Eco529: Macro, Money, and Finance

Lecture 06: Money

Markus Brunnermeier

Princeton University

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Outline

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- Frictionless Benchmark
- Adding Financial Frictions
- Adding Monetary Frictions
- Separating Money $\mathcal{M}$ and Gov. Bonds $\mathcal{B}$

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- Sims' Stepping on the Rake with Long-Maturity Bonds
- Quantitative Easing

3 Monetary Fiscal Connection

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- Sargent-Wallace's Unpleasant Monetary Arithmetic

4 Price Level Determination

- Fiscal Backing and the Fiscal Theory of the Price Level
- Bubble Theories and (In-)Determinacy
- "Pure" Unit of Account Theory

The 3 Roles of Money

- **Store of value**
  - fundamental cash flows due to backing
    e.g., commodity standard, exchange rate regime, fiscal backing
  - non-cash-flow benefits: helps overcome *intertemporal* financial frictions
    e.g., OLG (Samuelson), spatial separation (Townsend), uninsured idiosyncratic risk (Bewley)
  - store of value role not exclusive to money: non-monetary assets are substitute stores of values
- **Medium of exchange**
  - helps overcome monetary frictions = frictions in *intratemporal* exchange
  - key monetary friction: double coincidence of wants problem
  - makes money special relative to assets, which cannot serve as substitute media of exchange
- **Unit of account**
  - contractual values denominated in monetary unit
  - e.g., nominal goods prices (+ commitment to sell at quoted price), nominal debt contracts, nominal labor contracts

The Value of Money and the Price Level

- Core question of monetary economics: what determines the general level of nominal goods prices?
- Equivalently: what are the determinants of the value of money?
  - price level $\mathcal{P}_t$: price of real goods basket in units of money
  - real value of a single unit of money: $1/\mathcal{P}_t$
- Two aspects:
  1. which economic considerations justify the value of money in a given equilibrium?
  2. determinacy question:
     - does the model have a unique prediction for the value of money / price level? ($\sim$ equilibrium uniqueness)
     - more broadly, which economic forces lead to coordination on a specific monetary equilibrium?

Classification of Monetary Theories

1. **Backing theories**: value of money derives from fundamental cash flows that back it
   - store of value role (money is just another asset)
   - example: Fiscal Theory of the Price Level (FTPL)
2. **Bubble theories**: money valued because it can be passed on to others
   can be rational expectation if trading money overcomes market frictions:
   1. intertemporal financial frictions (e.g., incomplete markets)
      - emphasizes store of value role
      - examples: Samuelson, Townsend, Bewley, Brunnermeier-Merkel-Sannikov
   2. intratemporal monetary frictions (e.g., cash-in-advance constraint)
      - emphasizes medium of exchange role
      - example: (New) Monetarism
3. **Money as a pure unit of account**
   - value of money derives from role of money as a unit of account
   - not from the value of any monetary assets
   - example: New Keynesianism

Monetary Assets: Credit, Deposits, Cash, Reserves, Government Debt

In first two classes of theories, different assets may play the role of "money":

- Credit can substitute for
  - store of value assets (credit balances to keep track of resource distribution)
  - media of exchange (exchange goods against credit balance)

  imperfect credit prerequisite for bubble theories
- Bank deposits, cash, and central bank reserves all play a role in the payment system as media of exchange
- Government-provided outside money vs. inside money
  - outside money: positive net supply, backed by government fiscal capacity
  - inside money: zero net supply, backed by bank assets
- Cash & reserves (narrow outside money) vs. nom. government liabilities (broad outside money)
  - (primarily) narrow money provides medium of exchange services
  - but all nominal government liabilities
    - compete for the same backing real resources
    - serve as a store of value
    - are affected symmetrically by changes in the price level

A Unified Model of Money

- Next: develop and solve a simple model that illustrates several monetary theories
  - fiscal theory of the price level (backing theory)
  - money as a safe asset (store of value bubble theory)
  - money providing transaction services (medium of exchange bubble theory)
- For now, we disregard the determinacy question
  - we always select a specific equilibrium: the monetary steady state
  - will return to the determinacy question in the end

Outline

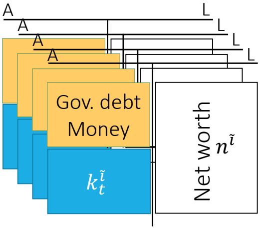
Model with Money

- Continuum of (heterogeneous) agents $\tilde{i} \in [0,1]$: choose $c_t^{\tilde{i}}$, $\theta_t^{\tilde{i}}$, $\iota_t^{\tilde{i}}$ to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log c_t^{\tilde{i}} + f(\mathcal{G}_t K_t)\right) dt\right]$$

$$s.t.\ \frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}\mathrm{d}t + \mathrm{d}r_t^{\mathcal{MB}} + (1-\theta_t^{\tilde{i}})(\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - \mathrm{d}r_t^{\mathcal{MB}})\ \&\ n_t^{\tilde{i}} \geq 0$$

- Each agent operates physical capital $k_t^{\tilde{i}}$
  - output (net of investment & transaction cost):
    $y_t^{\tilde{i}}\mathrm{d}t = (ak_t^{\tilde{i}} - \iota_t^{\tilde{i}}k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}})k_t^{\tilde{i}})\mathrm{d}t$
  - $\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta\right)\mathrm{d}t + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}} + \mathrm{d}\Delta_t^{k,\tilde{i}}$,
    ($\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ idiosyncratic Brownian risk)
  - output tax $\tau_t a k_t^{\tilde{i}}\mathrm{d}t$



- No aggregate risk $\mathrm{d}Z_t$
- Government budget constraint: $\underbrace{(\mu_t^{\mathcal{MB}} - i_t^{\mathcal{MB}})}_{=:\breve{\mu}_t^{\mathcal{MB}}}\mathcal{MB}_t + \mathcal{P}_t K_t \underbrace{(\tau_t a - \mathcal{G})}_{=:s_t} = 0$

Frictions

Let us build up model step by step adding one friction at a time:

1. Frictionless benchmark: $\tilde{\sigma} = 0$, $\mathfrak{T} \equiv 0$
   - only tax backing present
2. Financial friction (intertemporal)
   - $\tilde{\sigma} > 0$ and incomplete markets friction: $d\tilde{Z}_t^{\tilde{i}}$-shocks uninsurable
   - money serves as a safe asset
3. Monetary friction (intratemporal)
   - transaction costs $\mathfrak{T}_t(\nu_t^{\tilde{i}})$ increasing in velocity $\nu_t^{\tilde{i}}$
   - reduced-form device for medium of exchange role
   - interpretation: transaction costs incurred in unmodeled supply chain

Outline

Assets, Aggregate Resource Constraint, and Markets

- Assets: capital and money
  - q_t^K capital price
  - $q_t^{\mathcal{MB}} := \frac{\mathcal{MB}_t}{\mathcal{P}_t K_t}$ value of money per unit of capital
  - $q_t := q_t^K + q_t^{\mathcal{MB}} = N_t / K_t$ wealth per unit of capital
  - $\vartheta_t := \frac{\mathcal{MB}_t/\mathcal{P}_t}{q_t^K K_t + \mathcal{MB}_t/\mathcal{P}_t} = \frac{q_t^{\mathcal{MB}}}{q_t^K + q_t^{\mathcal{MB}}}$ share of nominal wealth
  - Postulate Ito price processes
    $\mathrm{d}q_t^K / q_t^K = \mu_t^{q,K} \mathrm{d}t, \;\; \mathrm{d}q_t^{\mathcal{MB}} / q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}} \mathrm{d}t, \;\; \mathrm{d}\vartheta_t / \vartheta_t = \mu_t^{\vartheta} \mathrm{d}t$
  - SDF for each agent $\tilde{i}$: $\mathrm{d}\xi_t^{\tilde{i}} / \xi_t^{\tilde{i}} = -r_t dt$
- Aggregate resource constraints:
  - Output: $C_t + \iota_t K_t + \mathscr{G} K_t = a K_t$
  - Capital: $\int k_t^{\tilde{i}} \mathrm{d}\Delta k_t^{k,\tilde{i}} \mathrm{d}\tilde{i} = 0$
- Markets: Walrasian goods, money, and capital markets

Return Processes

- Return on capital

$$dr_t^{K,\tilde{i}}(\iota) = \left(\frac{a(1-\tau_t) - \iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t$$
$$= \left(\frac{a - \mathscr{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t$$

  - second line uses government budget constraint

$$\breve{\mu}_t^{\mathcal{MB}}\mathcal{MB}_t + \mathcal{P}_t K_t(\tau_t a - \mathscr{G}) = 0 \Leftrightarrow \tau_t a - \mathscr{G} = -\breve{\mu}_t^{\mathcal{MB}} q_t^{\mathcal{MB}}$$

- Return on money

$$dr_t^{\mathcal{MB}} = i_t^{\mathcal{MB}}dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t} \qquad = i_t^{\mathcal{MB}}dt + \frac{d(q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t)}{q_t^{\mathcal{MB}}K_t/\mathcal{MB}_t}$$
$$= \left(i_t^{\mathcal{MB}} + \mu_t^{q,\mathcal{MB}} + \mu_t^K - \mu_t^{\mathcal{MB}}\right)dt \qquad = \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \breve{\mu}_t^{\mathcal{MB}}\right)dt$$

Optimal Investment and Goods Market Clearing

- Optimal investment: Tobin's Q condition

$$q_t^K = \frac{1}{\Phi'(\iota_t^{\tilde{i}})} = 1 + \phi \iota_t^{\tilde{i}}$$

  - in particular, all agents choose same investment rate: $\iota_t^{\tilde{i}} = \iota_t$

- Goods market clearing

$$\rho q_t \cancel{K_t} + \iota_t \cancel{K_t} + \mathscr{G} \cancel{K_t} = a \cancel{K_t}$$

- Solve for q_t, use $q_t^K = (1 - \vartheta_t) q_t$, plug into optimal investment condition:

$$(1 - \vartheta_t) \frac{a - \iota_t - \mathscr{G}}{\rho} = (1 - \vartheta_t) q_t = 1 + \phi \iota_t \qquad \Rightarrow \qquad \iota_t = \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho}$$

  - where $\check{a} = a - \mathscr{G}$

Intermediate Conclusion: Equilibrium Asset Prices in Terms of ϑ_t

- Plugging ι_t expression back into $q_t = (\check{a} - \iota_t)/\rho$:

$$q_t = \frac{1 + \phi \check{a}}{1 - \vartheta_t + \phi \rho}$$

- q_t, q_t^K, $q_t^{\mathcal{MB}}$, ι_t only depend on the nominal wealth share ϑ_t:

$$q_t = \frac{1 + \phi \check{a}}{1 - \vartheta_t + \phi \rho} \qquad \iota_t = \frac{(1 - \vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi \rho}$$

$$q_t^K = (1 - \vartheta_t)\frac{1 + \phi \check{a}}{1 - \vartheta_t + \phi \rho} \qquad q_t^{\mathcal{MB}} = \vartheta_t \frac{1 + \phi \check{a}}{1 - \vartheta_t + \phi \rho}$$

- Hence, ϑ_t is the key variable in this model!
  - in equilibrium, $\vartheta_t = \theta_t$ (asset market clearing)
  - so ϑ_t should be determined by portfolio choice

Characterizing ϑ_t: Portfolio Choice Conditions

- Portfolio choice conditions

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = r_t = \frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}]}{dt}$$

- Substitute in return expressions

$$-\check{\mu}_t^{\mathcal{MB}} + \cancel{\mu_t^K} + \mu_t^{q,\mathcal{MB}} = \frac{a - \iota_t - \mathcal{G} + q_t^{\mathcal{MB}} \check{\mu}_t^{\mathcal{MB}}}{q_t^K} + \cancel{\Phi(\iota_t) - \delta} + \mu_t^{q,K}$$

- Use
  - $a - \iota_t - \mathcal{G} = \rho q_t$ (goods market clearing)
  - $q_t^K = (1 - \vartheta_t) q_t$, $q_t^{\mathcal{MB}} = \vartheta_t q_t$ (def. of ϑ_t)
  - $\mu_t^{\vartheta} = (1 - \vartheta_t)(\mu_t^{q,\mathcal{MB}} - \mu_t^{q,K})$ (Itô)

  and solve for μ_t^{ϑ}:

$$\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{MB}}$$

Characterizing ϑ_t: The Money Valuation Equation

- By definition, $\mu_t^{\vartheta} = \mathbb{E}_t[d\vartheta_t]/\vartheta_t$, so last equation can be written as

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}}\right)\vartheta_t dt$$

  - this version is preferable because it remains valid for $\vartheta_t = 0$
  - but note that our derivation has assumed $\vartheta_t > 0$ (otherwise $dr_t^{\mathcal{MB}}$ is not well-defined)
- This is the *money valuation equation* that characterizes portfolio demand for money
- For interpretation, integrate forward in time

(this would be a BSDE in a stochastic setting, i.e., a forward-looking choice condition, so we need to integrate forward in time)

$$\vartheta_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}(-\check{\mu}_{t'}^{\mathcal{MB}})\vartheta_{t'}dt'\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\frac{s_{t'}}{q_{t'}}dt'\right]$$

  - portfolio demand for money arises from expectations of future primary surpluses ($s_t K_t$)
  - these represent real payouts/cash flows made to holders of money

Frictionless Benchmark: Steady State Equilibria

- Let's assume a steady state equilibrium with constant ϑ (and hence q, q^K, $q^{\mathcal{MB}}$, ι) (this is really a balanced growth path because K_t grows at constant rate $g = \Phi(\iota) - \delta$)
- Imposing steady state implies:
  - by the money valuation equation ($\mu_t^{\vartheta} = 0$)

$$\breve{\mu}^{\mathcal{MB}} = -\rho$$

  - by the government budget constraint

$$s = -\breve{\mu}^{\mathcal{MB}} q^{\mathcal{MB}} = \rho q^{\mathcal{MB}} \Rightarrow q^{\mathcal{MB}} = \frac{s}{\rho}$$

- Some of the remaining equilibrium quantities are then:

$$\vartheta_t = \frac{s(1+\phi\rho)}{s+\rho(1+\phi\check{a})} \qquad q^K = \frac{1+\phi(\check{a}-s)}{1+\phi\rho} \qquad \iota_t = \frac{\check{a}-s-\rho}{1+\phi\rho}$$

Remark: we need $0 < s < \check{a} + \frac{1}{\phi}$ for both assets to have positive value.

Some Observations from Frictionless Benchmark

1. The value of money depends on fiscal backing
   - need positive primary surpluses ($s > 0$) for money to be valued
   - higher s results in higher $q^{\mathcal{MB}}$ and ϑ
2. The following are inversely related to the value of money
   - the value of capital assets (q^K)
   - the investment rate (ι)
   - the growth rate of the economy ($g = \Phi(\iota) - \delta$)
3. The nominal interest rate paid on money does not matter for the real allocation
   - raising $i^{\mathcal{MB}}$ while maintaining $\check{\mu}^{\mathcal{MB}} = -\rho$ raises $\mu^{\mathcal{MB}} = i^{\mathcal{MB}} - \rho$ one for one
   - this affects the inflation rate ($\pi := \mu^{\mathcal{P}} = \mu^{\mathcal{MB}} - g$) but no real variables

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  - $\vartheta_t := \frac{\mathcal{MB}_t/\mathcal{P}_t}{q_t^K K_t + \mathcal{MB}_t/\mathcal{P}_t} = \frac{q_t^{\mathcal{MB}}}{q_t^K + q_t^{\mathcal{MB}}}$ share of nominal wealth
  - Postulate Ito price processes
    $\mathrm{d}q_t^K / q_t^K = \mu_t^{q,K}\mathrm{d}t,\ \ \mathrm{d}q_t^{\mathcal{MB}} / q_t^{\mathcal{MB}} = \mu_t^{q,\mathcal{MB}}\mathrm{d}t,\ \ \mathrm{d}\vartheta_t/\vartheta_t = \mu_t^{\vartheta}\mathrm{d}t$
  - SDF for each agent $\tilde{i}$: $\mathrm{d}\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t dt - \tilde{\varsigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}}$
- Aggregate resource constraints:
  - Output: $C_t + \iota_t K_t + \mathscr{G} K_t = aK_t$
  - Capital: $\int k_t^{\tilde{i}} \mathrm{d}\Delta k_t^{k,\tilde{i}} \mathrm{d}\tilde{i} = 0$
- Markets: Walrasian goods, money, and capital markets

Return Processes

- Return on capital

$$dr_t^{K,\tilde{i}}(\iota) = \left(\frac{a(1-\tau_t)-\iota}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}\, d\tilde{Z}_t^{\tilde{i}}$$
$$= \left(\frac{a-\mathscr{G}-\iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\check{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}\, d\tilde{Z}_t^{\tilde{i}}$$

- Return on money

$$dr_t^{\mathcal{MB}} = i_t^{\mathcal{MB}} dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t}$$
$$= \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \check{\mu}_t^{\mathcal{MB}}\right) dt$$

Optimal Investment and Goods Market Clearing

Exactly as in previous model:

- Optimal investment

$$q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi\iota_t$$

- Combining with market clearing implies

$$\iota_t = \frac{(1-\vartheta_t)\check{a} - \rho}{1 - \vartheta_t + \phi\rho}$$

- Implied asset prices

$$q_t = \frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho} \qquad q_t^K = (1-\vartheta_t)\frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho} \qquad q_t^{\mathcal{M}\mathcal{B}} = \vartheta_t\frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho}$$

- Hence, the key variable to determine is the nominal wealth share ϑ_t

Characterizing ϑ_t: Portfolio Choice Conditions

- Portfolio choice conditions

$$-\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,\mathcal{MB}} = \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = r_t$$

$$\frac{a - \mathscr{G} - \iota}{q_t^K} + \frac{q_t^{\mathcal{MB}}}{q_t^K}\breve{\mu}_t^{\mathcal{MB}} + \Phi(\iota) - \delta + \mu_t^{q,K} = \frac{\mathbb{E}_t[dr_t^K]}{dt} = r_t + \tilde{\varsigma}_t\tilde{\sigma}$$

- New element: idiosyncratic risk premium $\tilde{\varsigma}_t\tilde{\sigma}$ on capital
  - due to log utility $\tilde{\varsigma}_t = \tilde{\sigma}_t^n = (1-\theta_t)\tilde{\sigma}$
  - by asset market clearing then $\tilde{\varsigma}_t = (1-\vartheta_t)\tilde{\sigma}$
- After same steps as before, we obtain

$$\mu_t^{\vartheta} = \rho + \breve{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2$$

The Money Valuation Equation with Idiosyncratic Risk

- The money valuation equation is now

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}} - (1 - \vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t dt$$

- In integral form

$$\vartheta_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\left(-\check{\mu}_{t'}^{\mathcal{MB}} + (1 - \vartheta_{t'})^2\tilde{\sigma}^2\right)\vartheta_{t'} dt'\right]$$

- Money may be valued for two reasons:
  - because of cash flows from fiscal backing ($-\check{\mu}_t^{\mathcal{MB}}$)
  - because it is a safe asset that facilitates idiosyncratic risk sharing ($(1 - \vartheta_t)^2\tilde{\sigma}^2$)

    (we will explore the safe asset features in more detail in the next lecture)

Idiosyncratic Risk Model: Steady State Equilibria

- Possible values for $\breve{\mu}^{\mathcal{MB}}$ consistent with $\mu_t^{\vartheta} = 0$

$$-\rho \leq \breve{\mu}^{\mathcal{MB}} < -\rho + \tilde{\sigma}^2$$

- For any such value, there are two possible steady-state equilibria:

	Non-Monetary	**Monetary (Store of Value)**
ϑ	$\vartheta = 0$	$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}$
$q^{\mathcal{MB}}$	$q^{\mathcal{MB}} = 0$	$q^{\mathcal{MB}} = \frac{(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}})(1 + \phi \check{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi \rho \tilde{\sigma}}$
q^K	$q^K = \frac{1 + \phi \check{a}}{1 + \phi \rho}$	$q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}}(1 + \phi \check{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi \rho \tilde{\sigma}}$
ι	$\iota = \frac{\check{a} - \rho}{1 + \phi \rho}$	$\iota = \frac{\check{a}\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} - \tilde{\sigma}\rho}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi \rho \tilde{\sigma}}$
s	$s = 0$	$s = -\breve{\mu}^{\mathcal{MB}} \frac{(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{MB}}})(1 + \phi \check{a})}{\sqrt{\rho + \breve{\mu}^{\mathcal{MB}}} + \phi \rho \tilde{\sigma}}$

Bubbles and Seigniorage

- If $\tilde{\sigma}^2 > \rho$, $\breve{\mu}^{\mathcal{MB}} = 0$ is a possible choice in previous solution
  - then, in the monetary steady state, money is still valued
  - ... but there is no fiscal backing ($s = -\breve{\mu}^{\mathcal{MB}} q^{\mathcal{MB}} = 0$)
  - money is a (rational) *bubble*: intrinsic value is zero but market value is positive
- We can push this idea further and even pick $\breve{\mu}^{\mathcal{MB}} > 0$
  - feasible so long as $\breve{\mu}^{\mathcal{MB}} < \tilde{\sigma}^2 - \rho$
  - then the government runs permanent primary deficits, $s < 0$
- Permanent deficits are possible because the government can generate seigniorage by "mining the bubble"
  - print new money that dilutes the claims of existing money holders to the aggregate bubble
  - bubble mining here acts as a tax on risk sharing (lowers ϑ_t, raises risk exposures)

Observations from Frictionless Benchmark Revisited

1. The value of money depends on fiscal backing and idiosyncratic risk
   - do not necessarily need positive primary surpluses ($s > 0$) for money to be valued
   - higher s or higher $\tilde{\sigma}$ result in higher $q^{\mathcal{MB}}$ and ϑ
2. The following are inversely related to the value of money
   - the value of capital assets (q^K)
   - the investment rate (ι)
   - the growth rate of the economy ($g = \Phi(\iota) - \delta$)

   these observations remain correct (only depend on goods market clearing)
3. The nominal interest rate paid on money does not matter for the real allocation
   - raising $i^{\mathcal{MB}}$ while maintaining $\breve{\mu}^{\mathcal{MB}} = -\rho$ raises $\mu^{\mathcal{MB}} = i^{\mathcal{MB}} - \rho$ one for one
   - this affects the inflation rate ($\pi := \mu^{\mathcal{P}} = \mu^{\mathcal{MB}} - g$) but no real variables

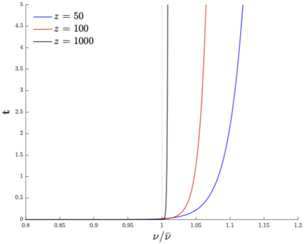
   these observations remain also correct

Outline

Adding Monetary Friction: Transaction Costs

- Recall: output produced by $\tilde{i}$ net of investment and transaction costs

$$y_t^{\tilde{i}}\mathrm{d}t = (ak_t^{\tilde{i}} - \iota_t^{\tilde{i}}k_t^{\tilde{i}} - \mathfrak{T}_t(\nu_t^{\tilde{i}})k_t^{\tilde{i}})\mathrm{d}t$$



- We now add the left-out details:
  - $\nu_t^{\tilde{i}}$ denotes output velocity of $\tilde{i}$'s money holdings:

$$\nu_t^{\tilde{i}} := \frac{\mathcal{P}_t a k_t^{\tilde{i}}}{m_t^{\tilde{i}}} = \frac{1-\theta_t^{\tilde{i}}}{\theta_t^{\tilde{i}}}\frac{a}{q_t^K}$$

    where $m_t^{\tilde{i}}$ denotes the money holdings of individual $\tilde{i}$
  - transaction costs are given by

$$\mathfrak{T}_t(\nu) = \frac{a}{(\mathfrak{z}-1)\,\bar{\nu}}\left[\left(\frac{\nu}{\bar{\nu}}\right)^{\mathfrak{z}-1} - \left(\frac{\nu_t^{\text{eq}}}{\bar{\nu}}\right)^{\mathfrak{z}-1}\right]$$

  - ν_t^{eq} is velocity of everyone else in equilibrium
- Limit case $\mathfrak{z} \to \infty$: cash-in-advance constraint

$$\nu_t^{\tilde{i}} \leq \bar{\nu} \Leftrightarrow \mathcal{P}_t a k_t^{i} \leq \bar{\nu} m_t^{\tilde{i}}$$

Return Processes

- Return on capital

$$dr_t^{K,\tilde{i}}(\iota,\nu) = \left(\frac{a(1-\tau_t) - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}\, d\tilde{Z}_t^{\tilde{i}}$$

$$= \left(\frac{a - \mathscr{G} - \iota - \mathfrak{T}_t(\nu)}{q_t^K} + \frac{q_t^M}{q_t^K}\check{\mu}_t^{\mathcal{M}} + \Phi(\iota) - \delta + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}\, d\tilde{Z}_t^{\tilde{i}}$$

- Return on money

$$dr_t^{\mathcal{MB}} = i_t^{\mathcal{MB}} dt + \frac{d(1/\mathcal{P}_t)}{1/\mathcal{P}_t}$$

$$= \left(\mu_t^{q,\mathcal{MB}} + \mu_t^K - \check{\mu}_t^{\mathcal{MB}}\right) dt$$

Optimal Investment and Goods Market Clearing

Exactly as in previous model:

- Optimal investment

$$\iota_t = \frac{(1-\vartheta_t)\check{a} - \rho}{1-\vartheta_t + \phi\rho}$$

- Implied asset prices

$$q_t = \frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho} \qquad q_t^K = (1-\vartheta_t)\frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho} \qquad q_t^{\mathcal{MB}} = \vartheta_t \frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho}$$

Portfolio Choice

Note: portfolio choice is nonstandard because θ_t enters net worth return nonlinearly via velocity. Therefore, we solve this explicitly using the stochastic maximum principle.

$$H_t = e^{-\rho t} \log c_t - \xi_t c_t + \xi_t n_t \left((1-\theta_t) \frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} + \theta_t \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} \right) - \tilde{\varsigma}_t \xi_t n_t (1-\theta_t) \tilde{\sigma}$$

Maximize H_t with respect to θ_t, ν_t subject to the constraint

$$\theta_t \nu_t = (1-\theta_t) \frac{a}{q_t^K}$$

Denoting the Lagrange multiplier by $\lambda_t^{\mathcal{MB}} \xi_t n_t$, the first-order conditions are:

$$\theta_t : \quad \frac{\mathbb{E}_t[dr_t^K(\iota_t, \nu_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \tilde{\varsigma}_t \tilde{\sigma} + \lambda_t^{\mathcal{MB}} \left(\nu_t + \frac{a}{q_t^K} \right)$$

$$\nu_t : \quad (1-\theta_t) \frac{\partial \mathbb{E}[dr_t^K(\iota_t, \nu_t)]/dt}{\partial \nu_t} + \lambda_t^{\mathcal{MB}} \theta_t = 0$$

θ-FOC and Money Valuation Equation

$$\frac{\mathbb{E}_t[dr_t^K(\iota_t,\nu_t)]}{dt} = \frac{\overbrace{a-\mathcal{G}-\iota_t-\mathfrak{T}_t(\nu_t)}^{=\rho/(1-\vartheta_t)}}{q_t^K} + \frac{\overbrace{q_t^{\mathcal{MB}}}^{=\vartheta_t/(1-\vartheta_t)}}{q_t^K} \quad \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,K}$$

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \quad - \check{\mu}_t^{\mathcal{MB}} + \Phi(\iota_t) - \delta + \mu_t^{q,\mathcal{MB}}$$

Take the difference:

$$\frac{\mathbb{E}_t[dr_t^K(\iota_t,\nu_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{MB}}]}{dt} = \frac{\rho}{1-\vartheta_t} + \frac{\check{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t}$$

Plug into FOC:

$$\frac{\rho}{1-\vartheta_t} + \frac{\check{\mu}_t^{\mathcal{MB}}}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \tilde{\varsigma}_t\tilde{\sigma} + \lambda_t^{\mathcal{MB}}\left(\nu_t + \frac{a}{q_t^K}\right) = (1-\vartheta_t)\tilde{\sigma}^2 + \frac{\lambda_t^{\mathcal{MB}}\nu_t}{1-\vartheta_t}$$

Define $\Delta i_t := i_t - i_t^{\mathcal{MB}} = \lambda_t^{\mathcal{MB}}\nu_t$. Intuitively, Δi_t represents a liquidity premium - the spread between a frictionless nominal interest rate and the return on money. Solve for $\mathbb{E}_t[d\vartheta_t]$:

(i_t: shadow nominal rate i_t on nominal asset without transaction services)

$$\boxed{\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \Delta i_t\right)\vartheta_t dt}$$

ν-FOC and Quantity Equation

- From capital return and functional form $\mathfrak{T}_t(\nu) = \frac{a}{(\mathfrak{z}-1)\bar{\nu}}\left[\left(\frac{\nu}{\bar{\nu}}\right)^{\mathfrak{z}-1} - \left(\frac{\nu_t^{\text{eq}}}{\bar{\nu}}\right)^{\mathfrak{z}-1}\right]$,

$$\frac{\partial \mathbb{E}[dr_t^K(\iota_t,\nu_t)]/dt}{\partial \nu_t} = -\frac{a}{q_t^K}\frac{1}{\bar{\nu}^2}\left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}-2} = -\frac{\vartheta_t}{1-\vartheta_t}\frac{1}{\bar{\nu}}\left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}-1}$$

  Plug this expression (and $\theta_t = \vartheta_t$) into ν_t-FOC:

$$\lambda_t^{\mathcal{MB}} = \frac{1}{\bar{\nu}}\left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}-1} \Rightarrow \boxed{\Delta i_t = \lambda_t^{\mathcal{MB}}\nu_t = \left(\frac{\nu_t}{\bar{\nu}}\right)^{\mathfrak{z}}}$$

- Solving for ν_t, plugging into definition of ν_t, and aggregating yields the quantity equation

$$\mathcal{P}_t Y_t = \underbrace{(\Delta i_t)^{\frac{1}{\mathfrak{z}}}\,\bar{\nu}}_{\nu_t}\,\mathcal{MB}_t$$

- *Remark*: in the CIA limit , $\mathfrak{z}\to\infty$, two possible cases

$$\begin{cases} \nu_t < \bar{\nu} & \Delta i_t = 0 \\ \nu_t = \bar{\nu} & \Delta i_t \geq 0 \end{cases}$$

Steady State Equilibrium

- Assume $\check{\mu}_t^{\mathcal{MB}} = \check{\mu}^{\mathcal{MB}}$ is constant and consider steady state ($\mu_t^{\vartheta} = 0$)
  1. Money Valuation Equation

$$\rho + \check{\mu}^{\mathcal{MB}} = (1-\vartheta)^2 \tilde{\sigma}^2 + \Delta i$$

  2. Quantity Equation

$$\Delta i = \left(\frac{\nu}{\bar{\nu}}\right)^{\mathfrak{z}} = \left(\frac{1}{\bar{\nu}} \frac{1-\vartheta+\phi\rho}{\vartheta} \frac{a}{1+\phi\check{a}}\right)^{\mathfrak{z}}$$

Remark: last equality follows from equations derived previously

$$\nu_t = \frac{1-\vartheta_t}{\vartheta_t} \frac{a}{q_t^K}, \qquad q_t^K = (1-\vartheta_t) \frac{1+\phi\check{a}}{1-\vartheta_t+\phi\rho}$$

- Combining the two equations yields nonlinear equation for steady-state ϑ
- No closed-form solution except in special cases, e.g.
  - no transaction costs ($\bar{\nu} \to \infty$) (as analyzed previously)
  - cash-in-advance limit ($\mathfrak{z} \to \infty$) (will consider this one next)

Special Case: Cash in advance constraint ($\mathfrak{z} \to \infty$)

Two cases:

1. $\Delta i = 0, \nu < \bar{\nu}$: valuation equation (store of value role) determines ϑ,

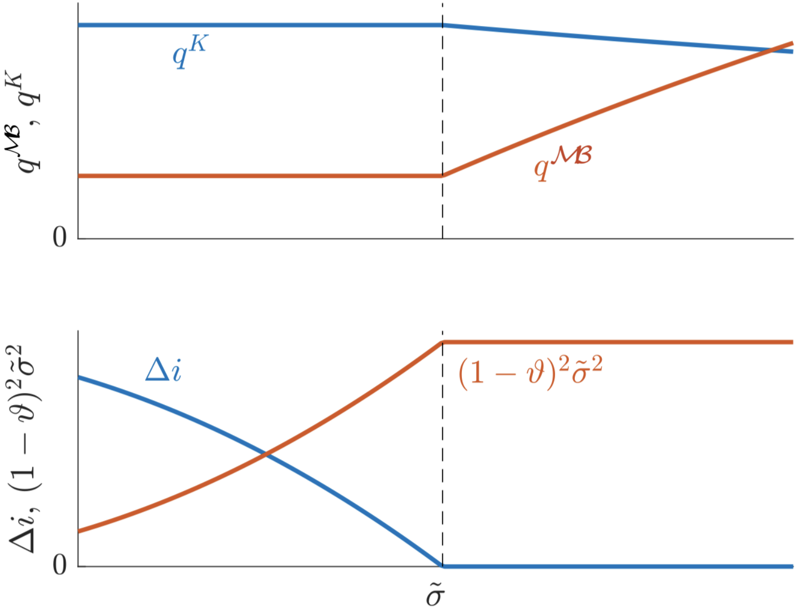
$$\rho + \check{\mu}^{\mathcal{MB}} = (1-\vartheta)^2 \tilde{\sigma}^2$$

2. $\Delta i > 0, \nu = \bar{\nu}$: quantity equation (medium of exchange role) determines ϑ,

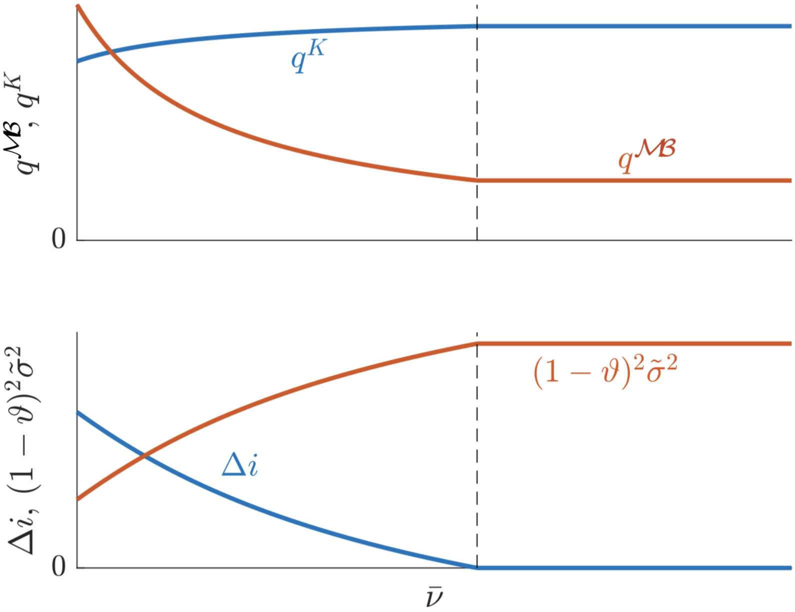
$$\frac{1}{\bar{\nu}} \frac{1 - \vartheta + \phi\rho}{\vartheta} \frac{a}{1 + \phi \check{a}} = 1$$

	Medium of Exchange	**Store of Value**
ϑ	$\vartheta = \frac{(1+\phi\rho)a}{a+(1+\phi\check{a})\bar{\nu}}$	$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \check{\mu}^{\mathcal{MB}}}}{\tilde{\sigma}}$
Δi	$\Delta i = \rho + \check{\mu}^{\mathcal{MB}} - \left(\frac{\bar{\nu}+\phi(\check{a}\bar{\nu}-a\rho)}{a+(1+\phi\check{a})\bar{\nu}}\right)^2 \tilde{\sigma}^2$	$\Delta i = 0$
$q^{\mathcal{MB}}$	$q^{\mathcal{MB}} = \frac{a}{\bar{\nu}}$	$q^{\mathcal{MB}} = \frac{(\tilde{\sigma} - \sqrt{\rho+\check{\mu}^{\mathcal{MB}}})(1+\phi\check{a})}{\sqrt{\rho+\check{\mu}^{\mathcal{MB}}} + \phi\tilde{\sigma}\rho}$
q^K	$q^K = \frac{1+\phi(\check{a} - a\rho/\bar{\nu})}{1+\phi\rho}$	$q^K = \frac{\sqrt{\rho+\check{\mu}^{\mathcal{MB}}}(1+\phi\check{a})}{\sqrt{\rho+\check{\mu}^{\mathcal{MB}}} + \phi\tilde{\sigma}\rho}$
ι	$\iota = \frac{\check{a} - \rho(1+a/\bar{\nu})}{1+\phi\rho}$	$\iota = \frac{\check{a}\sqrt{\rho+\check{\mu}^{\mathcal{MB}}} - \tilde{\sigma}\rho}{\sqrt{\rho+\check{\mu}^{\mathcal{MB}}} + \phi\tilde{\sigma}\rho}$

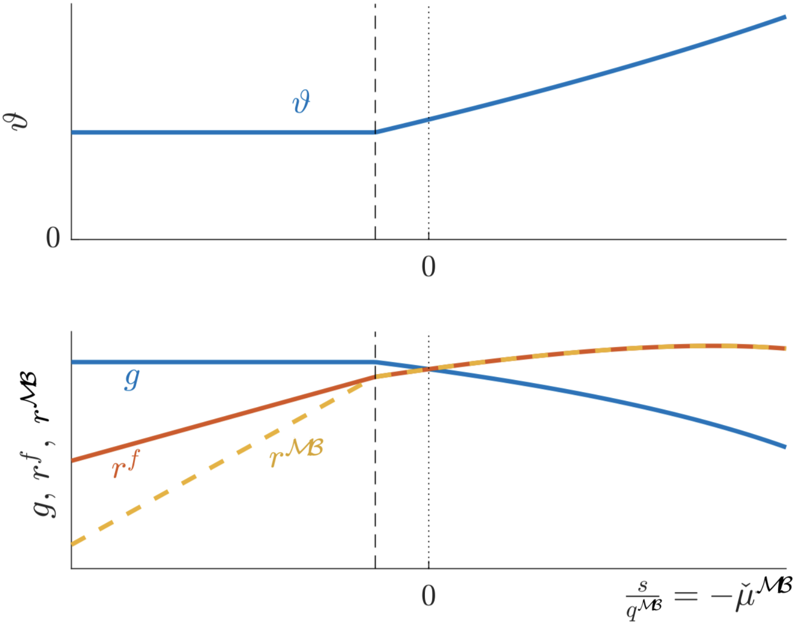
Comparative Statics w.r.t. Financial Friction ($\tilde{\sigma}$)



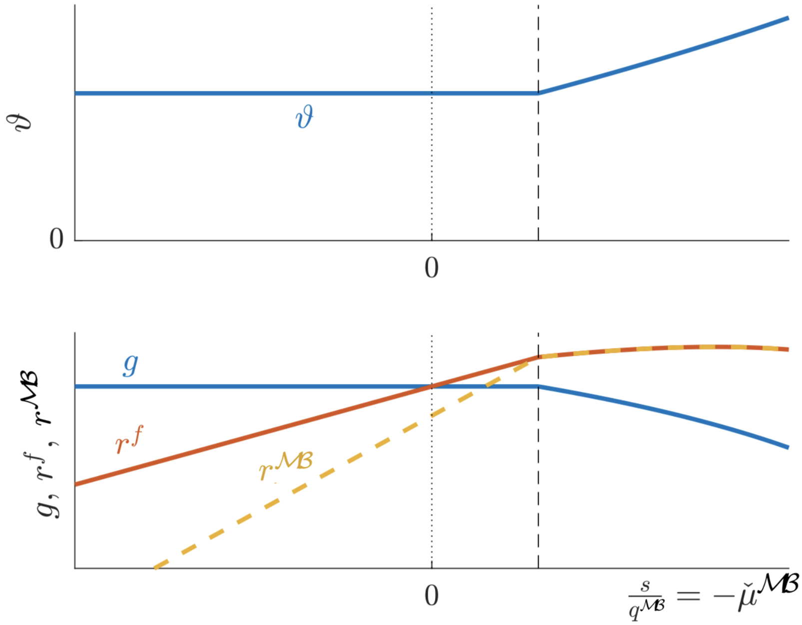
Comparative Statics w.r.t. Monetary Friction ($\bar{\nu}$)



Comparative Statics w.r.t. Fiscal Backing ($s/q^{\mathcal{MB}} = -\check{\mu}^{\mathcal{MB}}$)



Comparative Statics w.r.t. Fiscal Backing – Smaller $\bar{\nu}$



Determinants of Value of Money, Sources of Seigniorage

- Consider again the integral form of the money valuation equation

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \left(-\breve{\mu}_{t'}^{\mathcal{MB}} + (1-\vartheta_{t'})^2 \tilde{\sigma}^2 + \Delta i_{t'} \right) \vartheta_{t'} dt' \right]$$

- This emphasizes three sources of the value of money:
  1. cash flows from fiscal backing
  2. risk sharing benefits from money as a safe asset (store of value)
  3. transaction benefits from money as a medium of exchange
- Again, fiscal backing may actually be negative ($\breve{\mu}^{\mathcal{MB}} > 0$)
  - then money may still be valued if other benefits are sufficiently strong
  - the government then extracts seigniorage revenue from issuing more money
  - money is then a (rational) bubble

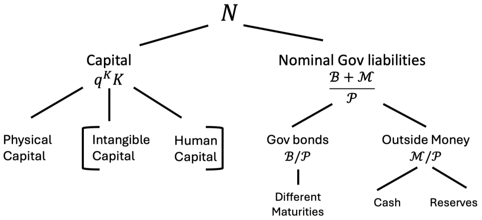
Money and Growth: Tobin Effect

- Observation from all three variants of the model: investment & growth depend negatively on money portfolio demand (ϑ_t)
- Intuition: money crowds out real investment
  - consumption demand depends on total wealth ($C_t = \rho(q^K + q^{\mathcal{MB}})K_t$)
  - but money is unproductive: higher $q^{\mathcal{MB}}$ increases wealth without raising output ($Y_t = aK_t$)
  - since output is fixed, investment must fall to meet increased consumption demand, reducing future capital and thus future output
- Formalizes argument by Tobin (1965) that portfolio choice between monetary and capital assets is a key determinant of real investment
- Aside: Tobin effect distinguishes outside money from bank-created inside money (compare Merkel, 2020)

Outline

Money and Nominal Government Debt

- Previous model: money is the only government liability
- More realistic: government issues money $\mathcal{M}_t$ and nominal bonds $\mathcal{B}_t$
  - both serve as a store of value
  - but only $\mathcal{M}_t$-component of govt. liabilities is medium of exchange



- Model analysis is the same as in the baseline model, except that we need to reinterpret some variables:
  - we need to reinterpret some variables:
    - $q_t^{\mathcal{MB}} \to q_t^{\mathcal{M}} + q_t^{\mathcal{B}}$ (value of all government liabilities)
    - $\vartheta_t \to \frac{q_t^{\mathcal{M}}+q_t^{\mathcal{B}}}{q_t^{\mathcal{M}}+q_t^{\mathcal{B}}+q_t^K}$ (nominal wealth share)
    - $\check{\mu}_t^{\mathcal{MB}} \to \frac{\mathcal{M}_t\check{\mu}_t^{\mathcal{M}}+\mathcal{B}_t\check{\mu}_t^{\mathcal{B}}}{\mathcal{M}_t+\mathcal{B}_t}$ (average dilution rate of nom. liabilities)
  - we need to allow for time-varying transaction benefits:

    $\bar{\nu}_t$ [money only model] $= \left(\frac{\mathcal{M}_t}{\mathcal{B}_t+\mathcal{M}_t}\right)^{1-1/\mathfrak{z}} \bar{\nu}$ [bond and money model]
  - we need to derive new valuation equations:

    $\mu_t^{\vartheta} = \rho + \check{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \vartheta_t^{\mathcal{M}}\Delta i_t$ (Govt. Liability Valuation Equation)

    $\frac{\mathcal{B}_0+\mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0[\int_0^T e^{-r^f t} s_t K_t \mathrm{d}t] + \mathbb{E}_0[\int_0^T e^{-r^f t}\Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t}\mathrm{d}t] + \mathbb{E}_0[e^{-r^f T}\frac{\mathcal{B}_T+\mathcal{M}_T}{\mathcal{P}_T}]$ (FTPL)

Derivation for Govt. Liab. Valuation Equation and FTPL

Long-Term Government Bonds

- We can further distinguish money and bonds by lengthening *bond duration*
- In previous extension, bonds have infinitesimal duration
  $\Rightarrow$ nominal bond price $= 1$
- With long-duration bonds, the nominal bond price can differ from 1
- Turns out to not matter a lot: the maturity composition of government bonds is irrelevant for
  - the real allocation
  - the equilibrium path of ϑ_t
  - ... but it does matter for nominal quantities, the price level, and inflation
- Modigliani-Miller intuition: the underlying "assets" backing bonds (taxes and safe asset services) are independent of maturity structure, hence so should be the total bond value

Outline

Monetary Policy

1. **“Pure” Monetary/Interest Rate Policy $i_t^{\mathcal{B}}$**
   (no “fiscal implications”, $\breve{\mu}_t^{\mathcal{MB}}$ remains unchanged)
   - ***i*-policy (Neo-Fisherian)**
     unexpected permanent increase in $i_t^{\mathcal{B}}$ and $i_t^{\mathcal{M}}$ without a change in Δi_t at $t = 0$
     $\Rightarrow$ at $t = 0$: ϑ_0 and $\mathcal{P}_0$ unchanged, $\breve{\mu}_t^{\mathcal{MB}}$ constant, i.e. $\mu_t^{\mathcal{MB}}$ increases
     $\Rightarrow$ at $t > 0$: increase in inflation (one-for-one), super-neutrality of money (growth)
   - **Δ*i*-policy (Monetarism)**
     unexpected permanent increase in Δi_t and no change in $i_t^{\mathcal{MB}}$, which is defined as $\frac{\mathcal{M}_t i_t^{\mathcal{M}} + \mathcal{B}_t i_t^{\mathcal{B}}}{\mathcal{M}_t + \mathcal{B}_t}$ in the case with separated money and bonds
     $\Rightarrow$ at $t = 0$: ϑ jumps to a new permanently higher level, $\mathcal{P}_0$ drops
     $\Rightarrow$ at $t > 0$: $\mu_t^{\mathcal{MB}}$ is constant, $\pi = i_t^{\mathcal{MB}} - g$ rises due to Tobin effect
2. **“Non-pure” Interest Rate Policy with Fiscal Reaction**
   (with “fiscal implications”, $\breve{\mu}_t^{\mathcal{MB}}$ changes)
   - ***i*-policy**
     $\Rightarrow$ Fiscal policy adjusts taxes to keep $\mu_t^{\mathcal{MB}}$ constant, then
     Neo-Fisherian policy $\breve{\mu}_t^{\mathcal{MB}}$ has directionally same effect as monetary tightening
     (increase in taxes in order to compensate for lost seigniorage income)

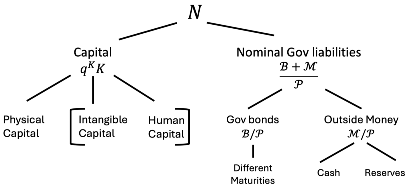
Monetary Policy Implementation

- **Interest on Reserves:**
  - Adjust $i_t^{\mathcal{M}}$, keep $\frac{\mathcal{M}}{\mathcal{M}+\mathcal{B}}$ constant
  - Implement **Neo-Fisherian policy**
- **Open Market Operation:**
  - Keep $i_t^{\mathcal{M}}$ constant, adjust $\frac{\mathcal{M}}{\mathcal{B}+\mathcal{M}}$
  - Implement **Monetarist policy**
    (mixed with some Neo-Fisherian elements since $i^{\mathcal{M}}$ and not $i^{\mathcal{M}\mathcal{B}}$ is kept fixed)

Outline

Introducing Long-Term Government Bonds

- Long-term bond
  - Yields fixed coupon rate $\underline{i}$ on face value $F^{(\underline{i},m)}$ with maturity m
  - Matures at random time with arrival rate $1/m$
  - Nominal price of the bond $P_t^{\mathcal{B}(\underline{i},m)}$
  - Nominal value of all bonds outstanding of a certain maturity: $\mathcal{B}_t^{(m)} = P_t^{\mathcal{B}(\underline{i},m)} F^{(\mathbf{i},m)}$
  - Nominal value of all bonds $\mathcal{B}_t = \sum_m \mathcal{B}_t^{(m)}$



- Special bonds
  - $\mathcal{B}_t^{(0)}$, note $P_t^{\mathcal{B}(0)} = 1$ (price is independent of i_t since coupon is floating rate)
  - $\mathcal{B}_t^{(\infty)}$: Consol bond

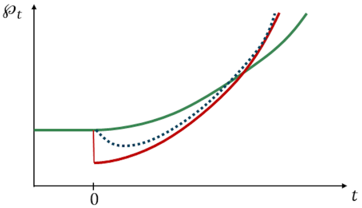
Proposition

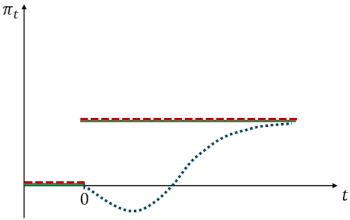
Maturity composition of $\mathcal{B}^{(m)}$ is irrelevant for real allocation and equilibrium path of ϑ_t ... but it matters for nominal quantities, the price level and inflation.

- Modigliani-Miller intuition (in one sector model) (as s-backing is unchanged)

Sims' Stepping on the Rake: "Bond Reevaluation Effect"

- Unexpected permanent increase in $i_t^{(0)}$ at $t = 0$ for all $t > 0$
  $\Rightarrow$ nominal value $\mathcal{B}_t^{(m>0)}$ of any long-term bond declines
  "Pure i-MoPo": keep $\breve{\mu}^{\mathcal{MB}}$ constant, i.e., "debt growth" increases,
  ϑ_t is constant and so is $q_t^{\mathcal{B}}$ (aside $s_t/q_t^{\mathcal{B}}$ also stays constant)
  - At $t = 0$ on impact: as all $\mathcal{B}_0^{(m>0)}$ decline $\Rightarrow$ $\mathcal{P}_0$ has to jump down
  - For $t > 0$: inflation π_t is higher like in Neo-Fisherian setting
    (with price stickiness like dotted curve)





- In sum, "Stepping on the Rake" only changes inflation (price drop) at $t = 0$.
  ... only with price stickiness (price drop down is smoothed out).

Outline

Quantitative Easing (QE)

- Assume $\mu_t^{\mathcal{M}} = \mu_t^{\mathcal{B}}$ for all t
- At $t = 0$ QE in form of an unexpected swap of $\mathcal{B}^{(0)}$-bonds (T-Bill) for money $\mathcal{M}$

T-Bill QE Proposition

T-Bill QE leads to positive price level jump.

Suppose $\mathcal{P}_t$ reacts less, so that real balances $\frac{\mathcal{M}_t}{\mathcal{P}_t}$ expand
$\Rightarrow$ Relaxes CIA constraint and
$\Rightarrow$ permanently lowers Δi (if CIA was binding beforehand)
$\Rightarrow$ lowers "money seigniorage"
$\Rightarrow$ upward jump in the price level (inflation) by

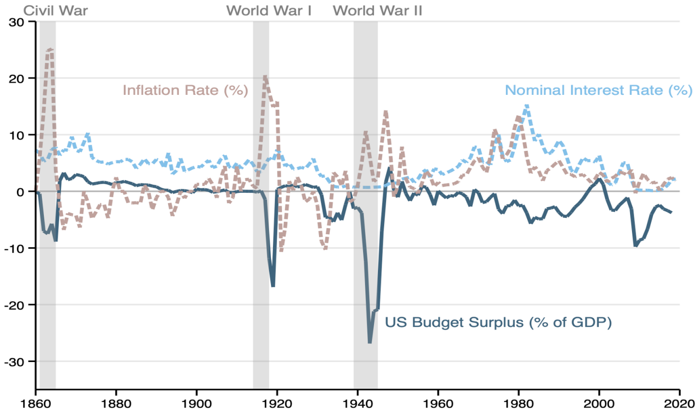
$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} s_s K_s \mathrm{d}s + \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} \mathrm{d}s + \mathbb{E}_t \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$$

The quantity equation (with fixed velocity) $\frac{\mathcal{M}_t}{\mathcal{P}_t} = \frac{C_t}{\nu}$ would also lead to upward jump of the price level.

Outline

Inflation–Fiscal Link

- Friedman (1961): "Inflation is always and everywhere a **monetary phenomenon**"
- Sims (1994): "In a fiat-money economy, inflation is a **fiscal phenomenon**, even more fundamentally than it is a monetary phenomenon".

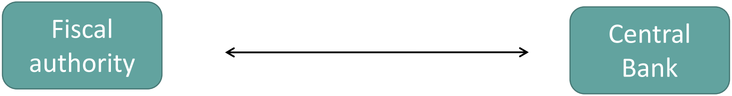


Source: FRED, MeasuringWorth.com, Mitchell (1908)

Remark: Two Inflation-Fiscal Connection

- **FTPL Channel**
  Issue additional bonds to finance new economic stimulus
  + don't change future primary surpluses $s_t K_t$
  $\Rightarrow$ dilutes value of existing bonds (as # of bonds is higher)
  $\Rightarrow$ Inflation
- **Short-run Aggregate Demand Channel**
  Issue additional bonds to finance new economic stimulus
  + Commit to increase $s_t K_t$, so that bond value is not diluted
  ($\Rightarrow$ FTPL Channel is switched off)
  (extra bonds are financed by extra future $s_t K_t$)
  If economic model is:
  - Ricardian $\Rightarrow$ stimulus is neutralized by future taxes
  - Non-Ricardian $\Rightarrow$ stimulus can boost demand/output
    (if there is a negative output gap e.g. in NK models)

Fiscal and Monetary Interaction



- **Monetary dominance**
  - Monetary tightening leads fiscal authority to reduce fiscal deficit
- **Fiscal dominance**
  - Interest rate increase does not reduce primary fiscal deficit
  - . . . only lead to higher inflation

Game of chicken

See YouTube video 4, minute 4:15

Outline

Sargent and Wallace's Unpleasant Monetary Arithmetic

- With medium of exchange role of $\mathcal{M} \to$ but $\tilde{\sigma} = 0$ to avoid possibility of bubble mining.
- Sargent and Wallace (SW) point out that "*even in an economy that satisfies monetarist assumptions [...] monetary policy cannot permanent control [...] inflation*"
  - They consider an economy in which $\mathcal{P}_t$ is fully determined by money demand $\nu \mathcal{M}_t = \mathcal{P}_t Y_t$
  - But the fiscal authority is "dominant": sets *deficits* independently of monetary policy actions
- SW emphasize seigniorage from money creation
  - Fiscal needs determine the total present value of *seigniorage*.
  - If monetary authority provides less, lower seigniorage today raises future government debt.
  - Required fiscal backing remains and the shortfall must be made up later via money printing.
  - **Tight money now means higher inflation eventually (Unpleasant Arithmetic)**.
- Controlling inflation is not always within the central bank's hands. Even when money demand determines the price level, fiscal policy can dominate in the long run.

Sargent and Wallace (1981)

Outline

The Determinacy Question

- So far: analysis of value of money restricting attention to monetary steady states
  - but this might not be the only equilibrium
  - in fact, for constant $\breve{\mu}^{\mathcal{MB}}$-policies: a second, non-monetary steady state exists
- Important question in monetary economics: under which conditions is the equilibrium unique?
- Why does this matter?
  - want to use model to analyze comparative statics, policy actions, transmission mechanisms, etc.
  - but this is difficult if there are multiple equilibria
  - which equilibria should we compare?
  - "intrinsic" effects of policy actions vs. effects of changing coordination

Notions of Uniqueness

- Strong notion: unique rational expectations (RE) equilibrium
- Various weaker notions in monetary literature:
  - locally unique RE equilibrium: no other equilibrium remains always nearby
    - requires non-negligible change in private-sector beliefs to coordinate on different one
  - unique Markov-perfect / minimum state variable equilibrium: no other equilibrium as function of minimal state space
    - without aggregate risk and time trends: steady state uniqueness
  - unique asymptotically monetary equilibrium: for all other RE equilibria, value of money vanishes in the long run
    - only equilibrium consistent with expectation that value of money will remain bounded away from zero
- Here: let's focus on strong notion and third weak notion

Remark: Government Policy Paths versus Rules

- Determinacy may depend on government policy
- For many questions, it is sufficient to specify policy along the equilibrium path
- However, for determinacy, this is insufficient:
  - we need to contemplate what the government would do if markets coordinated on different outcomes
  - to do so, we need a full government policy rule (or strategy) that specifies how the government would act at off-equilibrium nodes of the game tree
- Once we specify policy rules, we have to be careful that they are feasible also off-equilibrium, e.g.:
  - the government cannot violate its flow budget constraint at off-equilibrium prices
  - the government cannot commit to fund a primary deficit (negative taxes) in states in which money is worthless

Outline for Determinacy Analysis

- In the following: analyze determinacy in the money model
- To simplify matters:
  - assume no physical investment and no government expenditure, $\phi \to \infty$, then wealth per unit of capital is constant:

$$q_t = q = \frac{a}{\rho}$$

  - keep only one motive for holding money active at a time (backing, safety, transactions)
- Recall that money valuation equation

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}} - (1-\vartheta_t)^2\tilde{\sigma}^2 - \Delta i_t\right)\vartheta_t dt$$

  must hold in any RE equilibrium
  - in addition, any solution with $\vartheta_t \in [0,1]\ \forall t \geq 0$ corresponds to a valid equilibrium
  - $\vartheta_t < 0$ and $\vartheta_t > 1$ inconsistent with free disposal of money or capital

Outline

Fiscal Theory: Determinacy with Fiscal Backing

- Return to frictionless benchmark, $\tilde{\sigma} = 0$, $\mathfrak{T} \equiv 0$
- Suppose the fiscal authority follows the following policy rule:
  - set constant taxes $\tau > 0$ after any history
  - implies that also primary surplus-capital ratio $s_t = \tau a$ is constant and positive
- Money valuation equation simplifies to

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}_t^{\mathcal{MB}}\right)\vartheta_t dt = \left(\rho\vartheta_t - \frac{s_t}{q}\right)dt = \rho\left(\vartheta_t - \tau\right)dt$$

- This has a unique solution contained in $[0, 1]$:

$$\vartheta_t = \vartheta^{ss} := \tau$$

  - if $\vartheta_t > \vartheta^{ss}$, $\mathbb{E}_t[d\vartheta_t] > 0 \rightarrow$ solution eventually > 1
  - if $\vartheta_t < \vartheta^{ss}$, $\mathbb{E}_t[d\vartheta_t] < 0 \rightarrow$ solution eventually > 1
- Conclusion (*Fiscal Theory of the Price Level*): fiscal backing can generate a determinate value of money

FTPL: The Role of Fiscal Policy

- The previous logic generalizes if we replace constant s by any path of *positive* s_t
  - positive is essential: the government must expend real resources to provide backing
  - strictly speaking, $s_t > 0$ for all t not needed, positive present value is sufficient
- But the nature of the fiscal rule matters
  - A rule that fixes $\check{\mu}^{\mathcal{MB}} \leq -\rho$ instead of s is consistent with continuum of RE equilibria:
$$\mathbb{E}_t[d\vartheta_t] = \left(\rho + \check{\mu}^{\mathcal{MB}}\right) \vartheta_t dt \Leftrightarrow \vartheta_t = \vartheta_0 e^{(\rho + \check{\mu}^{\mathcal{MB}})t}$$
  - A rule that adjusts taxes to "keep debt sustainable", e.g., $\tau_t = \tau^0 + \alpha(\vartheta_t - \tau^0)$ ($\alpha > 1$), leads to indeterminacy:
$$\begin{aligned} \mathbb{E}_t[d\vartheta_t] &= \rho\left(\vartheta_t - \tau_t\right) dt = \rho\left(1-\alpha\right)\left(\vartheta_t - \tau^0\right) dt \\ \Leftrightarrow \qquad \vartheta_t &= \tau^0 + e^{-\rho(\alpha-1)t}(\vartheta_0 - \tau^0)\vartheta_0 e^{-(\alpha-1)t} \end{aligned}$$
- Latter case is the baseline assumption in NK literature $\rightarrow$ neutralizes effect on fiscal backing on determinacy

Outline

Bubble Theory: Global Indeterminacy in Models

- Suppose $s = \check{\mu}^{\mathcal{MB}} = 0$ and either of the following
  - (a) there is idiosyncratic risk $\tilde{\sigma} > \sqrt{\rho}$
  - (b) there are transaction costs $\mathfrak{T}_t(\nu) > 0$
- We focus on case (a) for concreteness, case (b) is similar

  (with some complications, see lecture notes)
- The money valuation equation is then

$$\mathbb{E}_t[d\vartheta_t] = \underbrace{\left(\rho - (1-\vartheta_t)^2\tilde{\sigma}^2\right)}_{\text{strictly increasing in } \vartheta_t}\vartheta_t dt$$

- This has a continuum of solutions contained in $[0, 1]$
  - the non-monetary steady state, $\vartheta_t = 0$
  - the monetary steady state, $\vartheta_t = \vartheta^{ss} := \frac{\tilde{\sigma}-\sqrt{\rho}}{\tilde{\sigma}}$
  - a nonstationary equilibrium for each $\vartheta_0 \in (0, \vartheta^{ss})$ that features $\vartheta_t > 0$ for all t but $\vartheta_t \to 0$ as $t \to 0$

Global Indeterminacy: Intuition

- Conclusion from last slide: RE equilibrium is not unique $\rightarrow$ indeterminacy
- This is because money does not provide intrinsic value
- Instead, it generates services from trading it:
  - as safe asset: provides risk sharing because it is *sold* to smooth idiosyncratic shocks
  - as medium of exchange: provides transaction services because it is used to *pay* for goods
- Value for individual therefore depends on resale value in exchange
  - but resale value depends on value for buyer
  - which in turn depends on resale value in next transaction

  $\vdots$

$\rightarrow$ In bubble theories, value of money depends on *social coordination*: infinite chain of beliefs how others will value it in future transactions

Bubble Theories and Weak Determinacy

- Despite this indeterminacy, there is a good reason to select $\vartheta_t = \vartheta^{ss}$
  - it is the only equilibrium with asymptotically valued money, $\lim_{t\to\infty} \vartheta_t > 0$
  - to sustain any other equilibrium, agents must believe there is eventual (hyper-)inflation that erodes the value of money
- Aside, $\vartheta_t = \vartheta^{ss}$ has also other properties that sets it apart:
  - it is locally unique and the only RE equilibrium that is
  - it is a minimum state variable equilibrium and the only one in which money has value
  - it is the only equilibrium that survives if the is a positive probability of some (arbitrarily small) fiscal backing in the future

Outline

A Model without Money as an Asset

- Take the frictionless benchmark and set $\mathcal{MB}_t = 0$ (which implies $\tau = s = 0$)
- Then $\vartheta = 0$ and all remaining model equations remain valid
- The real side of this model is trivial:
  - capital grows at a constant rate g
  - agents consume $C_t = aK_t$ (there is no idiosyncratic risk)
  - the real interest rate is $r = \rho + g$
- We can still add money as a unit of account by adding a *zero net supply* nominal bond
  - nominal interest rate i_t controlled by the central bank
  - portfolio choice leads to a Fisher equation (without risk)

$$i_t = r + \pi_t, \qquad \pi_t := \mu_t^{\mathcal{P}}$$

- Question: is there a unique equilibrium price level path $\mathcal{P}_t$?
  - answer: it depends on i-policy (and the notion of uniqueness)

Indeterminacy under Exogenous Interest Rates

- Suppose the central bank sets an exogenous time path for i_t
- Then by the Fisher equation

$$\pi_t = i_t - r = i_t - \rho - g$$

  is determined
- But the initial price level $\mathcal{P}_0$ is not
- In addition, even π_t is only determined among all perfect foresight equilibria
  - there are additional sunspot RE equilibria with different inflation (and price volatility)

(Local) Determinacy with Wicksellian Feedback Rules

- Let's instead assume the central bank follows a price level feedback rule

$$i_t = i_t^0 + \phi_{\mathcal{P}} \log \mathcal{P}_t, \qquad \phi_{\mathcal{P}} > 0$$

  - i_t^0 is an exogenous (bounded) intercept path
  - $\phi_{\mathcal{P}} \log \mathcal{P}_t$ incorporates feedback from observed price levels to i_t
- This is called a *Wicksellian interest rate rule* (Wicksell 1898)
- Combining this rule with $d\mathcal{P}_t = \pi_t \mathcal{P}_t dt$ and the Fisher equation yields

$$d \log \mathcal{P}_t = d\mathcal{P}_t / \mathcal{P}_t = \left(i_t^0 - r + \phi_{\mathcal{P}} \log \mathcal{P}_t \right) dt$$

$$\Rightarrow \qquad \log \mathcal{P}_t = e^{\phi_{\mathcal{P}} t} \left(\log \mathcal{P}_0 - \log \mathcal{P}_0^* \right) - \int_t^\infty e^{-\phi_{\mathcal{P}}(s-t)} (i_s^0 - r) ds, \qquad \log \mathcal{P}_0^* := -\int_0^\infty e^{-\phi_{\mathcal{P}} t} (i_t^0 - r) dt$$

- All but one solutions (the one with $\mathcal{P}_0 = \mathcal{P}_0^*$) lead to unbounded $\mathcal{P}_t$ & π_t
  - there is nothing wrong with these unbounded solutions economically
  - but if we add as an additional selection rule that we seek bounded solutions, then there is a unique $\mathcal{P}_t$ solution
  - in addition, that one is the only locally unique one

(Local) Determinacy with Taylor Rules

- Contemporary literature: inflation instead of price level feedback (Taylor 1993)

$$i_t = i_t^0 + \phi_\pi \pi_t, \qquad \phi_\pi > 1$$

- These do *not* work in continuous time without additional inertia, e.g.
  - interest rate smoothing
  - long-term nominal bonds
  - sticky prices
- With such inertia, such a rule can determine the path of inflation in the same way as a Wicksellian rule
  - i.e., we need to add the selection criteria "bounded inflation"
- But it will still not determine the price *level* unless prices are sticky

Appendix: Derivation for Govt. Liab. and FTPL Equation

$$
\begin{aligned}
H_t =& e^{-\rho t}\log c_t - \xi_t c_t \\
&+ \xi_t n_t \left\{ (1-\theta_t)\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{dt} + \theta_t \underbrace{\left[(1-\theta_t^{\mathcal{M}})\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} + \theta_t^{\mathcal{M}}\frac{\mathbb{E}_t[dr_t^{\mathcal{M}}]}{dt}\right]}_{\frac{\mathbb{E}_t[dr_t^{\mathcal{M}\mathcal{B}}]}{dt}:=} \right\} \\
&- \xi_t n_t \tilde{\varsigma}_t (1-\theta_t)\tilde{\sigma} \\
&+ \lambda_t^{\mathcal{M}} \xi_t n_t \left[\theta_t \theta_t^{\mathcal{M}} \nu_t - (1-\theta_t)\frac{a}{q_t^K}\right]
\end{aligned}
$$

First order conditions w.r.t:

$$
\theta_t^{\tilde{i}}: \quad \frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}\mathcal{B}}]}{\mathrm{d}t} = \tilde{\varsigma}\tilde{\sigma} + \lambda_t^{\mathcal{M}}\left(\nu_t\theta_t^{\mathcal{M}} + \frac{a}{q_t^K}\right)
$$

$$
\theta_t^{\mathcal{M}\tilde{i}}: \quad \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{\mathrm{d}t} = \lambda_t^{\mathcal{M}}\nu_t
$$

$$
\nu_t^{\tilde{i}}: \quad (1-\theta_t)\frac{\partial\mathbb{E}[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]/\mathrm{d}t}{\partial\nu_t} + \lambda_t^{\mathcal{M}}\theta_t\theta_t^{\mathcal{M}} = 0
$$

Recall Return Equation and Take Differences

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}(\iota_t,\nu_t)]}{dt} = \frac{a - \mathscr{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K} + \frac{q_t^{\mathcal{M}}\check{\mu}_t^{\mathcal{M}} + q_t^{\mathcal{B}}\check{\mu}_t^{\mathcal{B}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} \quad (1)$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{dt} = \check{\mu}_t^{\mathcal{B}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{B}}} = i_t^{\mathcal{B}} - \pi_t \quad (2)$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{dt} = \check{\mu}_t^{\mathcal{M}} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^{\mathcal{M}}} = i_t^{\mathcal{M}} - \pi_t \quad (3)$$

- Take difference (2) and (3): $\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{M}}]}{\mathrm{d}t} = \Delta i_t$
- Take weighted sum of (2) and (3):

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{MB}}]}{\mathrm{d}t} = \underbrace{\vartheta_t^{\mathcal{B}}\check{\mu}_t^{\mathcal{B}} + \vartheta_t^{\mathcal{M}}\check{\mu}_t^{\mathcal{M}}}_{\check{\mu}_t^{\mathcal{MB}}} + \vartheta_t^{\mathcal{B}}\check{\mu}_t^{q^{\mathcal{B}}} + \vartheta_t^{\mathcal{M}}\check{\mu}_t^{q^{\mathcal{M}}} + \Phi(\iota_t^{\tilde{i}}) - \delta \quad (4)$$

- Take difference of (1) and (4)

$$\frac{a - \mathscr{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K} + \frac{1}{1-\vartheta_t}\check{\mu}_t^{\mathcal{MB}} + \mu_t^{q^K} - \underbrace{\vartheta_t^{\mathcal{B}}\mu_t^{q^{\mathcal{B}}} + \vartheta_t^{\mathcal{M}}\mu_t^{q^{\mathcal{M}}}}_{=-\mu_t^{\vartheta}/(1-\vartheta_t)}$$

Government Liability Valuation Equation

- Plug into FOC w.r.t. θ_t:

$$\underbrace{\frac{a - \mathscr{G} - \iota_t^{\tilde{i}} - \mathfrak{t}(\nu_t^{\tilde{i}})}{q_t^K}}_{\substack{=\rho/(1-\vartheta_t) \\ \text{by goods-mkt clearing}}} + \frac{1}{1-\vartheta_t}\check{\mu}_t^{\mathcal{MB}} - \frac{\mu_t^{\vartheta}}{1-\vartheta_t} = \underbrace{\tilde{\varsigma}_t\tilde{\sigma}}_{\substack{=(1-\vartheta_t)\tilde{\sigma}^2 \\ \text{by log utility}}} + \lambda_t^{\mathcal{M}} \underbrace{\left(\theta_t^{\mathcal{M}}\nu_t + \frac{a}{q_t^K}\right)}_{\substack{=\frac{\vartheta_t^{\mathcal{M}}}{1-\vartheta_t}\nu_t \\ \text{by volatility def}}}$$

- Plug into FOC w.r.t. $\vartheta_t^{\mathcal{M}}$: $\Delta i_t = \lambda_t^{\mathcal{M}}\nu_t$

Government Liability Valuation Equation:

$$\mu_t^{\vartheta} = \rho - (1-\vartheta_t)^2\tilde{\sigma}^2 + \check{\mu}_t^{\mathcal{MB}} - \vartheta_t^{\mathcal{M}}\Delta i_t$$

FTPL-Equation with $\mathcal{B}$ and $\mathcal{M}$

- Money valuation equation for log utility $\gamma = 1$:

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t (\underbrace{\rho + \overbrace{g}^{\Phi(\iota)-\delta} - g - (1-\vartheta_t)^2 \tilde{\sigma}^2}_{=r^f - g} + \breve{\mu}_t^{\mathcal{MB}} - \vartheta^{\mathcal{M}} \Delta i_t)$$

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \vartheta_t N_t$$

$$\Rightarrow d\left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right) = \left(r^f - \cancel{g} + \breve{\mu}^{\mathcal{MB}} - \vartheta^{\mathcal{M}} \Delta i + \overset{\frac{d\mathcal{N}_t}{\mathcal{N}_t} = gdt}{\overset{\downarrow}{\cancel{g}}}\right)\left(\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right) dt$$

- Integrate forward:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T e^{-r^f t} \underbrace{(-\breve{\mu}_t^{\mathcal{MB}} + \vartheta_t^{\mathcal{M}} \Delta_i) \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}}_{=sK_t + \frac{\mathcal{M}_t}{\mathcal{P}_t} \Delta_i} \mathrm{d}t + e^{-r^f T} \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\right]$$

FTPL Equation:

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0\left[\int_0^T e^{-r^f t} s_t K_t \mathrm{d}t\right] + \mathbb{E}_0\left[\int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t\right] + \mathbb{E}_0\left[e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]$$

FTPL-Equations with $\mathcal{B}$ and $\mathcal{M}$: Joint and Separately

- Two ways to write FTPL equation

$$\frac{\mathcal{B}_0 + \mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} s_t K_t \mathrm{d}t + \mathbb{E}_0 \int_0^T e^{-r^f t} \Delta i_t \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}$$

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} s_t K_t \mathrm{d}t + \mathbb{E}_0 \int_0^T e^{-r^f t} \mu_t^{\mathcal{M}} \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{B}_T}{\mathcal{P}_T}$$

- Take difference:

$$\frac{\mathcal{M}_0}{\mathcal{P}_0} = \mathbb{E}_0 \int_0^T e^{-r^f t} (\Delta i_t - \mu_t^{\mathcal{M}}) \frac{\mathcal{M}_t}{\mathcal{P}_t} \mathrm{d}t + \mathbb{E}_0 e^{-r^f T} \frac{\mathcal{M}_T}{\mathcal{P}_T}$$

(may contain bubble term when take $T \to \infty$)

Back

Sargent and Wallace (1981)

- Assume that in equilibrium
  1. the payment constraint is always binding
  2. surpluses satisfy $s_t = \underline{s}, \underline{s} \leq 0$ (constant deficit-GDP ratio)
  3. $\nu > \rho$ (given log-utility)
- Then nominal wealth shares must satisfy:

$$\vartheta_t \vartheta_t^{\mathcal{M}} = \rho/\nu \quad \text{(from goods market clearing condition)}$$

$$\vartheta_t \vartheta_t^{\mathcal{B}} = \int_t^{\infty} \rho e^{-\rho(t'-t)}(s_{t'} + \mathcal{s}_{t'})\mathrm{d}t' = \underbrace{\underline{s}}_{<0} + \int_t^{\infty} \rho e^{-\rho(t'-t)} \mathcal{s}_{t'} \mathrm{d}t'$$

- Suppose after time $T < \infty$ the fiscal authority can take control of $\mu_t^{\mathcal{M}}$.
- Fiscal authority chooses seigniorage to keep debt-GPD ratio constant, i.e.

$$\mathcal{s}_t = \hat{\mathcal{s}}(\vartheta_T^{\mathcal{B}}) := -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}}, \quad t \geq T$$

(there are limites on feasible seigniorage but let's ignore this for simplicity)

- For $t \leq T$, the monetary authority chooses (constant) $\mu^{\mathcal{M}}$ independently
  - Also $\mathcal{s}_t = \mu^{\mathcal{M}} q_t^{\mathcal{M}} = \mu^{\mathcal{M}}(a - \mathcal{g})/\nu =: \mathcal{s}$ is controlled by the monetary authority
- **"Unpleasant Arithmetic" Proposition**:
  Tight money now means higher inflation eventually.
  - The (constant) inflation rate over $[T, \infty)$ is strictly decreasing in $\mu^{\mathcal{M}}$ over $[0, T]$

Why Does the Sargent-Wallace Proposition Hold?

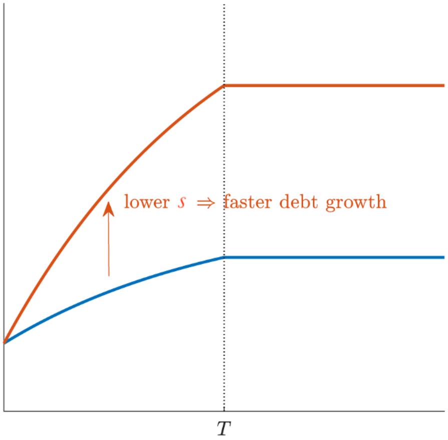
- Iterating government liabilities valuation equation forward in time:

$$\vartheta_T \vartheta_T^{\mathcal{B}} = \vartheta_0 \vartheta_0^{\mathcal{B}} - \int_0^T \rho e^{-\rho t}(\underline{s} + \mathcal{s})\mathrm{d}t$$

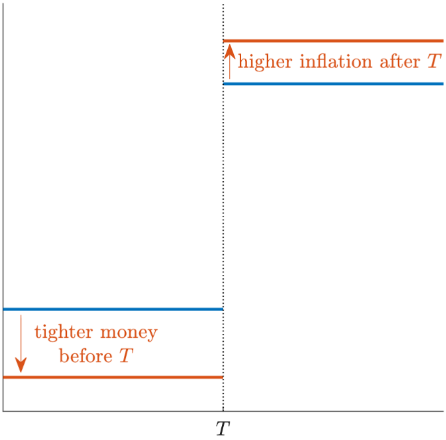
- Lower money $\mu_t^{\mathcal{M}}$ over $[0, T] \Rightarrow$ lower seigniorage transfers $\mathcal{s} = \mu^{\mathcal{M}}(a - \mathcal{g})/\nu$ $\Rightarrow$ debt grows faster
- Higher debt at T: need larger seigniorage thereafter to cover interest payments:
  - recall $\hat{\mathcal{s}}(\vartheta_T^{\mathcal{B}}) = -\underline{s} + \vartheta_T \vartheta_T^{\mathcal{B}}$ is increasing in $\vartheta_T^{\mathcal{B}}$

Illustration of Unpleasant Arithmetic

Real Value of Debt $(\vartheta_t, \vartheta_t^{\mathcal{B}}, q_t^{\mathcal{B}})$



Seigniorage and Inflation $(\mathcal{s}_t, \pi_t)$



Monetary Dominance

- Suppose $T = \infty$: monetary authority is always in control of the money supply
- Is there an equilibrium? (suppose also $\mathscr{s} \neq \vartheta_0 \vartheta_0^{\mathcal{B}} - \underline{s}$)
  - not with constant deficit$/K_t$-ratio $s_t = \underline{s}$
  - but: a constant deficit is not necessarily feasible policy
- Two cases
  1. if $\mathscr{s} > \vartheta_t \vartheta_t^{\mathcal{B}} - \underline{s}$, $s_t = \underline{s} < 0$ remains feasible
     - but fiscal authority will absorb money over time, effective money suppply is smaller than $\mathcal{M}_t$
     - fiscal authority controls inflation
       (e.g. if real debt to K_t ratio is kept constant, outcomes as if $\mathscr{s} = \vartheta_0 \vartheta_0^{\mathcal{B}} - \underline{s}$)
  2. if $\mathscr{s} < \vartheta_t \vartheta_t^{\mathcal{B}} - \underline{s}$, s_t has to rise to avoid default on nominal bonds
     - fiscal authority effectively faces an "intertemporal budget constraint"
     - e.g. smallest constant primary surpluses (per K_t is $s = \vartheta_0 \vartheta_0^{\mathcal{B}} - \mathscr{s}$)

Back