1}} x^{\frac{2\mu}{\sigma^2}-2}$$

Net Worth Trap. Figure 5.10 plots the stationary distribution for different σ. The net worth trap is that the stationary distribution, for certain values of σ (in this case $\sigma = 0.05$), is double humped shape. The system lives a lot of time in the center, but also lives a lot of time in the low η regime, or the crisis regime. Without the leverage constraint this phenomenon does not occur.

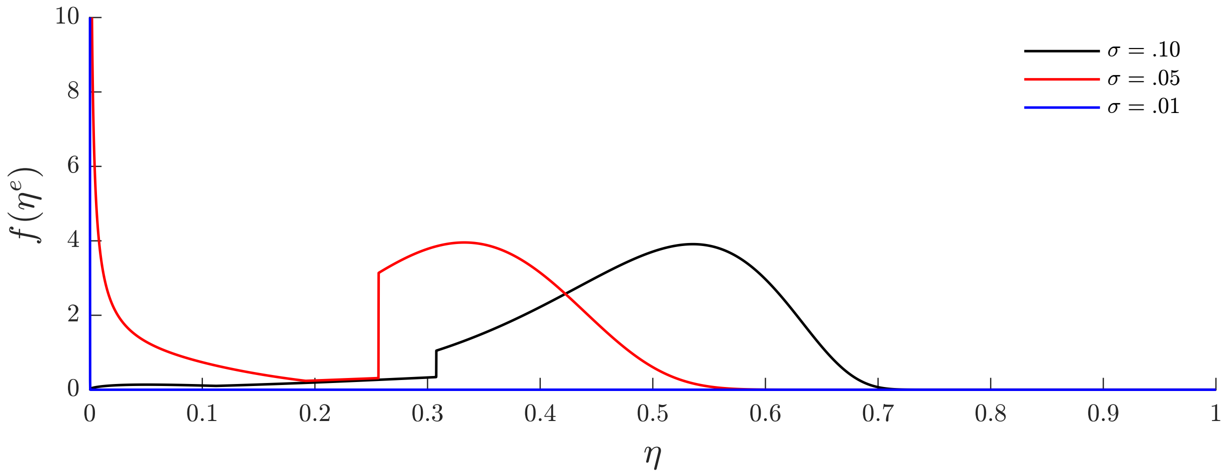


Figure 5.10: Stationary Distribution for Different σ. Fundamental volatility: $\sigma = .1$, $\sigma = .05$, $\sigma = .01$

For low values of σ, like 0.01 in this case, the stationary distribution is degenerate. Once experts become under capitalized they cannot escape the crisis regime and are pushed to hold no wealth. So when does the invariant distribution exist? The asymptotic solution (as $\eta \to 0$) follows

$$f(\eta) \sim \left(\frac{2\mu(0)}{\sigma^2(0)} - 1\right) \eta^{\frac{2\mu(0)}{\sigma^2(0)}-2}.$$

We then have the following cases,

- $\frac{2\mu(0)}{\sigma^2(0)} \geq 2$: $f(\eta)$ is finite at $\eta = 0$.
- $2 \geq \frac{2\mu(0)}{\sigma^2(0)} > 1$: $f(\eta)$ is infinite at $\eta = 0$, but still normalizeable ($\int f d\eta < \infty$).
- $1 \geq \frac{2\mu(0)}{\sigma^2(0)}$: $f(\eta)$ is infinite at $\eta = 0$, and stationary distribution does not exist.

5.4.1 Implementation in MATLAB: log-utility

Following is main code executing the algorithm under the assumption of log-utility.

```
%% Parameter values
rho_0 = 0.04; rho_e_d = 0.01;rho_h_d = 0.01;
rho_e = rho_0 + rho_e_d; rho_h = rho_0 + rho_h_d;
zeta = 0.05; delta = 0.05; a_e = 0.11; a_h = 0.03;
```

```
sigma = 0.1;alpha = .8; phi = 10;ell = .55;

N = 10001;

%% Grids
eta = linspace(0.0,1,N); deta = eta(2)-eta(1);

rho = rho_e.*eta+ rho_h.*(1-eta);

q = (1+phi* a_e)./(ones(N,1)+phi*rho'); q=q';
iota = (ones(N,1)*a_e - rho')./(ones(N,1)+phi.*rho');iota = iota';
chi = zeros(1,N);sig_q = zeros(1,N);kappa = zeros(1,N);

q(1) = (1+a_h*phi)/(1+rho_h*phi);
iota(1)=(a_h-rho_h)/(1+rho_h*phi);

tor = 1e-5;
max_it = 100;
ind_lv = 1;
ind_fl = 1;
flag = 0;
%% Model solution: Newton's method
for i = 2:N
    ind_fl = ind_fl +1;
    iter = 0;
    error = 1.0;
    etai = eta(i);
    F = @(x)[(a_e-a_h)/x(1)-alpha*((alpha*x(2)-etai)/etai/(1-etai))*...
        (sigma+x(3))^2;
        x(2)*a_e + (1-x(2))*a_h-(x(1)-1)/phi-x(1)*rho(i);...
        (x(1)-q(i-1))/deta* (alpha *x(2)-etai)*(sigma+x(3))-x(1)*x(3)];
    J = @(x) [-(a_e-a_h)/x(1)^2, -alpha^2*(sigma +x(3))^2/etai/(1-etai), ...
        -2*alpha*(alpha*x(2)-etai)/etai/(1-etai)*(sigma+x(3));
        -1/phi - rho(i), a_e - a_h, 0;...
        1/deta*(alpha*x(2)-etai)*(sigma+x(3))-x(3), alpha/deta*...
        (x(1)-q(i-1))*(sigma+x(3)), -1/deta*(x(1)-q(i-1))...
        *(alpha*x(2)-etai)-x(1)];
    z0 = [q(i-1), kappa(i-1),sig_q(i-1)];
    while error > tor
        iter = iter + 1;
        z1 = z0 - reshape(J(z0)\F(z0),[1,3]);
        error = norm(z1-z0)/norm(z0);
        z0 = z1;
        %disp(error);
        if iter>max_it
            disp("HAVE trouble!")
            break
        end
    end
    if z0(2)>etai/(alpha-ell)
        % if leverage constraint is violated, solve kappa by leverage
        % constraint
```

```
        kappai= etai/(alpha-ell); % alpha * kappa = ell * kappa + eta
        kappa(i) = kappai;
        q(i) = (a_e * kappai + a_h * (1-kappai) +1/phi)/(1/phi + rho(i));
        chi(i) = alpha * kappa(i);
        sig_q(i) = sigma/(1-(q(i)-q(i-1))/deta/q(i) * (chi(i)-etai))-sigma;
        iota(i)= (q(i)-1)/phi;
        ind_lv  = ind_lv+1;
    else
        q(i) = z0(1);
        kappa(i) = z0(2);
        chi(i)=alpha*kappa(i);
        sig_q(i) = z0(3);
        iota(i)= (q(i)-1)/phi;
    end

    if kappa(i)>1
        flag = i;
        break
    end
end

for i = flag:N
    etai = eta(i);
    q(i)=(1+phi*a_e)/(1+phi*rho(i));
    iota(i)=(a_e - rho(i))/(1+phi*rho(i));
    kappa(i)=1;
    if etai <alpha
        ind_fl = ind_fl +1;
        chi(i) =alpha;
        sig_q(i) = sigma *(q(i)-q(i-1))/deta/q(i)*(alpha-etai)/...
            (1-(q(i)-q(i-1))/deta/q(i)*(alpha-etai));
    else
        chi(i)=etai;
        sig_q(i)=0;
    end
end
 sig_q(1)=sig_q(2);

%% Compute drift and volatility of eta
mu_eta = (ones(1,N)-eta).*(((chi.^2-chi.*eta)./eta.^2-(eta-chi).*...
    (ones(1,N)-chi)./(ones(1,N)-eta).^2) ...
    .*(sigma.*ones(1,N)+sig_q).^2-(rho_e-rho_h).*ones(1,N)+(rho_h_d.*zeta.*ones(1,N)
    .*(1-eta)-rho_e_d.*(1-zeta).*ones(1,N).*eta)./(eta.*(ones(1,N)-eta)));
sig_eta = (sigma*ones(1,N)+sig_q).*(chi-eta)./eta;

%% Compute stationary distribution
M = buildM(eta ,mu_eta .*eta,sig_eta.*eta);

g = ones(N,1);
Mt = M';
g(2:N) = - (Mt(2:N,2:N)\(Mt(2:N,1))*g(1));
g = g/sum(g)/deta;
```

5.5 Net Worth Trap & Volatility Paradox Interaction

The net worth trap is based on the volatility paradox interaction with the leverage constraint. The leverage constraint depresses μ^η and σ^η when η is close to 0 since experts are constrained and cannot take on any risk. Furthermore, there is higher volatility of q and η in the fire-sale region outside the binding leverage constraint.

Regulation, like the Basel Accords, is important for keeping volatility low and maintaining stability in the economy. However, a leverage constraint which is not varying in η can induce a net worth trap, making it hard for experts to recover from low wealth shares.

Bibliography

Dumas, Bernard and Elisa Luciano, *The economics of continuous-time finance*, MIT Press, 2017.

Part III

Immersion Chapters

Chapter 6

A More General Macro-model with Endogenous Risk Dynamics

In Chapter 4, we studied a macro model with endogenous risk dynamics under log-utility. In this chapter, we present a generalization to other utility functions, namely constant relative risk aversion (CRRA) utility and Epstein-Zin (EZ) utility. We begin with the CRRA case.

6.1 CRRA Utility and Value Functions

Now with CRRA utility, sector i agents now have the following optimization problem

$$\begin{aligned} \max_{\{\iota_t^i, \boldsymbol{\theta}_t^i, c_t^i\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho^i t} \frac{c_t^{1-\gamma}}{1-\gamma} \mathrm{d}t \right] \\ \text{s.t.} \quad & \frac{\mathrm{d}n_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}\mathrm{d}t + \theta_t^{i,K}\mathrm{d}r_t^{i,K}(\iota_t^i) + \theta_t^{i,OE}\mathrm{d}r_t^{OE} + \theta_t^{i,D} r_t \mathrm{d}t \qquad (6.1) \\ & n_0^i \text{ given,} \end{aligned}$$

Once we deviate from the assumption of log-utility, we no longer have readily available expressions for prices of risk and consumption-to-wealth ratios. We can however express these variables in terms of agents' value functions, which we discuss in this Section.

Martingale approach works well in an endowment economy, where the consumption stream is exactly the endowment stream, hence the marginal utility is given exogenously by stochastic discount factor. While for our production economy case, we mix the martingale approach with value function method.

By the same arguments presented in Section 3.3.1, CRRA utility implies that c_t^i/n_t^i-ratio is invariant in n_t^i, that is[1]

$$V^i(n_t^i;\boldsymbol{\eta_t},K_t)=\frac{u\left(\omega^i(\boldsymbol{\eta_t},K_t)n_t^i\right)}{\rho^i}=\frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma},\qquad \frac{c_t^i}{n_t^i}=(\rho^i)^{1/\gamma}(\omega_t^i)^{1-1/\gamma}. \tag{6.2}$$

Let's first take a look at a special case. For constant investment opportunities $\omega_t^i=\omega$, c_t^i/n_t^i is constant, and hence $\mu_t^{c^i}=\mu_t^{n^i}$, $\sigma_t^{c^i}=\sigma_t^{n^i}$. Furthermore, by Itô's lemma,

$$\frac{\mathrm{d}(c_t^i)^{-\gamma}}{(c_t^i)^{-\gamma}}=\left[-\gamma\mu_t^{c^i}+\frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2\right]\mathrm{d}t-\gamma\sigma_t^{c^i}\mathrm{d}Z_t.$$

Because $\xi_t^i=e^{-\rho^i t}u'(c_t^i)=e^{-\rho^i t}(c_t^i)^{-\gamma}$, Itô's product rule implies that

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i}=-\rho^i\mathrm{d}t+\frac{\mathrm{d}(c_t^i)^{-\gamma}}{(c_t^i)^{-\gamma}}=\left[-\rho^i-\gamma\mu_t^{c^i}+\frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2\right]\mathrm{d}t-\gamma\sigma_t^{c^i}\mathrm{d}Z_t.$$

Recall the SDF process (4.2), and we now see that $\varsigma_t^i=\gamma\sigma_t^{c^i}=\gamma\sigma_t^{n^i}$ and

$$r_t=\rho^i+\gamma\mu_t^{c^i}-\frac{\gamma(1+\gamma)}{2}\left(\sigma_t^{c^i}\right)^2. \tag{6.3}$$

Consider a self-financing strategy that reinvests consisting of an agent's net worth with

[1]The value function for individuals i donates $V^i(n_t^i;\boldsymbol{\eta_t},K_t)$, where $(\boldsymbol{\eta_t},K_t)$ are state variables. For n-sector problem, $\boldsymbol{\eta_t}$ is a $n-1$ vector, while in this chapter $\boldsymbol{\eta_t}$ is a scalar η_t^e. For simplicity, later we use the notations $V_t^i=V^i(n_t^i;\boldsymbol{\eta_t},K_t),\omega_t^i=\omega^i(\boldsymbol{\eta_t},K_t),v_t^i=v^i(\boldsymbol{\eta_t})$.

consumption reinvested. By construction, the value of this strategy $p_t^{n^i}$ follows

$$\frac{\mathrm{d}p_t^{n^i}}{p_t^{n^i}} = \frac{\mathrm{d}n_t^i}{n_t^i} + \frac{c_t^i}{n_t^i}\mathrm{d}t.$$

The martingale approach tells us that $\xi_t^i p_t^{n^i}$ follows a martingale and the following asset pricing equation holds

$$\mu_t^{p^{n^i}} - r_t = \varsigma_t^i \sigma_t^{p^{n^i}} \iff \mu_t^{n^i} + \frac{c_t^i}{n_t^i} - r_t = \varsigma_t^i \sigma_t^{n^i} = \frac{(\varsigma_t^i)^2}{\gamma}.$$

The net worth then follows

$$\frac{\mathrm{d}n_t^i}{n_t^i} = \mu_t^{n^i}\mathrm{d}t + \sigma_t^{n^i}\mathrm{d}Z_t = \left[r_t + \frac{(\varsigma_t^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right]\mathrm{d}t + \frac{\varsigma_t^i}{\gamma}\mathrm{d}Z_t.$$

Since $\mu_t^{c^i} = \mu_t^{n^i}$, $\sigma_t^{c^i} = \sigma_t^{n^i}$, (6.3) implies

$$r_t = \rho^i + \gamma\left[r_t + \frac{(\varsigma_t^i)^2}{\gamma} - \frac{c_t^i}{n_t^i}\right] - \frac{\gamma(1+\gamma)}{2}\frac{(\varsigma_t^i)^2}{\gamma^2} \implies \frac{c_t^i}{n_t^i} = \rho^i + \frac{\gamma-1}{\gamma}\left[r_t - \rho^i + \frac{(\varsigma_t^i)^2}{2\gamma}\right]. \tag{6.4}$$

This will turn out to be a useful relationship for the next chapter, but now we focus on more general investment opportunity processes.

For arbitrary opportunity processes ω_t^i, we still have $\varsigma_t^i = \gamma\sigma_t^{c^i}$ and that $\xi_t^i p_t^{n^i}$ follows a martingale. By Itô's product rule,

$$\frac{\mathrm{d}(\xi_t^i p_t^{n^i})}{\xi_t^i p_t^{n^i}} = \frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i} + \frac{c_t^i}{n_t^i}\mathrm{d}t.$$

Rewrite (6.2) as

$$(c_t^i)^{-\gamma} = \frac{1}{\rho^i}(\omega_t^i)^{1-\gamma}(n_t^i)^{-\gamma} \iff e^{\rho^i t}\underbrace{e^{-\rho^i t}(c_t^i)^{-\gamma}}_{\xi_t^i} n_t^i = \underbrace{\frac{1}{\rho^i}(\omega_t^i)^{1-\gamma}(n_t^i)^{1-\gamma}}_{(1-\gamma)V_t^i}. \tag{6.5}$$

Hence,

$$\frac{\mathrm{d}V_t^i}{V_t^i} = \frac{\mathrm{d}(e^{\rho^i t}\xi_t^i n_t^i)}{e^{\rho^i t}\xi_t^i n_t^i} = \left(\rho^i - \frac{c_t^i}{n_t^i}\right)\mathrm{d}t + \underbrace{\frac{\mathrm{d}(\xi_t^i p_t^{n^i})}{\xi_t^i p_t^{n^i}}}_{\text{Martingale}}. \tag{6.6}$$

Unfortunately, we can not use Itô's formula on V_t^i to get the drift of $\mathrm{d}V_t^i/V_t^i$, as $n_t^i(\eta_t)$ and $\omega_t^i(\eta_t)$ are not differentiable when $q_t(\eta_t)$ has a kink[2]. Instead, we can de-scale the value function with regard to K_t and define the "de-scaled value function" v_t^i:

$$V_t^i = \frac{1}{\rho^i}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{1-\gamma} = \underbrace{\frac{\left(w_t^i n_t^i/K_t\right)^{1-\gamma}}{\rho^i}}_{v_t^i :=}\frac{K_t^{1-\gamma}}{1-\gamma}, \tag{6.7}$$

By such a de-scaling, we separate two state variables η_t^i and K_t[3], hence can work on them independently.

By Itô's product rule,

$$\frac{\mathrm{d}V_t^i}{V_t^i} = \frac{\mathrm{d}\left[v_t^i K_t^{1-\gamma}\right]}{v_t^i K_t^{1-\gamma}} = \left[\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i}\right]\mathrm{d}t + [\cdots]\mathrm{d}Z_t.$$

Recall (6.6), the drift of V_t^i equals

$$\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = \rho^i - \frac{c_t^i}{n_t^i}.$$

This gives us the following backward stochastic differential equation (BSDE)

$$\frac{\mathrm{d}v_t^i}{v_t^i} = \left[\rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)(\Phi(\iota_t)-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2 - (1-\gamma)\sigma\sigma_t^{v^i}\right]\mathrm{d}t + \sigma_t^{v^i}\mathrm{d}Z_t. \tag{6.8}$$

This is a BSDE that we can solve using standard numerical methods (Tourin, 2011). But before that, we need to study the evolution of the state η_t^i in order to pin down terms

[2]As we will see in section 6.4, this is indeed the case.

[3]We will see later v_i only depends on state variable η_t^i, and is twice differentiable in η_t^i. Besides, state variable K_t is easy to handle due to scale invariance.

like c_t^i/n_t^i and ι_t^i.

Price of risk ς_t^i. The value function (6.2) implies

$$
\begin{aligned}
V^i(n_t^i;\boldsymbol{\eta_t},K_t) &= \frac{u\left(\omega^i(\boldsymbol{\eta_t},K_t)n_t^i\right)}{\rho^i} \\
\Rightarrow \frac{\partial V^i(n_t^i;\boldsymbol{\eta_t},K_t)}{\partial n_t^i} &= \frac{\left(\omega^i(\boldsymbol{\eta_t},K_t)\right)^{1-\gamma}}{\rho^i}(n^i)^{-\gamma} = \underbrace{\frac{(\omega_t^i n_t^i/K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=}\left(\frac{K_t}{n_t^i}\right)^{1-\gamma}(n_t^i)^{-\gamma}
\end{aligned}
$$

Applying the envelop condition $\frac{\partial V_t}{\partial n_t} = u'(c_t)$,

$$
\frac{\partial V_t^i}{\partial n_t^i} = v_t^i\left(\frac{K_t}{n_t^i}\right)^{1-\gamma}(n_t^i)^{-\gamma} = (c_t^i)^{-\gamma} = \frac{\partial u(c_t^i)}{\partial c_t^i}.
$$

In equilibrium $N_t^i = n_t^i$ and $C_t^i = c_t^i$, plugging in $N_t^i = \eta_t^i q_t K_t$, the condition ends up becoming

$$
\frac{C_t^i}{K_t} = \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.9}
$$

Applying Itô's quotient rule and comparing the volatility terms, we have[4]

$$
\sigma_t^{c^i} - \sigma = \frac{1}{\gamma}\left(-\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q\right).
$$

The prices of risk are then

$$
\varsigma_t^i = \gamma\sigma_t^{c^i} = -\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q + \gamma\sigma. \tag{6.10}
$$

[4]Note that $\sigma_t^K = \sigma$ because $K_t = \sum_i \kappa^i k_t^i$ and $\sigma_t^{k^i} = \sigma,\ \forall i$.

Consumption propensity C_t^i/N_t^i. Note that we can express C_t^i/N_t^i in terms of η_t, q_t and v_t^i. Plug in $K_t = N_t^i/\eta_t^i q_t$ and rewrite (6.9) as

$$\frac{C_t^i}{N_t^i} = \frac{c_t^i}{n_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma-1}}{(v_t^i)^{1/\gamma}}. \tag{6.11}$$

On the aggregate level,

$$\frac{C_t}{N_t} = \sum_i \eta_t^i \frac{C_t^i}{N_t^i} = \frac{1}{q_t}\sum_i \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.12}$$

6.1.1 Value function iteration

To apply the finite difference method, we *postulate* that $v_t^i = v^i(\eta_t^e, t)$ (Note in two-sector model we only use η^e as state variable). By Itô's formula, it follows

$$\frac{\mathrm{d}v_t^i}{v_t^i} = \frac{\partial_t v_t^i + (\eta_t^e \mu_t^{\eta^e})\partial_\eta v_t^i + \frac{1}{2}(\eta_t^e\sigma_t^{\eta^e})^2\partial_{\eta\eta}v_t^i}{v_t^i}\mathrm{d}t + \frac{(\eta_t^e\sigma_t^{\eta^e})\partial_\eta v_t^i}{v_t^i}\mathrm{d}Z_t.$$

Comparing with the BSDE (6.8), we get the growth equation

$$\begin{aligned}\partial_t v_t^i &+ \left[\eta_t^e \mu_t^{\eta^e}\right]\partial_\eta v_t^i + \left[\frac{1}{2}\left(\eta_t^e\sigma_t^{\eta^e}\right)^2\right]\partial_{\eta\eta}v_t^i \\ &= \left\{\rho^i - \frac{c_t^i}{n_t^i} - (1-\gamma)\left[(\Phi(\iota_t^i)-\delta) - \frac{1}{2}\gamma\sigma^2 + \sigma\left(\eta_t^e\sigma_t^{\eta^e}\right)\frac{\partial_\eta v_t^i}{v_t^i}\right]\right\}v_t^i, \end{aligned}\tag{6.13}$$

where

$$\mu_t^{\eta^i} = (\varsigma_t^i - \sigma - \sigma_t^q)\sigma_t^{\eta^i} - \sum_{i'}\eta_t^{i'}(\varsigma_t^{i'} - \sigma - \sigma_t^q)\sigma_t^{\eta^{i'}} - \left(\frac{C_t^i}{N_t^i} - \frac{C_t}{N_t}\right) - \rho_d^i\zeta^{\neg i} + \rho_d^{\neg i}\zeta^i\frac{N_t^{\neg i}}{N_t^i}, \tag{6.14}$$

$$\sigma_t^{\eta^i} = \frac{\chi_t^i - \eta_t^i}{\eta_t^i}(\sigma + \sigma_t^q). \tag{6.15}$$

In order to solve this PDE, we need to know all the terms in red. Luckily, we already have all the building blocks from the previous sections:

- Tobin's q gives us the investment rate ι_t^i:

$$\iota_t^i = \frac{1}{\phi}(q_t - 1). \tag{6.16}$$

- The price of risk ς_t^i is given by (6.10):

$$\varsigma_t^i = -\sigma_t^{v^i} + \sigma_t^{\eta^i} + \sigma_t^q + \gamma\sigma \qquad \text{where} \qquad \sigma_t^{v^i} = \frac{(\eta_t^e \sigma_t^{\eta^e})\partial_\eta v_t^i}{v_t^i}. \tag{6.17}$$

- The amplification equation yields σ_t^q:

$$\sigma_t^q = \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q). \tag{6.18}$$

- The first-order conditions to the planner's problems gives χ_t^i, κ_t^i. It can be shown that the FOCs (4.8)-(4.9) are equivalent to the following[5]

$$\min\left\{\frac{a^e - a^h}{q_t} - \alpha\left(-\frac{\partial_\eta v_t^e}{v_t^e} + \frac{\partial_\eta v_t^h}{v_t^h} + \frac{1}{(1-\eta_t^e)\eta_t^e}\right)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)^2, 1 - \kappa_t^e\right\} = 0, \tag{6.19}$$

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}. \tag{6.20}$$

- In (6.11) and (6.12), we have obtained the consumption ratios $c_t^i/n_t^i, C_t/N_t$ from the optimal consumption condition:

$$\frac{C_t^i}{N_t^i} = \frac{c_t^i}{n_t^i} = \frac{(\eta_t^i q_t)^{1/\gamma - 1}}{(v_t^i)^{1/\gamma}}, \qquad \frac{C_t}{N_t} = \sum_i \eta_t^i \frac{C_t^i}{N_t^i} = \frac{1}{q_t}\sum_i \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.21}$$

- Finally, goods market clearing jointly constraints q_t and κ_t^i:

$$\sum_i \kappa_t^i a^i - \iota_t = \sum_i \frac{C_t^i}{K_t} = \sum_i \left(\frac{\eta_t^i q_t}{v_t^i}\right)^{1/\gamma}. \tag{6.22}$$

[5] Proof will be added soon. For now, please see relevant parts of the lecture slides.

In the following algorithm, we first guess two functions $v^e(\eta_t^e, T), v^h(\eta_t^e, T)$ and use them as *terminal conditions*. We then solve the PDE (6.13) *backwards* on a discretized time grid. In each step (time t), we solve for time-t equilibrium quantities as functions of η_t using the Inner Loop procedure, introduced above.

More specifically, the algorithm goes as following

1. Start by guessing two functions $v^e(\eta^e, T), v^h(\eta^e, T)$ over a grid of η^e
2. Loop over $t = \{T, T - \Delta t, \cdots, 0\}$ until changes in v^e-functions are small. In each step, do the following:
   (a) Compute $\partial_\eta v_t^i$ by first-order differences
   (b) Perform the Inner Loop procedure, using appropriate conditions for capital and risk allocation (6.19 and 6.20 instead of 4.15 and 4.16)
   (c) compute $\mu_t^{\eta^e}(\eta_t^e), \sigma_t^{\eta^e}(\eta_t^e), \mu_t^{v^i}(\eta_t^e), \sigma_t^{v^i}(\eta_t^e)$ using equations (6.14) and (6.15)
   (d) make time-step – back in time – and update the $v_t^i(t, \cdot)$ functions to $v_t^i(t - \Delta t, \cdot)$

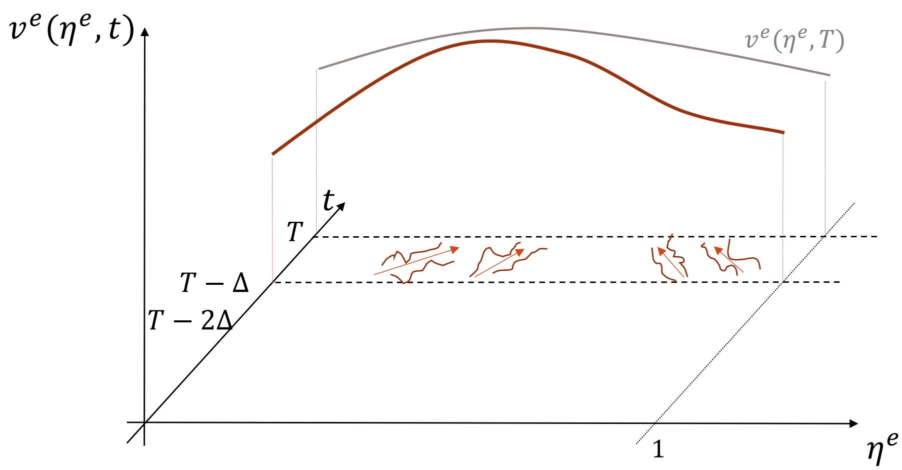


Figure 6.1: Visualization of the time step

6.2 Implementation in MATLAB

Following is main code executing the algorithm under the assumption of CRRA utility.

```
%% Parameters and grid
a_e = 0.11; a_h = 0.03;        % production rates
rho_0 = 0.04;                  % time preferences
rho_e_d = 0.01;rho_h_d = 0.01;% death rates
rho_e = rho_0 + rho_e_d;       % expert's discount rates
rho_h = rho_0 + rho_h_d;       % household's discount rates
zeta = 0.05;                   % probability of becoming an expert
delta = 0.05; sigma = 0.1;     % decay rate/volatility
phi = 10; alpha = 0.5;         % adjustment cost/equity constraint
dt = 100; tol = 1e-6;          % time step/convergence criterion
gamma = 2;                     % CRRA parameter

fsolveOptions = optimoptions('fsolve','Display','off','FunctionTolerance',1e-8);

N = 501;                            % grid size
eta = linspace(0.0001,0.999,N)'; % grid for \eta

%% Solution
deta = eta(2)-eta(1); % grid step

% Initial guess for the value functions
v_e = a_e^(-gamma)*eta.^(1-gamma);
v_h = a_e^(-gamma)*(1-eta).^(1-gamma);

vp_e = [v_e(2) - v_e(1); v_e(2:N) - v_e(1:N-1)]./deta;   % \partial{v_e}
vp_h = [v_h(2) - v_h(1); v_h(2:N) - v_h(1:N-1)]./deta;   % \partial{v_h}

ev_e = (eta./v_e).^(1/gamma);        % auxiliary variable for goods-market clearing
ev_h = ((1 - eta)./v_h).^(1/gamma); % auxiliary variable for goods-market clearing

%% Iteration

for i = 1:10000
    ev = ev_e + ev_h; % auxiliary variable for goods-market clearing
    vpv = -vp_e./v_e + vp_h./v_h + 1./(eta.*(1 - eta)); % auxiliary variable for kappa FOC

    % Solve for q(0), approximating value functions at the left boundary by
    % the values at the first grid point. Initial guess: q(0) under
    % log-utility
    q0 = fsolve(@(x) a_h - (x-1)/phi - x.^(1/gamma)*ev(1), (1 + a_h*phi)/(1 + rho_h*phi), fsolveOptions);

    % Inner loop
    [Q, SSQ, Kappa, Chi, Iota] = inner_loop_crra(eta, q0, ev, vpv, a_e, a_h, sigma, phi, alpha, gamma);
```

```

    S = (Chi - eta).*SSQ; % \sigma_{\eta^e} -- arithmetic volatility of \eta^e
    Phi = log(Q)/phi;     % Investment (net of costs)

    Sg_e = S./eta;      % \sigma^{\eta^e} -- geometric volatility of \eta^e
    Sg_h = -S./(1-eta); % \sigma^{\eta^h} -- geometric volatility of \eta^h

    vp_e = [v_e(2) - v_e(1); v_e(2:N) - v_e(1:N-1)]./deta;   % \partial{v_e}
    vp_h = [v_h(2) - v_h(1); v_h(2:N) - v_h(1:N-1)]./deta;   % \partial{v_h}

    ev_e = (eta./v_e).^(1/gamma);       % auxiliary variable for goods-market clearing
    ev_h = ((1 - eta)./v_h).^(1/gamma); % auxiliary variable for goods-market clearing

    Sv_e = S.*vp_e./v_e; % \sigma^{v^e}
    Sv_h = S.*vp_h./v_h; % \sigma^{v^h}

    VarS_e = -Sv_e + Sg_e + SSQ - (1-gamma)*sigma; % \varsigma^e -- experts' price of risk
    VarS_h = -Sv_h + Sg_h + SSQ - (1-gamma)*sigma; % \varsigma^h -- households' price of risk

    CN_e = ev_e.*Q.^(1/gamma - 1)./eta;       % experts' consumption-to-networth ratio
    CN_h = ev_h.*Q.^(1/gamma - 1)./(1 - eta); % households' consumption-to-networth ratio

    MU = eta .* (1-eta) .* ((VarS_e - SSQ).*(Sg_e + SSQ) -  (VarS_h - SSQ).*(Sg_h + SSQ) - (CN_e - CN_h) + (rho_h_d.*zeta.*(1-eta) - rho_e_d.*(1-zeta).*eta) ./ (eta.*(1-eta))); % \mu_{\eta^e} -- arithmetic drift of \eta^e

    S([1, N]) = 0;          % ensures volatiliuty is zero at the boundaries
    MU(1) = max(MU(1), 0);  % ensures drift at the left boundary is non-negative
    MU(N) = min(MU(N), 0);  % ensures drift at the right boundary is non-positive

    u_e = (CN_e + (1 - gamma)*(Phi - delta - gamma*sigma^2/2 + Sv_e*sigma)).*v_e; % flow term in experts' HJB
    u_h = (CN_h + (1 - gamma)*(Phi - delta - gamma*sigma^2/2 + Sv_h*sigma)).*v_h; % flow term in households' HJB

    v_e1 = update_v(v_e, eta, rho_e, u_e, MU, S, dt); % update experts' value function
    v_h1 = update_v(v_h, eta, rho_h, u_h, MU, S, dt); % update households' value function

    d = max(abs(v_e1 - v_e) + abs(v_h1 - v_h))/dt; % convergence
    if d <= tol
        break
    end

    v_e = v_e1;
    v_h = v_h1;
end
```

The inner loop is implemented in the function `inner_loop_crra.m`, which is identical to the one under log-utility, except for a couple of lines in the construction of the $F(\cdot)$ function for the Newton's method and its Jacobian J.

```
function [Q, SSQ, Kappa, Chi, Iota] = inner_loop_crra(eta, q0, ev, vpv, a_e, a_h, sigma, phi, alpha, gamma)

N = length(eta);
deta = [eta(1); diff(eta)]; % imposes the correct grid step for numerical derivative at \eta^e = 0

% variables
Q = ones(N,1);    % price of capital q
SSQ = zeros(N,1);   % \sigma + \sigma^q
Kappa = zeros(N,1); % capital fraction of experts \kappa

% Initiate the loop
kappa = 0; q_old = q0; q = q0; ssq = sigma;

% Iterate over eta
% At each step apply Newton's method to F(z) = 0 where z = [q, kappa, ssq]'
% Use chi = alpha*kappa
for i = 1:N
    % Compute F(z_{n-1})
    F = [kappa*(a_e - a_h) + a_h - (q-1)/phi - q^(1/gamma)*ev(i); % Compare with log-utility: C/N is no longer \rho
         ssq*(q - (q - q_old)/deta(i) * (alpha*kappa - eta(i))) - sigma*q;
         a_e - a_h -  q*alpha*(alpha*kappa - eta(i))*ssq^2*vpv(i)];  % Compare with log-utility: price of risk is no longer \sigma^n

    % Construct Jacobian J^{n-1}
    J = zeros(3,3);
    J(1,:) = [-1/phi - q^(1/gamma - 1)/gamma * ev(i), a_e - a_h, 0];
    J(2,:) = [ssq*(1 - (alpha*kappa - eta(i))/deta(i)) - sigma, ...
          -ssq*(q-q_old)/deta(i)*alpha, q - (q-q_old)/deta(i)*(alpha*kappa - eta(i))];
    J(3,:) = [-alpha*(alpha*kappa - eta(i))*ssq^2*vpv(i), ...
          -q*alpha^2*ssq^2*vpv(i), -2*q*alpha*(alpha*kappa - eta(i))*ssq*vpv(i)];

    % Iterate, obtain z_{n}
    z = [q, kappa, ssq]' - J\F;

    % If the new kappa is larger than 1, break
    if z(2) >= 1
        break;
    end

    % Update variables
    q = z(1); kappa = z(2); ssq = z(3);

    % save results
    Q(i) = q; Kappa(i) = kappa; SSQ(i) = ssq;
    q_old = q;
end
```

```

% Set kappa = 1, use chi = max(alpha, eta) and compute the rest
n1 = i;
for i = n1:N
    F = a_e - (q-1)/phi - q^(1/gamma)*ev(i); % Compare with log-utility: C/N is no longer \rho
    J = -1/phi - q^(1/gamma - 1)/gamma * ev(i);
    q = q - F/J;
    qp = (q - q_old)/deta(i);

    Q(i) =  q; Kappa(i) = 1;
    SSQ(i) = sigma/(1 - (max(alpha, eta(i)) - eta(i))*qp/q);
    q_old = q;
end

% Compute chi, iota
Chi  = max(alpha*Kappa, eta);
Iota = (Q - 1)/phi;
```

The time step is implemented in the function `update_v.m`:

```
function [v] = update_v(v, x, rho, u, mu, sig, dt)

N = length(x); % Grid size

% Constrict the M matrix
M = buildM(x, mu, sig);

B = (1 + dt*rho)*speye(N) - dt*M;
v = B\(u*dt + v); % update v
```

with matrix M constructed in function `biuldM.m`:

```
function [M] = buildM(x, mu, sig)
% Construct the M matrix using three vectors (dM, dD, dU), corresponding to
% the main diagonal, the diagonal below the main one, the diagonal above
% the main one:
% M = [dM(1) dU(1)    0     0  ....      0       0      0
%      dD(2) dM(2) dU(2)    0  ....      0       0      0
%         0  dD(3) dM(3) dU(3) ....      0       0      0
%      ....  ....  ....  ....  ....   ....    ....   ....
%         0     0     0     0  .... dD(N-1) dM(N-1) dU(N-1)
%         0     0     0     0  ....      0    dD(N)   dM(N)]
% ==========================================================================
% Input:
% x - grid (N-by-1), equally spaced
% mu - drift term (N-by-1)
% sig - volatility term (N-by-1)

N = length(x); % Grid size
dx = x(2)-x(1); % Grid step
dx2 = dx^2; % Grid step squared

```

```
% Constrict the diagonals
dD = -min(mu, 0)/dx + sig.^2/(2*dx2);
dM = -max(mu, 0)/dx + min(mu, 0)/dx - sig.^2/dx2;
dU = max(mu, 0)/dx + sig.^2/(2*dx2);

% Construct the M matrix
M = spdiags([dD dM dU],[1 0 -1],N,N)';
```

To ensure numerical stability, it is important to:

- Use implicit method in step 2.(b)
- Use an upwind scheme when taking derivatives in step (a):

$$\partial_\eta v^i_t(t,\eta^i(n)) = \begin{cases} \dfrac{v^i(t,\eta^i(n+1)) - v^i(t,\eta^i(n))}{\eta^i(n+1)-\eta^i(n)} & \text{if } \mu^{\eta^i}_t \eta^i_t > 0 \\ \dfrac{v^i(t,\eta^i(n)) - v^i(t,\eta^i(n-1))}{\eta^i(n)-\eta^i(n-1)} & \text{if } \mu^{\eta^i}_t \eta^i_t < 0 \end{cases} \tag{6.23}$$

A good initial guess is usually crucial to the success of a numerical procedure. Here are some common ways of choosing initial guesses for v^i:

- Take an arbitrary constant, e.g. a vector of ones. It is the easiest way, but doesn't always work / may take a long time to converge.
- Take a specific constant, namely the value at the boundary steady state ($\eta^e = 0$ or $\eta^e = 1$) where only one type exists (if that is a valid equilibrium) – this is typically also very easy.
- Assume there are no financial contracts and compute for each η^e the autarky value of the agent types, when the initial wealth distribution is described by η^e.
- Along the same lines, assume complete markets and compute first-best utility as a function of η^e (this certainly bounds utility from above).
- If the log utility model is simple to solve, solve it first. Use the consumption path of agents in that model, but compute the implied CRRA utility.

- If you have solved the model for different parameters that are "close", use that solution as an initial guess.

6.3 Epstein-Zin Preferences

The Epstein-Zin utility function allows to specify the elasticity of intertemporal substitution (EIS) and the degree of relative risk aversion (RRA) separately. It is defined in a recursive way:

$$\begin{aligned}
U_t &= \mathbb{E}_t\left[\int_t^\infty f(c_s, U_s)ds\right] \\
f(c,U) &= \frac{1-\gamma}{1-\psi^{-1}}\rho U\left(\left(\frac{c}{((1-\gamma)\rho U)^{1/(1-\gamma)}}\right)^{1-\psi^{-1}} - 1\right)
\end{aligned}$$

with EIS ψ and RRA γ. Setting $\gamma = \psi^{-1}$ recovers the CRRA utility function $U_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho t}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right]$. The model solution follows the same steps as under CRRA utility, with some differences that we highlight below.

1. Naturally, consumption-to-wealth ratio (6.2) is now given by:

$$\frac{c_t^i}{n_t^i} = (\rho^i)^\psi(\omega_t^i)^{1-\psi}$$

2. Stochastic discount factor now satisfies (compare with (6.5)):

$$e^{-\int_0^t \frac{\partial f^i(c_s,V_s)}{\partial U}ds}\xi_t^i n_t^i = (1-\gamma)V_t^i$$

   and hence, analogously to (6.6):

$$\frac{dV_t^i}{V_t^i} = \left(-\frac{\partial f^i(c_s,V_s)}{\partial U} - \frac{c_t^i}{n_t^i}\right)dt + \text{martingale}$$

   with $\frac{\partial f^i(c,U)}{\partial U} = \frac{\rho^i}{1-\psi^{-1}}\left[(\psi^{-1}-\gamma)\left(\frac{c}{((1-\gamma)\rho^i U)^{1/(1-\gamma)}}\right)^{1-\psi^{-1}} - (1-\gamma)\right]$. This affects the BSDE for v_t^i (6.8) by changing its drift $\mu_t^{v^i}$.

Finally, a particularly useful special case is when IES = 1 (as under log-utility), which implies:

$$
\begin{aligned}
\frac{c_t^i}{n_t^i} &= \rho^i \\
f(c,U) &= \rho U\left[(1-\gamma)\log c - \log((1-\gamma)\rho U)\right] \\
\frac{\partial f(c,U)}{\partial U} &= \rho\left[(1-\gamma)\log c - \log((1-\gamma)\rho U) - 1\right]
\end{aligned}
$$

6.4 Numerical Results

In this section, we demonstrate the solutions generated by the code in Section 6.2 and discuss their implications. We use CRRA utility and set the baseline parameters are as follows.

$\rho_0^{e,h}$	$\rho_d^{e,h}$	ζ^e	a^e	a^h	δ	σ	α	γ	ϕ
0.04	0.01	0.05	0.11	0.03	0.05	0.10	0.50	2	10

Figure 6.2 illustrates the equilibrium with baseline parameter values. Note that $q(\eta_t^e)$ indeed has a kink, which marks the boundry between the cirsis region near $\eta^e = 0$ and the normal region near $\eta^e = 1$. In the crisis region, $\kappa^e < 1$, and households hold some capital, while in the normal region, experts hold all capital in the economy. Unlike the model under log utility in Chapter 4, the price of capital under the CRRA utility is not flat in the normal region. This difference is not due to risk aversion, but rather the elasticity of intertemporal substitution (EIS) different from 1. The consumption to wealth ratio is dependent on investment opportunities. One can see the role of the EIS if one generalizes the CRRA utility functions to EZ with EIS as we outline in Section 6.3.

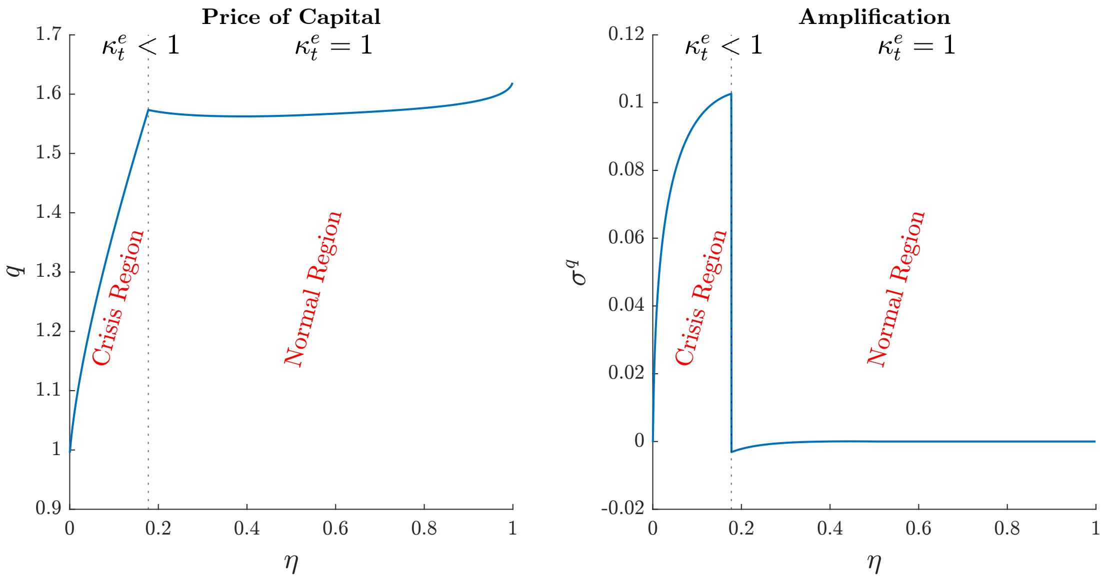


Figure 6.2: Equilibrium for the baseline set of parameters (η should read η^e)

6.5 Exercises

6.5.1 He and Krishnamurthy (2013)

The goal of this problem is to characterize equilibria in the model of He-Krishnamurthy ("Intermediary Asset Pricing") and to use the iterative method to compute equilibria. There are two agent types: experts and households. Households have log utility, while experts have CRRA utility with relative risk aversion γ, and both with discount rate ρ. New households are born continuously, and the newborn receive labor income at rate lK_t.

Aggregate capital follows the law of motion

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Capital produces dividend of aK_t. The price of capital per unit is denoted by q_t and follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t.$$

Only experts can hold capital, and they can finance capital by borrowing through risk-free debt and by issuing equity to households, but they must retain fraction of at least $\underline{\chi} = 1/(1+m)$ of risk.

(a) Write down the expression for dr_t^K for the return on capital.

Experts make optimal consumption and portfolio decisions: they choose how much to borrow and how much outside equity to issue (up to fraction $1 - \underline{\chi}$) to buy capital. Denote the required risk premium of experts by ς_t^e and recall that $\varsigma_t^e = -\gamma \sigma_t^{Ce}$, where σ_t^{Ce} is the volatility of aggregate consumption of experts. Denote the value function of a representative expert by

$$v_t^e \frac{K_t^{1-\gamma}}{1-\gamma}.$$

(b) Write down the law of motion of aggregate net worth of experts N_t^e as a function of the risk-free rate r_t, the experts' equity share $\underline{\chi}_t^e$, the experts' net worth share η_t^e, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' risk premium ς_t^e and process v_t^e. To write the law of motion of N_t^e, you need to express the experts' consumption rate C_t^e / N_t^e as a function of η_t^e, q_t and v_t^e.

(c) He and Krishnamurthy assume that inside and outside equity of experts earn the same returns. Thus, the experts' equity held by households earns the risk premium of ς_t^e, even though households' required risk premium is higher. Under this assumption, write down the law of motion of world wealth $q_t K_t$, as a function of the risk-free rate r_t, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' risk premium ς_t^e and output parameter a.

(d) From your answers to parts (b) and (c), derive the law of motion of the experts' wealth share $\eta_t = N_t^e / (q_t K_t)$.

(e) Write down the market-clearing condition for output. Hint: Recall that total world output is $(a + l)K_t$, including dividend and labor income of newborn households.

Next, you should determine the size of the "constrained region" where $\chi_t^e = \underline{\chi}$ and the size of the unconstrained region where $\chi_t^e > \underline{\chi}$. To do that, you should use the following assumptions of He and Krishnamurthy. Assume that fraction λ of households (i.e. the net worth share of these households is $(1 - \eta_t^e)\lambda$) are "debt" households who can only hold the risk-free asset. Fraction $1 - \lambda$ are "equity" households who can hold outside equity of experts and the risk-free asset. He and Krishnamurthy furthermore assume that equity households cannot use leverage, i.e. the risk of their net worth can be at most equal to the risk of experts' net worth (who hold their own inside equity). Assume (you can verify this later), that it is this constraint that determines the amount of equity that experts can issue.

(f) Derive the value of χ_t^e as a function of η_t^e implied by the constraint that equity households cannot use leverage.

The goal of the next questions is to formulate a procedure to compute equilibria using Matlab, using the value function iteration in section 6.2. You should use Matlab function `payoff_policy_growth.m` to perform the "time step" of the iterative procedure.

(g) Formulate a procedure for the static step. That is, suppose you are given function $v^e(t, \eta^e)$ for all η^e at time t. Find the price of capital $q(t, \eta^e)$ for all η^e at time t. Then, given this function, derive the law of motion of η^e. Provide an expression for $\mu_t^{v^e}$.

(h) Formulate a procedure for the time step. That is, for the function

```
F = payoff_policy_growth(X, R, MU, S, G, V, lambda0),
```

what values of `X, R, MU, S, G, V, lambda0` should you use?

(i) Program the iterative procedure using the terminal condition $v^e(T, \eta) = a^{-\gamma}(\eta^e)^{1-\gamma}$. Use $N = 1000$. Compute an example for the parameters of He and Krishnamurthy, $\rho = 0.04$, $g = 0.02$, $m = 4$, $a = 1$, $l = 1.84$, $\sigma = 0.09$, $\gamma = 2$ and $\lambda = 0.6$. (See Table 2 of HK - for these parameters you should be able to get convergence by setting lambda0 for payoff_policy_growth aggressively to 0.9).

Plot, as a function of η^e, the price of capital q, the risk-free rate r_t, the drift and volatility of $\eta_t \eta$ (i.e. $\sigma_t^{\eta^e} \eta^e$ and $\mu_t^{\eta^e} \eta^e$), the fraction of equity χ_t^e held by experts and the experts' consumption rate C_t^e / N_t^e.

(j) Replicate Figure 2 from He and Krishnamurthy, where the vertical axis displays the risk premium for capital, i.e. $\varsigma_t^e(\sigma + \sigma_t^q)$.

Bibliography

He, Zhiguo and Arvind Krishnamurthy, "Intermediary asset pricing," *American Economic Review*, 2013, *103* (2), 732–70.

Tourin, Agnes, "An Introduction to Finite Difference Methods for PDEs in Finance," in Nizar Touzi, ed., *Optimal Stochastic Target problems, and Backward SDE, Fields Institute Monographs*, Springer, 2011, pp. 201–212.

Chapter 7

A Model with Jumps

In Chapter 4, we studied a benchmark model with financial frictions and endogenous risk dynamics. Despite having highly non-linear dynamics, the model is driven by Brownian shocks, and the paths of all variables are *continuous*. Idiosyncratic death or type switching jumps (with log-utility) do not lead to jumps in endogenous prices and state variables.

Nevertheless, there is a long tradition of modeling both unanticipated and anticipated jumps in macro-finance models. Some notable examples include models of bank runs (Diamond and Dybvig, 1983), liquidity spirals (Brunnermeier and Pedersen, 2009), sudden stops (Calvo, 1998; Mendoza, 2010), currency attacks (Obstfeld, 1996; Morris and Shin, 1998), twin crises (Kaminsky and Reinhart, 1999), and the loss of safe asset status.

In this chapter, we illustrate how to incorporate self-fulfilling jumps into the model outlined in Chapter 4, based on a simplified version of Mendo (2020).

7.1 Jump Processes

Previously, we focused on Itô processes in the form

$$dX_t = \mu_t^X X_t dt + \sigma_t^X X_t dZ_t,$$

where the Brownian "shocks" $\mathrm{d}Z_t$ are i.i.d. and small, such that time paths are continuous. For non-normal shocks within dt one needs discontinuities. In this chapter, we allow for discontinuities by considering a more general class of processes with i.i.d. increments: Levy processes.

The Levy-Itô decomposition states that any Levy process L_t can be additively decomposed into three independent components: a linear time drift, a scaled Brownian motion and a Levy jump process, that is

$$\mathrm{d}L_t = a\mathrm{d}t + b\mathrm{d}Z_t + \mathrm{d}J_t,$$

where a, b are constants, $\mathrm{d}Z_t$ is a Brownian motion and $\mathrm{d}J_t$ is a Levy jump process. Processes driven by Levy-noise therefore look formally like Itô processes with an additional jump component

$$\mathrm{d}X_t = \mu_t^X X_t \mathrm{d}t + \sigma_t^X X_t \mathrm{d}Z_t + j_t^X X_{t-} \mathrm{d}J_t.$$

There is a fairly rich class of Levy jump processes $\mathrm{d}J_t$. Here we restrict our attention to Poisson processes.[1] Consider a Poisson process with intensity (or arrival rate) $\lambda > 0$: J_t takes only values in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and increments $J_{t+\Delta t} - J_t$ are Poisson distributed with parameter $\lambda \Delta t$. The following are some important properties of this Poisson process:

- J_t is weakly increasing. That is, J_t is locally constant or has a jump of size 1.
- Conditional on $J_t = n$, the random time to the next jump, $\tau := \inf\{s \geq 0 \mid J_{t+s} > n\}$, is exponentially distributed with parameter λ (i.e. the expected time to the next jump is constant and given by $1/\lambda$).
- The stochastic integral with respect to a Poisson process simply sums the values

[1] Note: this is not as restrictive as it may seem: general Levy jump processes can be written as integral with respect to Poisson random measures, a generalization of sums of integrals with respect to Poisson processes.

of the integrand at the jump times:

$$\int_0^T a_t \mathrm{d}J_t = \sum_{n=1}^{J_T} a_{\tau_n}$$

where τ_n is the time at which J_t jumps from $n-1$ to n.

To capture time-varying macroeconomic dynamics, we will allow for a slightly more general process, sometimes called a Cox process, where λ does not need to be a constant, but can be time- and state-dependent (λ_t).

It is important to note that, while a Brownian motion $\mathrm{d}Z_t$ is a martingale, a jump process J_t is not — it is expected to drift up. To get a martingale, we have to "compensate" the jump process by its intensity. In other words, $J_t - \int_0^t \lambda_s \mathrm{d}s$ is a martingale.[2]

The following Itô formulae holds for jump diffusions driven by Brownian and Poisson noise.

Consider geometric jump diffusions X_t, Y_t

$$\frac{\mathrm{d}X_t}{X_{t-}} = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + j_t^X \mathrm{d}J_t, \qquad \frac{\mathrm{d}Y_t}{Y_{t-}} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t + j_t^Y \mathrm{d}J_t,$$

where $\mathrm{d}Z_t$ is a standard Brownian motion and $\mathrm{d}J_t$ is a Poisson process. The following results hold.

Itô's lemma:

$$\begin{aligned}\mathrm{d}f(X_t) = &\left[f'(X_t)(\mu_t^X X_t) + \frac{1}{2} f''(X_t)(\sigma_t^X X_t)^2\right] \mathrm{d}t + \\ &f'(X_t)(\sigma_t^X X_t)\mathrm{d}Z_t + (f((1+j_t^X)X_{t-}) - f(X_{t-}))\mathrm{d}J_t\end{aligned}$$

Itô's power rule:

[2] More generally, if $X_t = \int_0^t a_s \mathrm{d}J_s$ (and a is "predictable", i.e. a_t uses information only up to right before time t, but does not contain information about potential jumps at time t), then $X_t - \int_0^t a_s \lambda_s ds$ is a martingale.

$$\frac{\mathrm{d}(X_t^\gamma)}{X_{t-}^\gamma} = (\gamma\mu_t^X + \gamma(\gamma-1)(\sigma_t^X)^2)\mathrm{d}t + \gamma\sigma_t^X \mathrm{d}Z_t + ((1+j_t^X)^\gamma - 1)\mathrm{d}J_t.$$

Itô's product rule:

$$\frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X\sigma_t^Y)\mathrm{d}t + (\sigma_t^X + \sigma_t^Y)\mathrm{d}Z_t + (j_t^X + j_t^Y + j_t^X j_t^Y)\mathrm{d}J_t.$$

Itô's quotient rule:

$$\frac{\mathrm{d}(X_t/Y_t)}{X_t/Y_t} = \left[\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)\right]\mathrm{d}t + (\sigma_t^X - \sigma_t^Y)\mathrm{d}Z_t + \frac{j_t^X - j_t^Y}{1+j_t^Y}\mathrm{d}J_t.$$

Notice that these equations are the same as our earlier rules for geometric Itô processes, but with new terms for the jump process. The new terms in the power, product and quotient rules can be expressed more simply as:

$$1 + j_t^{(X^\gamma)} = (1+j_t^X)^\gamma$$

$$1 + j_t^{XY} = (1+j_t^X)(1+j_t^Y)$$

$$1 + j_t^{X/Y} = \frac{1+j_t^X}{1+j_t^Y}$$

7.2 Model Setup

The model setup follows the structure outlined in Section 4.1 but with CRRA utility and without agents' death. The innovation comes from our postulated price process, which will now include a jump term. That is, the model has the same primitives as before, but we now allow for *self-fulfilling / sunspot* jumps.

Environment. Like before, there is no labor and the economy is populated by experts and households, $i \in \{e, h\}$. However, now households can also produce consumption goods but with an inferior technology. Agents can issue both equity and debt, but subject to certain financial frictions.

Experts. Experts have a CRS technology $y_t^e = a^e k_t^e$. Denote their consumption and investment rate by c_t^e, ι_t^e. Experts' capital stock evolves according to

$$\frac{\mathrm{d}k_t^e}{k_t^e} = (\Phi(\iota_t^e) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

Still, we have only aggregate risk in the environment. Experts have a CRRA utility function and they each maximize

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \frac{(c_t^e)^{1-\gamma}}{1-\gamma} \mathrm{d}t \right].$$

Households. Households also have a CRS technology $y_t^h = a^h k_t^h$ with $a^h \leq a^e$. Households' capital accumulation process is

$$\frac{\mathrm{d}k_t^h}{k_t^h} = (\Phi(\iota_t^h) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t.$$

We let households hold capital to capture fire-sales. Households are more patient than the experts, i.e., $\rho^h \leq \rho^e$. As we have discussed in section 3, assuming that households are more patient than the experts is a modeling trick to ensure that the experts do not hold all the capital in the long run. The households maximize

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \frac{(c_t^h)^{1-\gamma}}{1-\gamma} \mathrm{d}t \right].$$

Financial Friction. The financial friction is due to incomplete markets. Although experts are allowed to issue equity, they must hold at least α fraction of their risk. The balance sheets of the two sectors are as following:

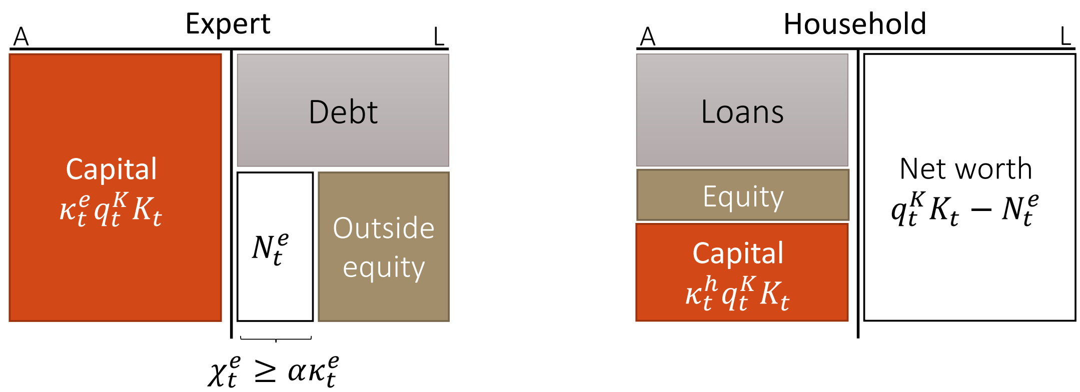


Figure 7.1: Balance sheets of experts and households

The skin-in-the-game constraint can be expressed as $\chi_t^e \geq \alpha\kappa_t^e$, where χ_t^e is the fraction of risk held by experts and κ_t^e is the fraction of capital held by experts.

Unanticipated Run on Experts. Now we investigate the risk of a sudden and unanticipated funding withdrawal, which corresponds to a *bank run* or *sudden stop* on experts. We ask: Can an unanticipated withdrawal of *all* external funding to experts be self-fulfilling?

An *unanticipated crash* occurs if, at some instant, the experts lose their external financing. In terms of the experts' share of aggregate net worth, this leads to the instantaneous transition:

$$\eta_t^e \longrightarrow 0.$$

Once that jump happens, experts are effectively bankrupt and can no longer operate. In normal times (i.e., *absent a run*), the model solution evolves continuously and does not jump to $\eta_t^e = 0$ unless there is a large, coordinated withdrawal. Thus, the event $\eta_t^e \to 0$ reflects a *self-fulfilling* or *sunspot* run rather than a typical shock.

To see when a price drop is sufficient to wipe out experts, note that the model determines the market price of capital $q\left(\eta_t^e\right)$ based on the experts' net worth share. If the price suddenly falls to $q(0)$ when experts lose funding (a "fire-sale" price), then experts' net worth will be fully destroyed if

$$\Big(q(\eta_t^e) - q(0)\Big)\ \underbrace{\Big(\theta_t^{e,K} + \theta_t^{e,OE}\Big)\,\eta_t^e}_{\chi_t^e}\,K_t \quad \geq \quad \eta_t^e\, q(\eta_t^e)\, K_t.$$

Equivalently, one can write

$$q(\eta_t^e)\left(1\ -\ \frac{\eta_t^e}{\chi^e(\eta_t^e)}\right)\ \geq\ q(0) \quad \Longleftrightarrow \quad q(\eta_t^e)\left(1\ -\ \frac{1}{\theta_t^{e,K}\ +\ \theta_t^{e,OE}}\right)\ \geq\ q(0),$$

where $\chi^e(\eta_t^e)$ is the fraction of total risk borne by the experts. When leverage and exposure (i.e., $\theta_t^{e,K} + \theta_t^{e,OE}$) are high, the price drop from $q(\eta_t^e)$ to $q(0)$ is large enough to eliminate all expert equity.

There are two types of runs:

1. **Funding supply run (households withdraw).** Households/depositors suddenly cut off credit to experts, causing forced liquidation and a crash in q.
2. **Funding demand run (experts fire-sell).** Even if credit remains available, experts may collectively choose to liquidate and repay debt preemptively, driving q down.

Both cases lead to the same outcome: a sudden jump in η_t^e to zero. The distinction lies in who triggers the liquidation (households or experts themselves).

Vulnerability Region The unanticipated crash leads to a *vulnerability region* for the experts' net worth share η_t^e:

- The price of capital $q\left(\eta_t^e\right)$ is *high* (not very low η_t^e).
- The experts hold a *high* fraction of risk/leverage (not very high η_t^e).

In such states, everyone fears that if others withdraw funding, the resulting price drop will annihilate experts' net worth. That belief can become self-fulfilling, causing the run to occur in equilibrium. Post-run, we set $\eta_t^e = 0$ permanently: the expert sector vanishes, and any residual capital is owned by households or newcomers at $q(0)$.

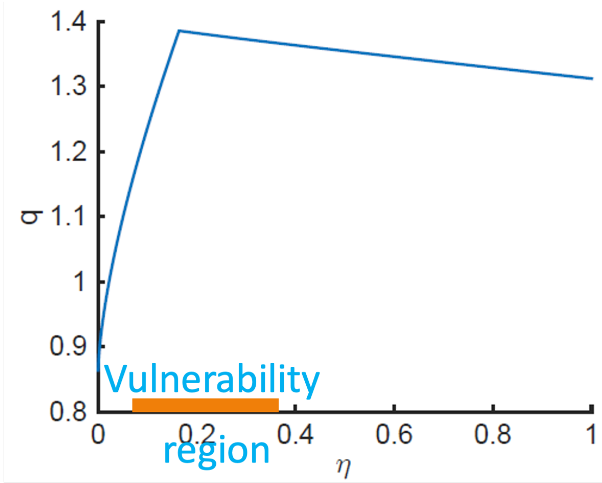


Figure 7.2: Vulnerability region

For numerical solutions, one cannot impose standard boundary conditions of the form $\mu^\eta(0) \geq 0$ or $\sigma^\eta(0) = 0$ because once $\eta^e = 0$, the experts cease to exist and the state is an *absorbing boundary*. The solution algorithm must explicitly allow for the possibility of a jump to zero and the absorbing state thereafter.

Model Advantage. One appealing feature of this framework is that, in the event of a run, the capital price collapses to a single, well-defined value, denoted by $q(0)$. Because experts are forced out of the market at that moment, the equilibrium price afterward is set entirely by the remaining sector (households). This "fire-sale" or "lowest possible" price highlights how a run can effectively *zero out* the experts' net worth, making $\eta_t^e = 0$ an absorbing state.

Modeling Challenges. While the model's ability to pin down a single post-run price $q(0)$ is tractable, it also gives rise to certain challenges. Several of these are discussed in Mendo (2020). We summarize three key issues:

First, experts are wiped out forever. Once the experts' net worth is destroyed by a jump, the baseline model does not let them rebuild. In an OLG structure, all agents face a Poisson death hazard ρ^d; a fraction ζ of newborn agents are designated as experts, so that some expert capacity is gradually reintroduced. Without such an OLG setup, the system simply continues without any expert sector once $\eta^e = 0$.

Second, anticipated runs lead to infinite marginal utility. If a run to $\eta_t^e = 0$ is anticipated, experts effectively foresee a state of infinitely high marginal utility (because losing all wealth is catastrophic). A common device is to impose a transfer, τK, that forces experts into bankruptcy after the run, thereby keeping their utility from blowing up and also ensuring that experts are definitively removed.

Third, bifurcation of experts into defaulters vs. survivors. Large jumps and non-convex default decisions can cause some experts to default while others do not, creating a continuum of outcomes. One modeling trick is to introduce a strong relative-performance penalty so that, if a run occurs, *all* experts default in unison. This prevents partial or selective default, which can greatly simplify the solution.

7.3 Solution Method

7.3.0 Postulate aggregates, price processes and obtain return processes

We introduce jumps by *postulating* that q_t follows

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t + j_t^q \mathrm{d}J_t.$$

As before, we can calculate the return rate to capital for both sectors, $r_t^{i,K}(\iota_t^i)$, using Itô's product rule to calculate the capital gains rate (in the absence of purchases or sales). The process is similar to (4.1), but with a new jump term

$$\mathrm{d}r_t^{i,K}(\iota_t^i) = \left[\underbrace{\frac{a^i - \iota_t^i}{q_t}}_{\text{Dividend yield}} + \underbrace{\Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q}_{\mathbb{E}[\text{Capital gain rate}]} = \frac{\mathrm{d}(q_t k_t)}{q_t k_t} \right] \mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t + j_t^q \mathrm{d}J_t. \tag{4.1'}$$

Similarly, the return on defaultable debt is

$$\mathrm{d}r_t^D = r_t \mathrm{d}t + j_t^{r^D} \mathrm{d}J_t.$$

Note that $j_t^{r^D}$ only reflects the default in debt and not the overall change in debt holdings at the time of a jump. We then *postulate* that the SDF ($\xi_t^i = e^{-\rho t}u'(c_t^i)$) follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t^{F,i}\mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t - \nu_t^i(\mathrm{d}J_t - \lambda \mathrm{d}t), \tag{4.2'}$$

where $r_t^{F,i}$ is the (shadow) risk-free rate, ς_t^i is the price of Brownian risk, ν_t^i is the price of jump risk and $\lambda \mathrm{d}t$ is the Sunspot arrival rate (which, as discussed above, ensures that the jump process J_t is a martingale). Note that in contrast to Chapter 4, the risk-free rate now depends on i and may vary across agents in the model. This is because the introduction of jumps means that the debt traded in the model is no longer risk-free.

7.3.1 For given SDF processes, derive individual equilibrium conditions

Optimal investment ι. As before, the choice of investment rate is a static and time-separable problem. An agent chooses ι_t^i to maximize her return $r_t^{i,K}(\iota_t^i)$. The first-order condition yields the Tobin's q equation

$$\frac{1}{q_t} = \Phi'(\iota_t^i).$$

This choice applies to all agents with the special functional form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, $\phi\iota_t = q_t - 1$.

Goods market clearing. The goods market clearing condition

$$(A(\kappa) - \iota_t)K_t = \sum_i C_t^i, \tag{7.1}$$

which is the same as condition (4.14) in Chapter 4.

Asset and risk allocation using the martingale approach. To derive the optimal portfolio choice, we can again use the martingale approach, with a slight modification to incorporate the newly added jumps.

Martingale approach with jumps.

Consider a portfolio choice problem in continuous time:

$$\begin{aligned}
\max_{\{c_t, \theta_t^e\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u(c_t) \mathrm{d}t \right] \\
\text{s.t.} \quad & \frac{\mathrm{d}n_t}{n_t} = -\frac{c_t}{n_t} \mathrm{d}t + \sum_j \theta_t^j \mathrm{d}r_t^j + \text{ labor income/endowment/taxes} \\
& n_0 \text{ given.}
\end{aligned}$$

n_t is the net worth of the agent and r_t^j denotes the return of asset j. Let x_t^A be the value of a self-financing trading strategy A where one reinvests all dividends. Define the SDF as $\xi_t^i = e^{-\rho t} u'(c_t^i)$. As before, $\xi_t x_t^A$ follows a martingale. Let

$$\frac{\mathrm{d}x_t^A}{x_t^A} = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t + j_t^A \mathrm{d}J_t.$$

Assume that the SDF follows

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t^{F,i} \mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t - \nu_t^i (\mathrm{d}J_t - \lambda \mathrm{d}t)$$

Using Itô's product rule,

$$\begin{aligned}
\frac{\mathrm{d}(\xi_t^i x_t^A)}{\xi_t^i x_t^A} =& (-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \nu_t^i \lambda) \mathrm{d}t + (\sigma^A - \varsigma_t^i) \mathrm{d}Z_t + \\
& (j_t^A - \nu_t^i - \nu_t^i j_t^A) \mathrm{d}J_t \\
=& (-r_t^{F,i} + \mu_t^A - \varsigma_t^i \sigma_t^A + \lambda j_t^A - \lambda \nu_t^i j_t^A) \mathrm{d}t + (\sigma^A - \varsigma_t^i) \mathrm{d}Z_t + \\
& (j_t^A - \nu_t^i - \nu_t^i j_t^A)(\mathrm{d}J_t - \lambda \mathrm{d}t).
\end{aligned}$$

Where $(\sigma^A - \varsigma_t^i) \mathrm{d}Z_t + (j_t^A - \nu_t^i - \nu_t^i j_t^A)(\mathrm{d}J_t - \lambda \mathrm{d}t)$ is a martingale, given the inclusion of the compensating $\lambda \mathrm{d}t$ term. Then, since $\xi_t x_t^A$ follows a martingale, its drift equals zero, giving us that

$$\mu_t^A + \lambda j_t^A = r_t^{F,i} + \varsigma_t^i \sigma_t^A + \lambda \nu_t^i j_t^A.$$

Where $r_t^{F,i}$ is the (shadow) risk-free rate, ς_t^i is the price of Brownian risk, $\varsigma_t^i \sigma_t^A$ is the required Brownian risk premium, $\lambda\nu_t^i$ is the price of Poisson upside risk if $j_t^A > 0$. For risk-neutral agents $\nu_t^i = 0$. For CRRA utility, $1 - \nu_t^i = (1 + j_t^\omega)^{1-\gamma}(1 + j_t^n)^{-\gamma}$ since SDF is $\xi_t = e^{-\rho t}\omega_t^{1-\gamma} n_t^{-\gamma}$. For log utility, $\nu_t^i = 1 - \frac{1}{1+j_t^n} = \frac{j_t^n}{1+j_t^n}$. For Epstein-Zin, ν_t^i is part of ω_t–process.

We should note that we have three risk assets in the model. $\mathrm{d}r^{e,K}$ is experts' return on capital, $\mathrm{d}r^{h,OE}$ is households' return on outside equity, and $\mathrm{d}r^{h,D}$ is households' return on debt. The debt is risky due to bankruptcy.

For any two self-financing strategies A, B, the martingale approach implies

$$\mu_t^A - \mu_t^B + \lambda(j_t^A - j_t^B) = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \lambda\nu_t^i(j_t^A - j_t^B).$$

Using the martingale approach on expert capital with outside equity issuance (after plugging in households' outside equity choice), we get

$$\begin{aligned}\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - \left[\frac{\chi_t^e}{\kappa_t^e} r_t^{F,e} + (1 - \frac{\chi_t^e}{\kappa_t^e}) r_t^{F,h}\right] + \lambda j_t^q \\ = \left[\frac{\chi_t^e}{\kappa_t^e}\varsigma_t^e + (1 - \frac{\chi_t^e}{\kappa_t^e})\varsigma_t^h\right](\sigma + \sigma^q) + \left[\frac{\chi_t^e}{\kappa_t^e}\nu_t^e + (1 - \frac{\chi_t^e}{\kappa_t^e})\nu_t^h\right]\lambda j_t^q\end{aligned}$$

Similarly, we can derive the household portfolio choice condition by taking the difference between the drift of household capital and defaultable debt

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t^{F,h} + \lambda(j_t^q - j_t^{r^D}) \leq \varsigma_t^h(\sigma + \sigma^q) + \nu_t^h\lambda(j_t^q - j_t^{r^D}),$$

This condition holds with equality if $\kappa^e < 1$ (i.e., if households hold a non-zero amount of capital).

Asset and risk allocation using the price-taking social planners problem.

As in Chapter 4, we can also solve for the equilibrium risk and asset allocation using the social planner's problem. In this environment, we can generalize the price-taking planner's theorem to include the choice of jump risk.

Theorem 7.1 (Price-Taking Planner's Theorem)**.** *A social planner that takes prices as given chooses a physical asset allocation, κ_t, Brownian risk allocation, χ_t, and jump risk allocation,*

ζ_t, that coincides with the choices implied by all individuals' portfolio decisions.

The planner's optimization problem is formulated as follows:

$$\begin{aligned}
&\max_{\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t} \frac{\mathbb{E}_t[\mathrm{d}r_t^N(\boldsymbol{\kappa}_t)]}{\mathrm{d}t} - \boldsymbol{\varsigma}_t\sigma(\boldsymbol{\chi}_t) - \lambda\boldsymbol{\nu}j(\boldsymbol{\zeta}_t) \\
&\text{s.t.} \quad F(\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t) \leq 0 \qquad \text{(Financial Frictions)}
\end{aligned}$$

In this formulation:

- $\boldsymbol{\kappa}_t$ represents the planner's choice of physical asset allocations across agents.
- $\boldsymbol{\chi}_t$ determines the allocation of Brownian risk, influencing the exposure of individual agents to continuous stochastic fluctuations.
- $\boldsymbol{\zeta}_t$ corresponds to the allocation of jump risk, which affects agents' exposure to discontinuous price movements.
- $\boldsymbol{\varsigma}_t = (\varsigma_t^1, \ldots, \varsigma_t^I)$ is a vector of risk price sensitivities for each agent.
- The function $\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1\sigma^N, \ldots, \chi_t^I\sigma^N)$ describes the effect of Brownian risk allocation on expected returns.
- The function $\boldsymbol{j}(\boldsymbol{\zeta}_t) = (\zeta_t^1 j^N, \ldots, \zeta_t^I j^N)$ captures the impact of jump risk allocation.
- The financial friction constraint, $F(\boldsymbol{\kappa}_t,\boldsymbol{\chi}_t,\boldsymbol{\zeta}_t) \leq 0$, ensures that allocations are subject to market imperfections and frictions.

For example, if we set $\chi_t = \zeta_t = \kappa_t$ as the financial friction, that means experts can't issue outside equity to offload Brownian or risky debt to offload Jump risk. Alternatively, imposing the constraint $\chi_t \geq \alpha\kappa_t$ represents a skin-in-the-game requirement, where the issuance of outside equity is restricted to a certain limit.

"Invariance" of relative capital demand.

One of the insights of Mendo (2020) is that self-fulfilling jumps do not influence the relative demand for capital of experts relative to households. In other words, the excess market return that experts demand to hold capital remains unaffected.

Subtracting the experts' pricing condition from that of households gives the following inequality:

$$\mu_t^{r^k,e} - \mu_t^{r^k,h} \geq \frac{\chi_t^e}{\kappa_t^e}(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) - \frac{\chi_t^e}{\kappa_t^e}\lambda(1 - \nu_t^h)\underbrace{\left(\frac{\partial j_t^D}{\partial \theta_t^{e,K}}(\theta_t^{e,K} - 1) + j_t^q - j_t^D\right)}_{=0}$$

Losses are distributed between experts and households through the use of defaultable debt. Since the losses of experts are capped by their net worth due to limited liability, any additional losses arising from increasing capital holdings, denoted as $\theta_t^{e,K}$, are absorbed by households.

The concept of "invariance" in this setting is dependent on the performance penalty, which approaches infinity as $\tau \to 0$. The presence of this penalty ensures that all experts follow the same default outcome. Without this penalty, experts would bifurcate—some choosing to default after a jump while others would not.

7.3.2 Evolution of state variable η_t

Drift of η_t. We calculate the drift of η_t by changing to the total wealth N_t numeraire. The change of numeraire approach is similar to the case without jumps, with an additional equation for ν_t.

As before, we change the numeraire from consumption goods to total wealth. Consider two assets:

- Asset A: sector = i's portfolio return in terms of total wealth:

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i/N}\right)\mathrm{d}t + \sigma_t^{\eta^i/N}\mathrm{d}Z_t + j_t^{\eta^i/N}\mathrm{d}J_t.$$

Expanding the equation, we get:

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i} - \underbrace{\frac{\rho^d \zeta^i (1-\eta_t^i) - (1-\zeta^i)\eta_t^i}{\eta_t^i}}_{\mu_t^{\mathrm{pop},i} :=} \right) \mathrm{d}t + \sigma_t^{\eta^i} \mathrm{d}Z_t + j_t^{\eta^i} \mathrm{d}J_t.$$

- Asset B: a benchmark asset that everyone can hold, such as a risk-free asset or money, measured in terms of total economy-wide wealth as the numeraire:

$$r_t^{bm} \mathrm{d}t + \sigma_t^{bm} \mathrm{d}Z_t.$$

Apply our martingale asset pricing formula in the total wealth N_t numeraire,

$$\mu_t^A - \mu_t^B + \lambda(j_t^A - j_t^B) = \hat{\varsigma}_t(\sigma_t^A - \sigma_t^B) + \lambda \hat{v}_t (j_t^A - j_t^B).$$

The martingale asset pricing formula gives

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^{bm} + \lambda \left(j_t^{\eta^i} - j_t^{bm} \right) - \mu_t^{\mathrm{pop},i} = (\varsigma_t^i - \sigma_t^N) \left(\sigma_t^{\eta^i} - \sigma_t^{bm} \right) + \lambda \hat{v}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right).$$

Summing across all types weighted by their respective shares in the total economy, we obtain the aggregate condition

$$\begin{aligned} &\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^{bm} + \underbrace{\lambda \sum_{i'}^{I} \eta_t^{i'} j_t^{\eta^{i'}} - \lambda j_t^{bm}}_{=0} - \underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\mathrm{pop},i'}}_{=0} \\ &\qquad = \sum_{i'}^{I} \eta_t^{i'} \hat{\varsigma}_t^{\eta^{i'}} (\sigma_t^{\eta^{i'}} - \sigma_t^{bm}) + \lambda \sum_{i'}^{I} \eta_t^{i'} \hat{v}_t^{\eta^{i'}} (j_t^{\eta^{i'}} - j_t^{bm}). \end{aligned}$$

where capital letters without superscripts denote economy-wide aggregates.

Subtracting this aggregate equation from the individual equation gives the net worth share dynamics

$$\mu_t^{\eta^i} + \lambda j_t^{\eta^i} = \frac{C_t}{N_t} - \frac{C_t^i}{N_t^i} + \hat{\varsigma}_t^i(\sigma^{\eta^i} - \sigma^{bm_t}) - \sum_{i'}^{I} \eta_t^{i'} \hat{\varsigma}_t^{\eta^{i'}} (\sigma_t^{\eta^{i'}} - \sigma_t^{bm})$$
$$+\lambda \hat{v}_t^i \left(j_t^{\eta^i} - j_t^{bm} \right) - \lambda \sum_{i'}^{I} \eta_t^{i'} \hat{v}_t^{\eta^{i'}} (j_t^{\eta^{i'}} - j_t^{bm}) + \mu_t^{\text{pop},i}.$$

For experts

$$\mu_t^{\eta^e} + \lambda j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e \left(\sigma_t^{\eta^e} - \sigma_t^{bm} \right) + (1-\eta_t^e)\hat{\varsigma}_t^h \left(\sigma_t^{\eta^h} - \sigma_t^{bm} \right)$$
$$+ (1-\eta_t^e)\lambda \hat{v}_t^e \left(j_t^{\eta^e} - j_t^{bm} \right) - (1-\eta_t^e)\lambda \hat{v}_t^h \left(j_t^{\eta^h} - j_t^{bm} \right) + \mu_t^{\text{pop},i}$$

In this context, the benchmark asset is risky debt. Since j_t^D is the return on risky debt jump in c-numeraire and j_t^N represents the wealth jump, apply quotient rule for jumps $\sigma_t^{bm} = -\sigma_t^N$ and $j_t^{bm} = \frac{j_t^{r^D} - j_t^N}{1+j_t^N}$.

$$\mu_t^{\eta^e} + \lambda j_t^{\eta^e} = \frac{C_t}{N_t} - \frac{C_t^e}{N_t^e} + (1-\eta_t^e)\hat{\varsigma}_t^e \left(\sigma_t^{\eta^e} + \sigma_t^N \right) + (1-\eta_t^e)\hat{\varsigma}_t^h \left(\sigma_t^{\eta^h} + \sigma_t^N \right) +$$
$$(1-\eta_t^e)\lambda \hat{v}_t^e \left(j_t^{\eta^e} - \frac{j^{r^D} - j^N}{1+j^N} \right) - (1-\eta_t^e)\lambda \hat{v}_t^h \left(j_t^{\eta^h} - \frac{j^{r^D} - j^N}{1+j^N} \right) + \mu_t^{\text{pop},e}$$

Volatility of η_t^i. We can calculate the volatility of η_t^i using Itô's quotient rule. Since $\eta_t^i = N_t^i / N_t$ we have

$$\sigma_t^{\eta^i} = \sigma_t^{N^i} - \sigma_t^N = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = (1-\eta_t^i)\sigma_t^{N^i} - \sum_{\neg i \neq i} \eta_t^{\neg i} \sigma_t^{N^{\neg i}}$$

Jumps in η_t^i. Similarly,

$$j_t^{\eta^i} = \frac{j_t^{N^i} - J_t^N}{1 + j_t^N} = \frac{j_t^{N^i} - \sum_{i'} \eta_t^{i'} J_t^{N^{i'}}}{1 + \sum_{i'} \eta_t^{i'} J_t^{N^{i'}}}$$

For 2 types example,

$$j_t^{\eta^e} = \frac{(1-\eta_t^e)(j_t^{N^e} - j_t^{N^h})}{1 + \eta^e j_t^{N^e} + (1-\eta_t^e) j_t^{N^h}}$$

7.3.3 BSDE functions

BSDE functions for CRRA. For CRRA, we generalize the result from earlier lecture by adding jump terms in value function BSDEs. Specifically, we have

$$\xi^i n_t^i = e^{-\rho^i t}(c_t^i)^{-\gamma}\frac{\omega_t^i n_t^i}{\rho^i} = e^{-\rho^i t}\frac{(\omega_t^i n_t^i)^{1-\gamma}}{\rho^i} = \underbrace{\frac{(\omega_t^i n_t^i / K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} K_t^{1-\gamma} e^{-\rho^i t}$$

when transforming $e^{\rho^i t}\xi^i(n_t^i; \boldsymbol{\eta}_t, K_t)$-process into $v^i(\boldsymbol{\eta}_t)$-process, so

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i} = \frac{\mathrm{d}\left(v_t^i K_t^{1-\gamma}\right)}{v_t^i K_t^{1-\gamma}} - \rho^i \mathrm{d}t.$$

By Itô's product rule:

$$\frac{\mathbb{E}_t[\mathrm{d}(\xi_t^i n_t^i)]}{\xi_t^i n_t^i} = \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} - \rho^i + \lambda j_t^{v^i}\right)\mathrm{d}t$$

Recall by consumption optimality for CRRA utility:

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i}\mathrm{d}t + \frac{c_t^i}{n_t^i}\mathrm{d}t \text{ follows a martingale}$$

Hence, $\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = \rho^i - \frac{c_t^i}{n_t^i} - \lambda j_t^{v^i}$

BSDE functions for Epstein-Zin. The SDF in the most general case of Epstein-Zin utility is given by:

$$\xi_t^i = e^{\left(\int_0^t \frac{\partial f}{\partial U}(c_s, V_s^i) ds\right)} \frac{\partial V_t^i}{\partial n_t^i}$$

By envelop condition:

$$\xi_t^i n_t^i = e^{\int_0^t \frac{\partial f}{\partial V}(c_s^i, V_s^i)\mathrm{d}s} \frac{1}{\rho^i}(\omega^i n_t^i)^{1-\gamma} = e^{\int_0^t \frac{\partial f}{\partial V}(c_s^i, V_s^i)\mathrm{d}s} \underbrace{\frac{(\omega_t^i n_t^i / K_t)^{1-\gamma}}{\rho^i}}_{v_t^i :=} K_t^{1-\gamma}$$

By Itô's product rule:

$$\frac{\mathbb{E}_t[\mathrm{d}(\xi_t^i n_t^i)]}{\xi_t^i n_t^i} = \left(\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \tfrac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} + \tfrac{\partial f}{\partial V}(c_t^i, V_t^i) + \lambda j_t^{v^i}\right)\mathrm{d}t.$$

Similarly, recall by consumption optimality for EZ utility:

$$\frac{\mathrm{d}(\xi_t^i n_t^i)}{\xi_t^i n_t^i}\mathrm{d}t + \frac{c_t^i}{n_t^i}\mathrm{d}t \text{ follows a martingale}$$

Hence, $\mu_t^{v^i} + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + (1-\gamma)\sigma\sigma_t^{v^i} = -\frac{\partial f}{\partial V}\left(c_t, V_t^i\right) - \frac{c_t^i}{n_t^i} - \lambda j_t^{v^i}$.

We still have to solve for $\mu_t^{v^i}, \sigma_t^{v^i}$. The numerical solution is the same as in Chapter 4, with minor changes such as adjusting μ_t^{η} and $\mu_t^{v^i}$, and keeping track of the vulnerability region together with all the jump loadings.

Discussions. If we deviate from EIS $= 1$, the consumption-wealth ratio of agents will vary with investment opportunities, as these depend on the precise specification of perceived run risk, even under log utility. This variation will, in turn, influence q through goods market clearing. However, if we maintain EIS $= 1$ but alter the level of risk aversion, the q-function will only be affected if capital is allocated differently for the same value of η. This is because, in such a case, the average consumption-wealth ratio in the economy remains unchanged, allowing goods market clearing to establish a one-to-one mapping between q and capital allocation. A key question is whether the "invariance" of capital demands holds solely due to the absence of hedging demands or whether it generalizes even in their presence. Without the full set of equations at hand, a reasonable conjecture is that this result lacks robustness, implying that capital allocation and q will still be affected even when EIS $= 1$.

7.4 Key Takeaways

In this chapter, we introduced *jumps* into a continuous-time macrofinance model and explored how sudden stops or runs can arise endogenously. Several technical tools were developed in the process. First, we extended the usual Itô's Lemma to handle discontinuous paths driven by Poisson processes or Lévy jumps. We also generalized the martingale approach for portfolio choice and asset pricing, ensuring agents' strategies and pricing conditions fully incorporate both Brownian and jump risk. A "price-taking" social-planner formulation then allowed us to solve for capital and risk allocation across experts and households. Finally, we changed numeraires to economy-wide net worth, carefully adjusting drift and volatility terms so that jumps in wealth shares are properly accounted for.

On the economic side, we saw that *defaultable debt* plays a key role in risk sharing. In extreme downturns, debt can default, offloading part of the experts' downside onto households, which leads to a form of *invariance* in how much capital experts wish to hold. Another striking observation is that there are often *no runs in very bad times*: once the expert sector's net worth is too low, the price can no longer drop enough to wipe them out, so the vulnerability region does not start at $\eta = 0$. Finally, we encountered a *volatility paradox* in the presence of jump risk. Paradoxically, a low-risk environment can breed higher leverage and thus amplify the impact of a jump, ultimately making a system more fragile.

Overall, incorporating *jump processes* in this way offers a tractable lens through which to study self-fulfilling crises, fire-sale prices, and abrupt collapses of intermediaries' balance sheets. By combining these new technical instruments with the economic insights outlined above, we obtain a unified framework for analyzing sudden stops and runs.

7.5 Exercises

7.5.1 Introducing a collateral (borrowing)/leverage constraint

$$-\theta_t^{e,D} \leq \ell \theta_t^{e,K}$$

(a) Show that with log-utility, $q(\eta)$ is not affected by jumps even with the leverage constraint.

(b) Show how jumps affect the risk premium in the "vulnerability region".

(c) Verify whether the occasionally binding leverage constraint lowers the drift μ_t^η for small η values.

(d) Describe conditions under which the leverage constraint leads to a bimodal stationary distribution of η.

(e) Show that with a sufficiently strict leverage constraint, the vulnerability region (when $j_t^q \chi_t \geq \eta_t$) shrinks or even disappears.

(f) Consider a setting in which the debt is fully collateralized, i.e., it is default-free. When entering the vulnerability region, the "worst price" can jump, and hence the debt capacity contracts discontinuously.

To solve this model, one needs an extra loop to determine the fixed point when the vulnerability region starts.

Bibliography

Brunnermeier, Markus K. and Lasse Heje Pedersen, "Market liquidity and funding liquidity," *The Review of Financial Studies*, 2009, *22* (6), 2201–2238.

Calvo, Guillermo A., "Capital flows and capital-market crises: the simple economics of sudden stops," *Journal of Applied Economics*, 1998, *1* (1), 35–54.

Diamond, Douglas W. and Philip H. Dybvig, "Bank runs, deposit insurance, and liquidity," *Journal of Political Economy*, 1983, *91* (3), 401–419.

Kaminsky, Graciela L and Carmen M. Reinhart, "The twin crises: the causes of banking and balance-of-payments problems," *American Economic Review*, 1999, *89* (3), 473–500.

Mendo, Fernando, "Risky low-volatility environments and the stability paradox," *Working Paper*, 2020.

Mendoza, Enrique G., "Sudden stops, financial crises, and leverage," *American Economic Review*, 2010, *100* (5), 1941–66.

Morris, Stephen and Hyun Song Shin, "Unique equilibrium in a model of self-fulfilling currency attacks," *American Economic Review*, 1998, pp. 587–597.

Obstfeld, Maurice, "Models of currency crises with self-fulfilling features," *European Economic Review*, 1996, *40* (3-5), 1037–1047.