Macrofinance

Lecture 05: Contrasting Financial Frictions

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Course Overview

1. Intro
2. Portfolio & Consumption Choice

Real Macrofinance Models with Heterogeneous Agents

3. Simple Real Macrofinance Models
4. Endogenous (Price of) Risk Dynamics
5. Contrasting Financial Frictions

Immersion Chapters

Money Models

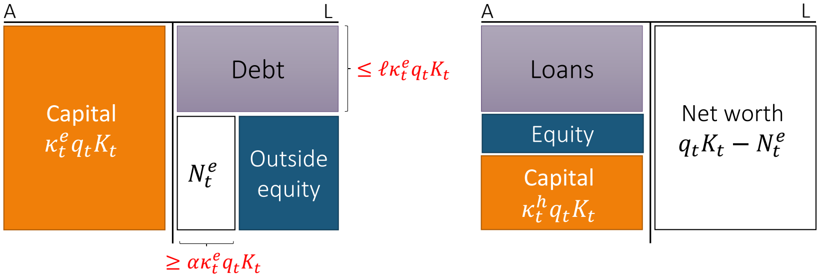
International Macrofinance Models

Main Takeaways

- **Toolboxes: Technical Innovations**
  - Several occasionally binding constraints
    - Short-sale constraint
    - Skin in the game constraint
    - Collateral constraint
  - Price setting social planner to find capital and risk allocation
  - Stationary distribution KFE solution
    - Closed form ODE solution (for one-dimensional η)
    - Simulation (multi-dimensional η)
    - Solve PDE: forward iteration
  - Fan charts
- **Economic Insights:**
  - "Net worth trap" (double-humped shaped distribution)
  - Interaction btw. net worth trap and volatility paradox

Two Sectors: Leverage + Skin-in-the-Game Constraint

- Expert sector 　　　　　　Household sector



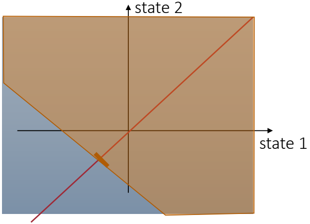
- Households can produce with capital.
  - Productivity $0 < a^h < a^e$
- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- The fraction of aggregate risk held by experts: $\chi_t^e = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}}$
- Experts can issue debt, and outside equity.

  Leverage Constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$.

  Skin in the Game Constraint: $OE_t^e \leqslant \underbrace{(1-\alpha)}_{\in[0,1-\ell]} \kappa_t^e q_t K_t$

Financial Frictions and Distortions

- Belief distortions
  - Match "belief surveys"
- Incomplete markets
  - "natural" leverage constraint (BruSan)
  - Costly state verification (*BGG*)
- + Leverage constraints
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period's price $Rb_t \leqslant q_{t+dt} k_t$ (*KM*)
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+dt}) k_t$ (*BruPed*)
- Search Friction (*DGP*)



Occasionally binding equity constraint

Two Sector Model Setup: Leverage + Skin-in-the Game

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},e}$
- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)\mathrm{d}t\right]$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^h
  $\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$
- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log(c_t^h)\mathrm{d}t\right]$

Friction: Can issue

- Risk-free debt,
  Leverage constraint: $-\theta_t^{e,D} \leqslant \ell\theta_t^{e,K}$ (occasionally binding)
- Outside equity,
  Skin-in-the-Game constraint: $-\theta^{e,OE} \leqslant (1-\alpha)\theta_t^{e,K}$ (occasionally binding)

Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given $\check{\rho}^i := C^i/N^i$-ratio and $\xi^i = SDF^i$ processes for each i — **finance block**
   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   Fisher separation theorem
   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach
2. Evolution of state variable η (and K) — **forward equation**
   (degenerated KFE as η is simply scalar)
3. Value functions — **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility
4. Numerical model solution
5. KFE: Stationary distribution, fan charts

1b. Overview: Different Approaches

- **Approach 1:** Portfolio Optimization θ
  - Optimization via Stochastic Maximum Principle: most general way, but requires setting up Hamiltonian.
  - Optimization via Martingale Approach: complicated when constraints interact in a non-trivial way (here w/o leverage constraint)
- **Approach 2:** Price-taking Social Planner Approach (κ, χ)

1b. Experts' θ-Choice: Stochastic Maximum Principle

- Experts' problem: (let $r_t^{e,j} := \mathbb{E}[dr_t^{e,j}]/dt$)

$$\max_{c_t^e, \iota_t^e, \theta_t^{e,K}, \theta_t^{e,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t\right] \; s.t.$$

$$\mathrm{d}n_t^e = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]\mathrm{d}t + n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

$$(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geqslant 0 \text{ (skin in the game)}, (1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leqslant 1 \text{ (leverage)}$$

- Denote the multiplier on leverage constraint as λ_t^ℓ, multiplier on skin in the game constraint as λ_t^s. The Hamiltonian can be constructed as $\mathcal{H}_t^e =$

$$e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} - \varsigma_t^e \xi_t^e \overbrace{n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e}$$
$$+\xi_t^e n_t^e \lambda_t^\ell\left(1 - (1-\ell)\theta_t^{e,K} - \theta_t^{e,OE}\right) + \xi_t^e n_t^e \lambda_t^\chi\left((1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE}\right)$$

- Objective function is linear in θ (divide through $\xi_t^e n_t^e$)
  $\Rightarrow$ bang-bang (indifferent or at a constraint)
- FOC w.r.t. c_t is separated/de-coupled from FOC w.r.t. θ_ts as well as ι_t^e
  $\Rightarrow$ Fisher Separation Theorem btw. $c_t^e, \theta_t^e, \iota_t^e$

1b. Households' θ-Choice: Stochastic Maximum Principle

- Households' problem:

$$\max_{c_t^h, \iota_t^h, \theta_t^{h,K}, \theta_t^{h,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t} u(c_t^h) \mathrm{d}t\right], \ s.t.$$

$$\mathrm{d}n_t^h = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K} - r_t) + \theta_t^{h,OE}(r_t^{h,OE}(\iota_t^h) - r_t)\right)\right]\mathrm{d}t + n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

$$\theta_t^{h,K} \geqslant 0 \text{ (household short sale constraint)}$$

- Denote the multiplier on the short selling constraint on capital as λ_t^h. The Hamiltonian can be constructed as:

$$\mathcal{H}_t^h = e^{-\rho^h t} u(c_t^h) + \xi_t^h \overbrace{\left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t) + \theta_t^{h,OE}(r_t^{h,OE} - r_t)\right)\right]}^{\mu_t^{n^h} n_t^h}$$

$$-\varsigma_t^h \xi_t^h \overbrace{n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^h} n_t^h} \ + \ \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Linear in θ_t and Fisher Separation Theorem

1b. θ-Choice: Stochastic Maximum Principle

- Experts' FOC w.r.t. θ:

$$\begin{cases} r_t^{e,K} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi & (1) \\ r_t^{OE} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi & (2) \end{cases}$$

- Households' FOC w.r.t. θ:

$$\begin{cases} r_t^{h,K} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) - \lambda_t^h & (3) \\ r_t^{OE} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) & (4) \end{cases}$$

- Take difference btw (1) and (3) as well as btw (2) and (4)

$$\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^h + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi,$$

$$0 = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi,$$

1b. θ-Portfolio Constraints: Figuring out λs

- Focus on the return gap $r_t^{OE} - r_t^{h,K}$ and $r_t^{e,K} - r_t^{OE}$

$$\begin{cases} r_t^{e,K} - r_t^{OE} = \alpha\lambda_t^{\chi} - \ell\lambda_t^{\ell} \\ r_t^{OE} - r_t^{h,K} = \lambda_t^{h} \end{cases}$$

- Household short selling constraint not binding: $\lambda_t^h = 0$
  - $\lambda_t^{\chi} = 0, \lambda_t^{\ell} > 0$ impossible because $r_t^{e,K} > r_t^{h,K}$
    $\Rightarrow$ whenever the leverage constraint binds, so does the skin-in-the-game constraint
  - $\lambda_t^{\chi} > 0, \lambda_t^{\ell} > 0$ and $\lambda_t^{\chi} > 0, \lambda_t^{\ell} = 0$ are possible
    $\Rightarrow$ whenever the skin-in-the-game constraint, the leverage constraint may/or may not bind
- Household short selling constraint binding: $\lambda_t^h > 0$
  - Define $\eta^{e,*}$ as smallest η_t^e such that $\lambda_t^h > 0$
  - $\lambda_t^{\ell} > 0$ impossible because $1/\eta_t^e < 1/\eta^{e,*}$
    $\Rightarrow$ Only skin-in-the-game may bind.
    Intuition: outside equity cannot generate higher return than physical capital

1b. θ-Portfolio to (κ, χ)-Asset/Risk Allocation Constraint

- First order condition (plug in for λs)

$$\frac{a^e - a^h}{q_t} \geqslant \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta\text{-risk premia}}, \quad \text{with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

$$\varsigma_t^e \geqslant \varsigma^h, \quad \text{with equality if } \chi_t^e > \alpha\kappa_t^e$$

- Constraints were translated from θ space to $(\kappa$-$\chi)$-space:
  Skin-in-the-game constraint $\Rightarrow \chi_t^e = \eta_t^{e,K}\theta_t^{e,K} + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\geqslant -(1-\alpha)\kappa_t^e} \geqslant \alpha\kappa_t^e,$

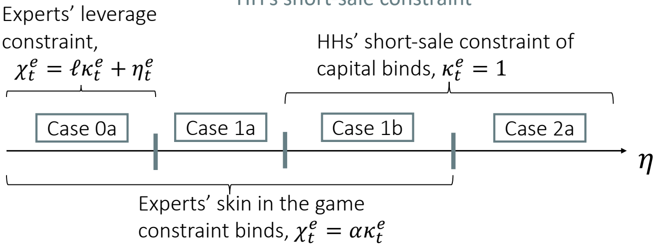
  Leverage constraint $\Rightarrow \chi_t^e = \eta_t^e\theta_t^{e,K} + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\leqslant(1-(1-\ell)\theta_t^{e,K})} \leqslant \ell\kappa_t^e + \eta_t^e$

1b. Occasionally Binding Constraints across η

Cases	0a	1a	1b	2a
leverage	$\chi_t^e = \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$
skin in game	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e > \alpha\kappa_t^e$
short-sale	$\kappa_t^e < 1$	$\kappa_t^e < 1$	$\kappa_t^e = 1$	$\kappa_t^e = 1$
Δa vs. $\Delta\varsigma$	$>$	$=$	$>$	$>$
outside equity	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$

complementary slackness conditions

Occasionally binding constraints:
Leverage constraint
Skin in the game constraint
HH's short-sale constraint



1b. θ-Choice: Martingale Approach (aside) (Relaxed Skin-in-the-Game, No Leverage Constraint)

- **Approach 1:** Portfolio Optimization
  - Step 1: Optimization e.g. via Martingale Approach – recall: $\boxed{\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A}$
    - Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q = r_t + \underbrace{[\varsigma_t^e \chi_t^e/\kappa_t^e + \varsigma_t^h(1 - \chi_t^e/\kappa_t^e)]}_{\text{new compared to lecture 04}}(\sigma + \sigma_t^q)$$

    - Of households' capital choice:

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \leqslant r_t + \varsigma_t^h(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1$$

  - Step 2: Capital market clearing to obtain asset/risk allocation κ_t^e, χ_t^e from portfolio weights θs
- **Approach 2:** Price-taking Social Planner Approach

1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

- Maximization within each {}-term = maximization over weighted sum
- Choose η-weighted sum of expert + HH maximization problem

$$\eta^e\{\ldots\} + \eta^h\{\ldots\}$$

- Why?
  - positive net supply assets become capital and risk shares (# of Brownian)
  - zero net supply assets cancel out

$$\overbrace{\eta_t^e\theta_t^{e,K}}^{\equiv\kappa_t^e}\mathbb{E}[dr_t^{e,K}]/\mathrm{d}t + \overbrace{\eta_t^h\theta_t^{h,K}}^{\equiv\kappa_t^h}\mathbb{E}[dr_t^{h,K}]/dt + \overbrace{(\eta_t^e\theta_t^{e,OE} + \eta_t^h\theta_t^{h,OE})}^{=0}\mathbb{E}[dr_t^{OE}]/dt$$

$$+\underbrace{(\eta_t^e\theta_t^{e,D} + \eta_t^h\theta_t^{h,D})}_{=0}r_t - \varsigma_t^e\underbrace{\eta_t^e(\theta_t^{e,K} + \theta_t^{e,OE})}_{\equiv\chi_t^e}\sigma_t^{r^K} - \varsigma_t^h\underbrace{\eta_t^h(\theta_t^{h,K} + \theta_t^{h,OE})}_{\equiv\chi_t^h}\sigma_t^{r^K}$$

- Translate portfolio constraints in capital and risk share constraints

1b. Portfolio to Asset/Risk Allocation Constraints

- Convert λ-constraints into κ, χ-constraints

  Skin-in-the-game constraint $\Rightarrow \chi_t^e = \eta_t^e \theta_t^e + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\geqslant -(1-\alpha)\kappa_t^e}$,

  Leverage constraint $\Rightarrow \chi_t^e = \eta_t^e \theta_t^e + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\leqslant (1-(1-\ell)\theta_t^{e,K})} \leqslant \ell \kappa_t^e + \eta_t^e$

- **Price-taking social planner's problem:**

$$\max_{\substack{\{\chi_t^e \in [\alpha\kappa_t^e, \kappa_t^e], \chi_t^h = 1-\chi_t^e, \\ \kappa_t^e, \kappa_t^h = 1-\kappa_t^e, \iota_t\}}} \left[\frac{\kappa_t^e a^e + \kappa_t^h a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta \right] - (\varsigma_t^e \chi_t^e + \varsigma_t^h \chi_t^h)\sigma_t^{r^K}$$

End of Proof. Q.E.D

- Linear objective (if frictions take form of constraints)
  - Price of risk adjust such that objective becomes flat *or*
  - Bang-bang solution hitting constraints
- First order condition w.r.t. κ^e (plug in relevant constraints depending on η)

$$\frac{a^e - a^h}{q_t} \geqslant \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta\text{-risk premia}}, \text{ with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

Cases	0a	1a	1b	2a
leverage	$\chi_t^e = \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$
skin in game	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e > \alpha\kappa_t^e$
short-sale	$\kappa_t^e < 1$	$\kappa_t^e < 1$	$\kappa_t^e = 1$	$\kappa_t^e = 1$
Δ-risk premia	$>$	$=$	$>$	$>$
risk-sharing	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$

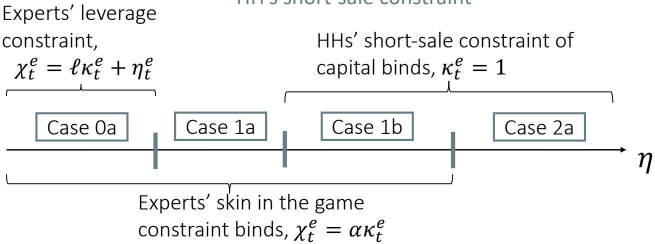
complementary slackness conditions

Occasionally binding constraints:

Leverage constraint

Skin in the game constraint

HH's short-sale constraint



1b. Price Taking Social Planner (General Theorem)

Theorem (Price-Taking Planner)

A social planner that takes prices as given chooses a physical asset allocation, κ_t and risk allocation, χ_t that coincides with the choices implied by all individuals' portfolio choices.

Notation:

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, ..., \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, ...\chi_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1 \sigma^N, ..., \chi_t^I \sigma^N)$$

Planner's problem:

$$\max_{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t} \mathbb{E}_t[dr_t^N(\boldsymbol{\kappa}_t)]/dt - \boldsymbol{\varsigma}_t \sigma(\boldsymbol{\chi}_t) \quad (= dr^F/dt \text{ in equilibrium})$$
$$s.t. \quad F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t) \leqslant 0 \qquad (\text{Financial Frictions})$$

Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
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   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   Fisher separation theorem
   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach
2. Evolution of state variable η (and K) $\Rightarrow$ **as in Lecture 04** **forward equation**
   (degerated KFE as η is only a scalar)
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
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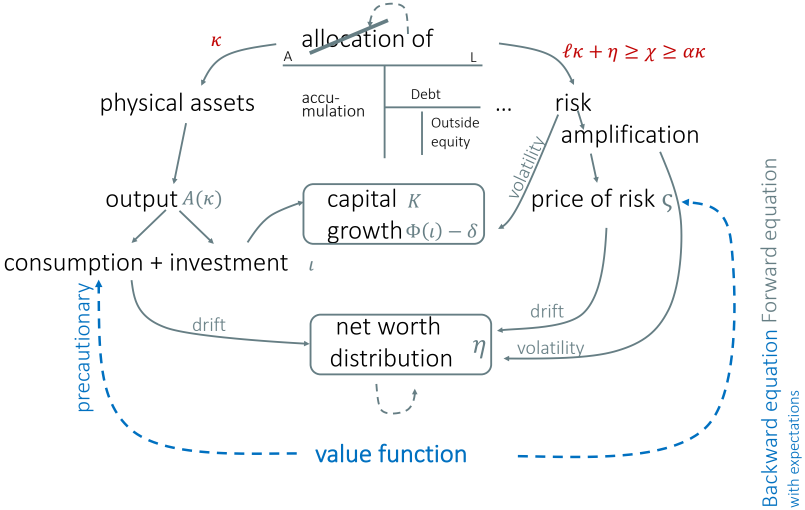
The Big Picture



Solving Macro Models Step-by-Step

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4a. Obtain κ for Goods Market Clearing (Outside Equity)

- Determination of κ_t
  - Based on difference in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$
  - For log utility: $(\sigma_t^{n^e} - \sigma_t^{n^h})(\sigma + \sigma_t^q) = \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$

    Since: $\eta_t^{\eta^e} = \frac{\kappa_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q), \eta_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$, and $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{\eta^e} - \sigma_t^{\eta^h}$
- Hence,

$$\boxed{\frac{a^e - a^h}{q_t} \geqslant \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.}$$

- Determination of χ_t^e:

$$\boxed{\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}}$$

- Determination of κ_t^e in the leverage constrained region:

$$\boxed{\kappa_t^e = \frac{\eta_t^e}{\alpha - \ell}}$$

4a. Investments and Capital Prices

- Replacing ι_t.
  - Recall from optimal re-investment $\Phi'(\iota) = 1/q_t$:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$$

- Recall from "amplification slide"

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}\frac{\chi_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

4a. Market Clearing

- Output good market:

$$(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t)K_t = C_t$$

$$\Rightarrow \quad \boxed{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t = q_t[\eta_t \rho^e + (1-\eta_t)\rho^h]}$$

- Capital market is taken care off by price-taking social planner approach.
- Risk-free debt market also taken care off by price taking social planner approach. (would be cleared by Walras Law anyways)

4b. Algorithm – Static Step

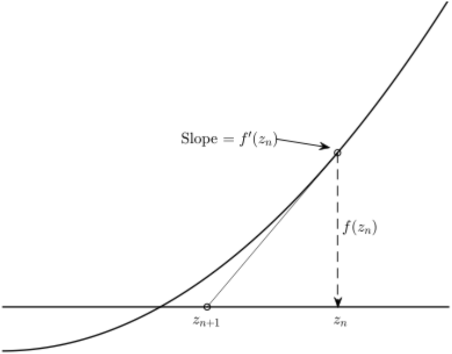
- We have five static conditions

1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e: $\frac{a^e - a^h}{q_t} \geqslant \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Planner condition for χ_t^e: $\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$
4. $\kappa_t^e a_t^e + (1 - \kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t\rho^e + (1 - \eta_t)]\rho^h$
5. $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)$
   $\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

- Start at $q(0)$, solve to the right, use different procedure for two η regions depending on κ:

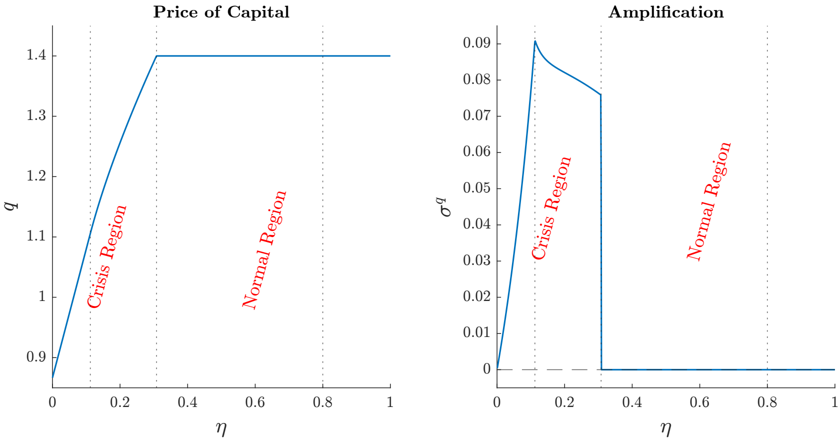
1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (4)
   - Plug in the Planner's condition of χ_t
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method
   - if $\chi_t^e \geqslant \ell\kappa_t^e + \eta_t^e$, replace κ_t^e by $\frac{\eta_t^e}{\alpha - \ell}$, solve (3) (4) (5) for $\chi(\eta^e)$,$q(\eta^e)$,$\sigma^q(\eta^e)$
2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (4) for $q(\eta^e)$
   (HINT: When constraint binds, we directly substitute in κ^e)

4b. Aside: Newton's Method



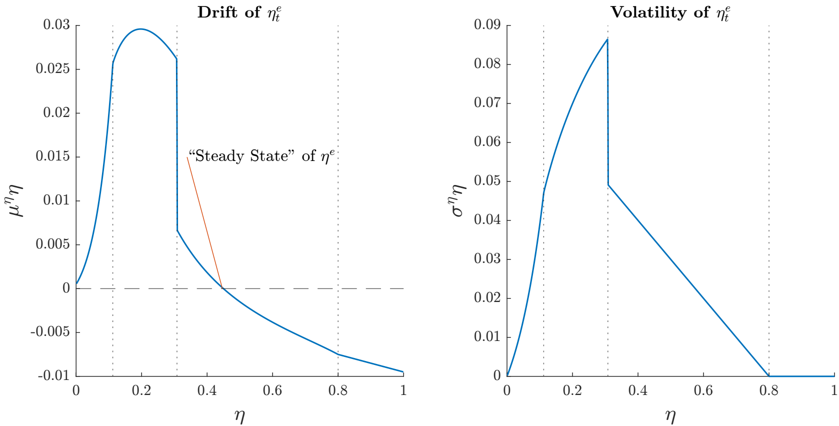
$$\boldsymbol{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, F(\boldsymbol{z}_n) = \begin{bmatrix} \kappa_t^e a_t^e + (1 - \kappa_t^e) a^h - \iota(q_t) - q_t[\eta_t \rho^e + (1-\eta_t)\rho^h] \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{Planner.} \end{bmatrix}$$

Capital Price and Volatility



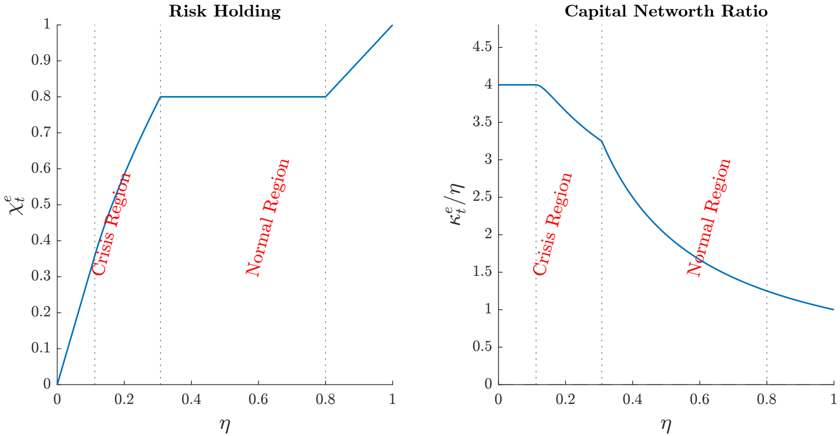
$$\rho^{e,h} = 0.05,\ \rho_0^{e,h} = 0.04,\ \rho_d^{e,h} = 0.01,\ \zeta^e = 0.05,\ \delta = 0.05,$$
$$a^e = 0.11,\ a^h = 0.03,\ \sigma = 0.10,\ \phi = 10,\ \alpha = 0.8,\ \ell = 0.55.$$

Net Worth Evolution: Drift & Volatility



$\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, $\delta = 0.05$,
$a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

Risk Allocation & Leverage

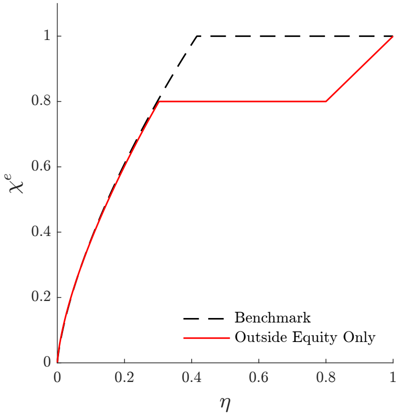


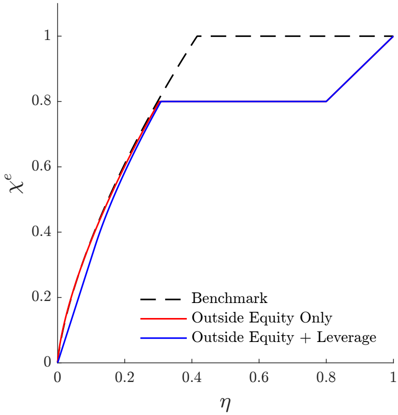
$\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, $\zeta^e = 0.05$, $\delta =$0.05,
$a^e = 0.11$, $a^h = 0.03$,$\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

Risk Allocation: Compare with $\alpha = 1$, $\ell = 1$

- allow some outside equity $\alpha = .8$

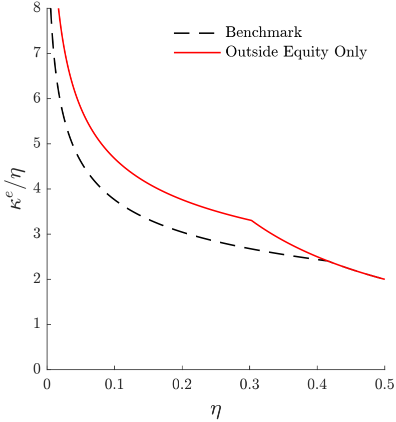
limit leverage $\ell = .55$

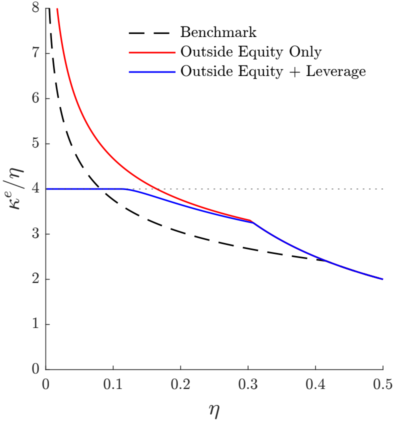




Leverage: Capital Net Worth Ratio

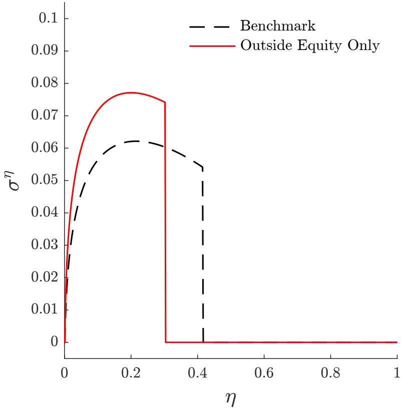
- allow some outside equity $\alpha = .8$ limit leverage $\ell = .55$

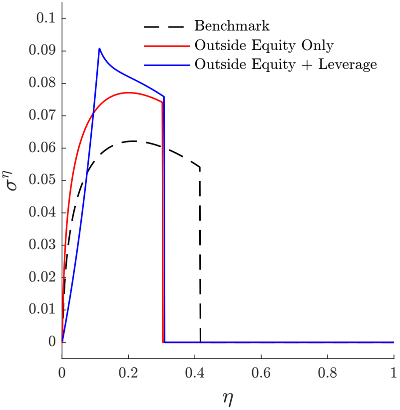




Price Volatility: Compare with $\alpha = 1$, $\ell = 1$

- allow some outside equity $\alpha = .8$ limit leverage $\ell = .55$

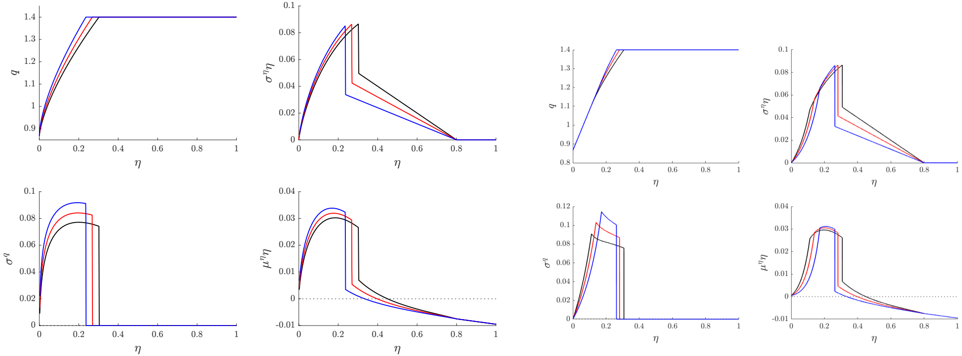




Volatility Paradox: $\alpha = 0.8$, $\ell = 1$ vs. $\ell = .55$

- σ^η (as well as $\sigma + \sigma^q$) stays roughly constant as σ varies (even when $\sigma \to 0$)

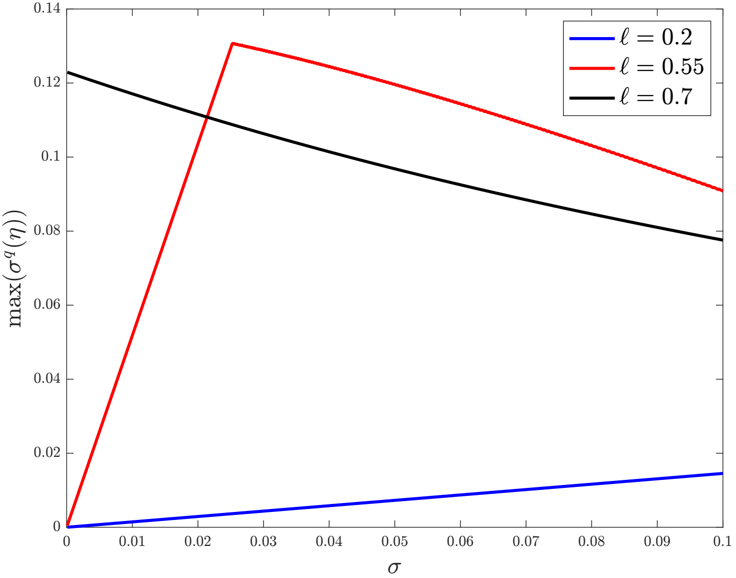
$\sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$



- arises in fire-sale region in which leverage constraint does not bind
- binding collateral/leverage constraints lowers volatility and drift

Volatility Paradox and Collateral/Leverage Constraint

- When collateral/leverage constraint binds, there is no Volatility Paradox (decline in exogenous risk σ also lowers maximum q-price volatility $\max \sigma^q$.



Solving Macro Models Step-by Step

0 Postulate aggregates, price processes and obtain return processes

1 For given $\check{\rho}^i := C^i/N^i$-ratio and $\xi^i = SDF^i$ processes for each i **finance block**

Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach

- a Real investment ι + Goods market clearing *(static)*
- b Fisher separation theorem
  Portfolio choice θ + asset market clearing or
  Asset allocation κ & risk allocation χ
  Toolbox 2: "Price-taking" social planner approach

2 Evolution of state variable η (and K) $\Rightarrow$ **as in Lecture 04** **forward equation**
(degenerated KFE as η is only a scalar)

3 Value functions **backward equation**

- a Value fcn. as fcn. of individual investment opportunities ω
  Special case: log-utility

4 Numerical model solution

5 KFE: Stationary distribution, Net worth trap (Lack of Resilience)
Fan chart

5. Kolmogorov Forward Equation

- How does density $f(\eta, t)$ evolve over time?
- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows:

$$\frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"
- Corollary: If stationary distribution $f(\eta)$ exists, it satisfies ODE:

$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2}$$

  - Closed form solution:

$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)} \mathrm{d}x\right)$$

5. Different Methods to Solve for Stationary Distribution

- KFE:

$$0 = \frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

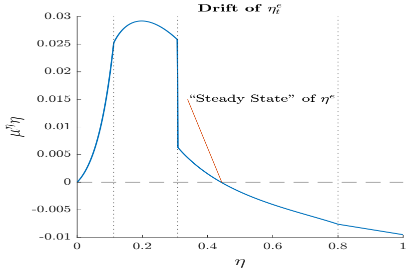
1. **Solve ODE** (closed form for one-dimensional state variable)

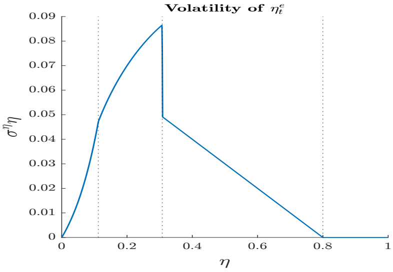
$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2}$$

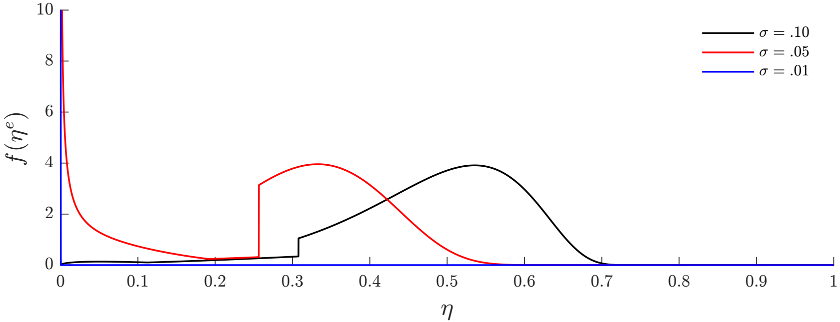
$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x\right), \quad \text{Const s.t. } \int_0^1 f(\eta)d\eta = 1$$

2. **Simulation** (useful for multi-dimensional state variables)
3. **Solve PDE**
   Since linear $\mu(\eta), \sigma(\eta)$ are known functions that do not depend on $f(\cdot)$, not so difficult
   - Discretize stationary KFE to obtain linear equation system
   - Make sure that the density integrates to 1
   - Iterate time-dependent KFE until convergence

5. Distribution of Wealth Shares of People Not Died







$$\rho^{e,h} = 0.05,\ \rho_0^{e,h} = 0.04,\ \rho_d^{e,h} = 0.01,\ \zeta^e = 0.05,\ \delta = 0.05,$$
$$a^e = 0.11,\ a^h = 0.03,\ \sigma = 0.10,\ \phi = 10,\ \alpha = 0.8,\ \ell = 0.55.$$

5. Method 1: Solve ODE

- KFE:

$$0 = \frac{\partial f(\eta,t)}{\partial t} = -\frac{\partial[f(\eta,t)\mu(\eta)]}{\partial\eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial\eta^2}$$

- **Solve ODE**

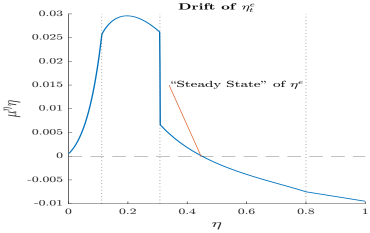
$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2}$$

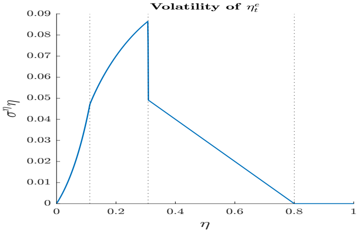
  Closed form solution for ODE is given by

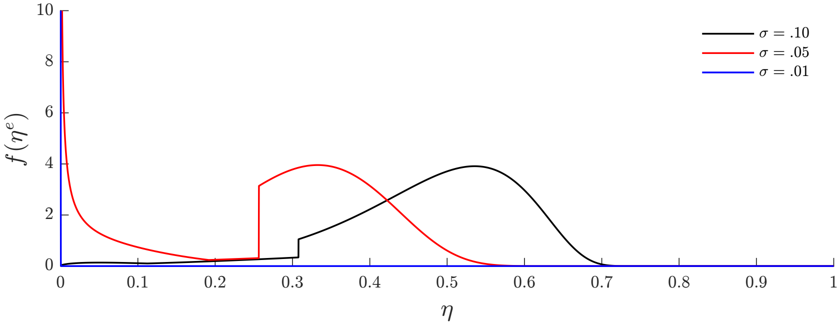
$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)}\exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x\right)$$

- 4 Remarks:
  - The drift term $\mu(x)$ is without the new-born term to get the distribution of wealth shares conditional on people who have not died.
  - Solve $f(\eta)$ starting from $\eta = 1$, since density at $f(0) \to \infty$.
  - Jump in σ leads to a jump in density due to $\frac{\text{Const}}{\sigma^2(\eta)}$-term.
    (Jump size is given approaching jump from the right vs. the left.)
  - Determine constant so that $\int_0^1 f(\eta)d\eta = 1$.

5. Distribution from Drift & Volatility







- How can the system spend so much time around $\eta = 0$
  even though the drift $\mu^\eta \eta$ is positive (non-negative) around $\eta = 0$?

5. Existence of Stationary Distribution

- Observation of comp statics $\Rightarrow$ stationary dist does not exist for $\sigma = 0.01$
- (Intuition:) When does invariant distribution exist? $\Rightarrow$ recurrency
  - Forces pull particle out when collapse.
  - "Bounce" back when hitting barrier.
- (Math:) Recall closed form solution:

$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^{\eta} \frac{2\mu(x)}{\sigma^2(x)} \mathrm{d}x\right)$$

  - $f(\eta) \geqslant 0$: probability cannot be negative.
  - $\int f(\eta)\mathrm{d}\eta = 1$: probability distribution is normalizable.

5. Detour: Existence of $f(\cdot)$ for Geometric BM

- **Reflected Geometric** Brownian Motion (Reflecting barrier at $x = d$):

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}Z_t - \mathrm{d}U_t, X_t \in (0, d]$$

- KFE:

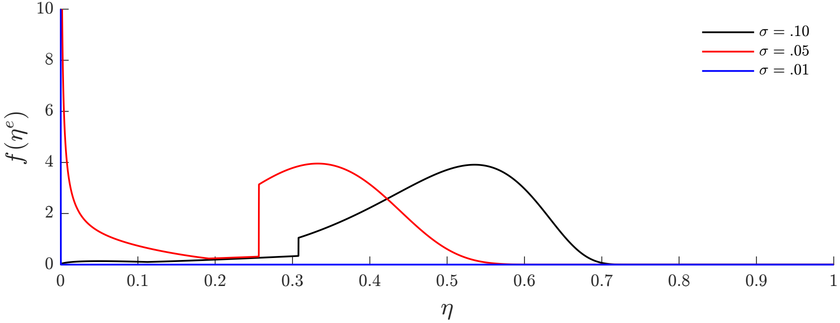
$$\frac{\partial f}{\partial t} = -\frac{\partial(\mu x f)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2 x^2 f)}{\partial x^2}$$

- Stationary distribution

$$f(x) = \frac{\text{Const}}{\sigma^2 x^2} \exp\left(\int_0^x \frac{2\mu y}{\sigma^2 y^2} \mathrm{d}y\right) = \frac{\frac{2\mu}{\sigma^2} - 1}{d^{\frac{2\mu}{\sigma^2}-1}} x^{\frac{2\mu}{\sigma^2}-2}$$

- Question: when $f(x)$ becomes a density?
- "Bouncing back" because of reflecting barrier at $x = d$.
- "Pulled back" by strong enough $\mu(x)$ at $x = 0$.

5. ... back to our Distribution



- Asymptotic solution ($\eta \to 0$):

$$f(\eta) \sim \left(\frac{2\mu(0)}{\sigma^2(0)} - 1\right) \eta^{\frac{2\mu(0)}{\sigma^2(0)}-2}$$

  - $\frac{2\mu(0)}{\sigma^2(0)} \geqslant 2$: $f(\eta)$ is finite at $\eta = 0$
  - $2 \geqslant \frac{2\mu(0)}{\sigma^2(0)} > 1$: $f(\eta)$ is infinite at $\eta = 0$, but still normalizeable ($\int f \mathrm{d}\eta < \infty$)
  - $1 \geqslant \frac{2\mu(0)}{\sigma^2(0)}$: $f(\eta)$ is infinite at $\eta = 0$, stationary distribution does not exist

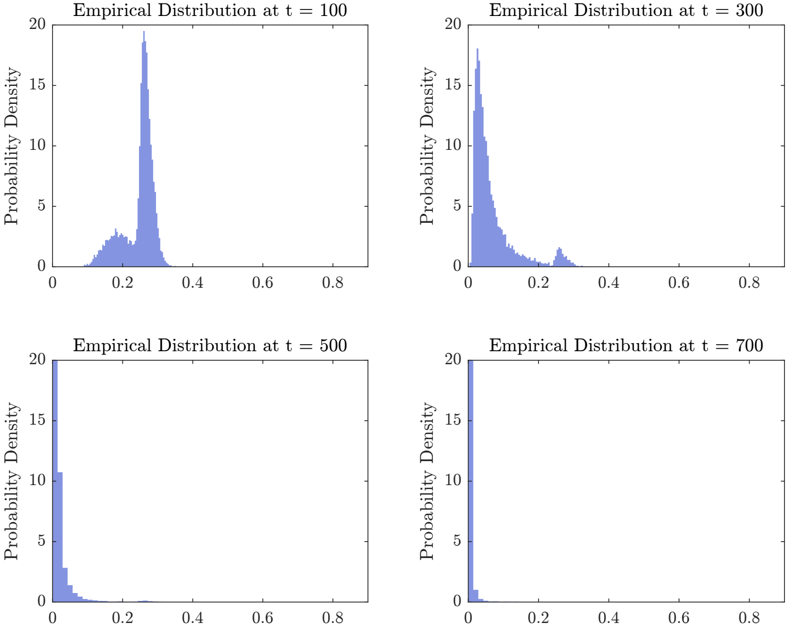
5. Method 2: Simulation

Algorithm Pseudo code

for $\eta = 0 : d\eta : 1$
Solve for $\kappa^e(\eta), \sigma^q(\eta), q(\eta)$ **do**
end for
Construct arithmetic drift & volatility $\mu_\eta = \mu^\eta . * \eta, \sigma_\eta = \sigma^\eta . * \eta$
Interpolate $\tilde{\mu}_\eta, \tilde{\sigma}_\eta$ Since new μ_η, σ_η may not lie on the grid
$\boldsymbol{\eta_0}$ initial distribution
for $t = 0 : dt : T_{max}$ **do**
    Sample $d\boldsymbol{Z_t}$ from $\boldsymbol{N}(0, dt)$
    Interpolate and compute $\tilde{\mu}_\eta(\boldsymbol{\eta_t})$, $\tilde{\sigma}_\eta(\boldsymbol{\eta_t})$
    $\boldsymbol{\eta}_{t+dt} = \boldsymbol{\eta}_t + \tilde{\mu}_\eta(\boldsymbol{\eta}_t)dt + \tilde{\sigma}_\eta(\boldsymbol{\eta}_t)d\boldsymbol{Z}_t$
end for

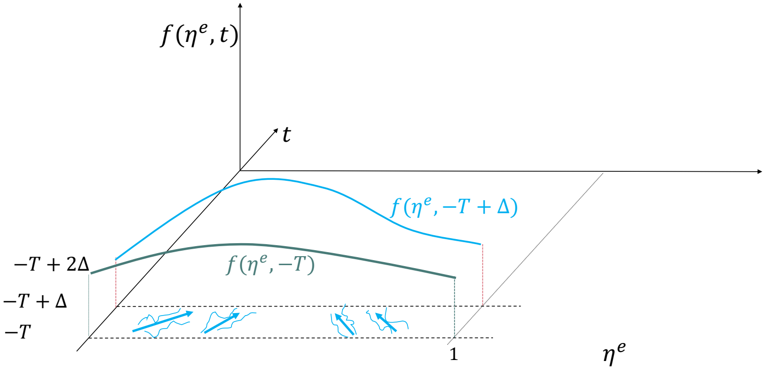
5. Method 2: Time Evolution Obtained via Simulation

- Given $\eta_0 = 0.6$, the (empirical) distribution evolves as:



Simulated distribution for $\sigma = 0.04, \eta_0 = 0.6$

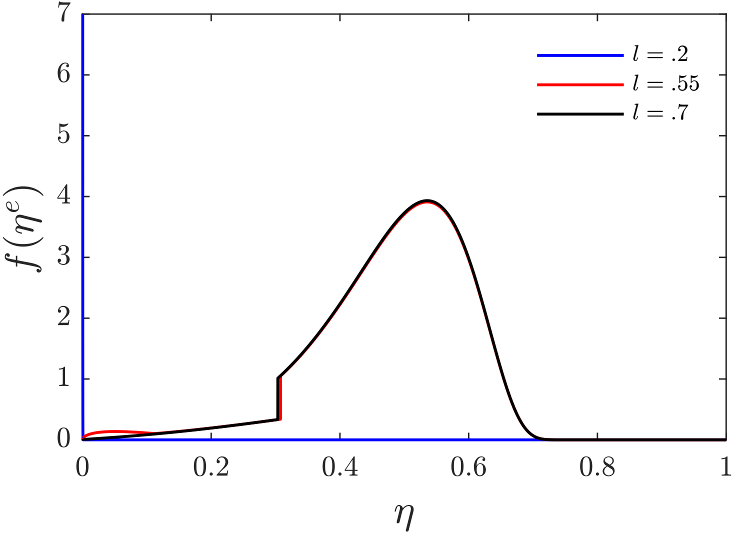
5. Method 3: Solving PDE via Forwards Iteration



- Obtain descaled function $f(\eta^e, -T + \Delta)$
- Repeat previous steps
- Initial $f(\cdot, \cdot)$ is the distribution of starting points across different economies
- Use transition matrix M' (see next lecture):
  - Probability coming from previous state η to current state
  - M' is the transpose of M transition matrix from backwards iteration

5. The Role of Leverage Constraint in Net Worth Trap

- "**Net worth Trap**" is induced by a too-tight leverage constraint, $\sigma = 0.1$



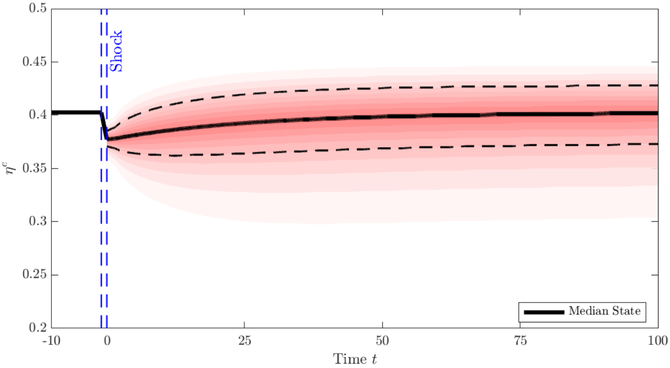
5. Interaction: Net Worth Trap & Volatility Paradox

- Net Worth Trap based on volatility paradox interaction with leverage constraint:
  - Leverage constraint depresses μ^η and σ^η
  - High volatility in fire-sale region outside binding leverage constraint
  - As η declines, does μ^η or $(\sigma^\eta)^2$ decline faster?

- Micro- and Macro-Prudential Regulation: Basel I, II, III
  - Basel I: fixed risk-weights and capital requirement
  - Basel II: risk-weights but not time-varying $\Rightarrow$ Net Worth Trap
  - Basel III: Countercyclical capital buffer: (contemporaneous, not past)

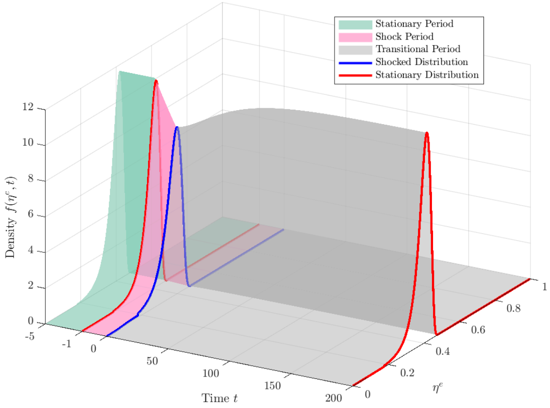
5.2 Fan Chart and Distributional Impulse Response

- ... the theory to Bank of England's empirical fan charts
- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock $dZ_t = -2.32dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



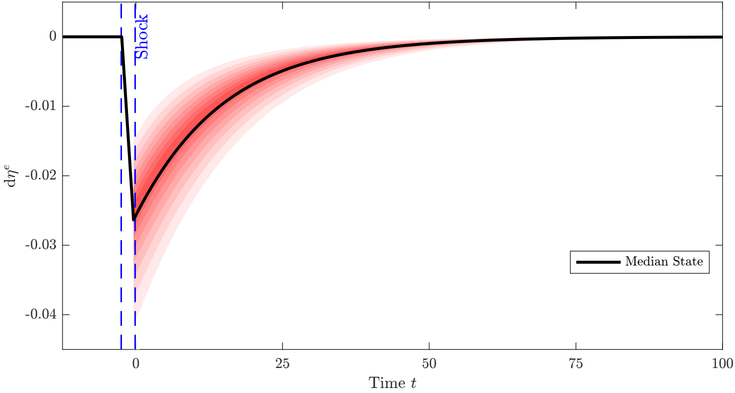
5.2 Density Diffusion

- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock $dZ_t = -2.32dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



5.2 Distributional Difference Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



5.3 The 3 Roles of KFE

- KFE characterizes the
  1. Stationary probability distribution of the state variable
  2. Density evolution of the system over time (distribution impulse fan charts)
     - Markov process maps probabilistic predictions for the initial state η_0 (i.e. density f_0) into probabilistic prediction for state η_t (i.e. density $f(\cdot, t)$)

  KFE as
  3. State equation (e.g. in Aiyagari-type models) [Step 2] describes the evolution of the cross-sectional distribution of net worth across a continuum of households (not the evolution of probability).
     - Mathematically identical (similar with jumps)
     - would have dZ_t-term

Main Takeaways

- **Toolboxes: Technical Innovations**
  - Several occasionally binding constraints
    - Short-sale constraint
    - Skin in the game constraint
    - Collateral constraint
  - Price setting social planner to find capital and risk allocation
  - Stationary distribution KFE solution
    - Closed form ODE solution (for one-dimensional η)
    - Simulation (multi-dimensional η)
    - Solve PDE: forward iteration
  - Fan charts
- **Economic Insights:** Binding leverage constraint (e.g. due to regulation)
  - Limits Volatility Paradox,
  - ... but destroys resilience due to "net worth trap" (double-humped shaped distribution)

Homework: Extra Exercise

- Generalize the analysis for the case in which
  the leverage parameter ℓ is a decreasing function η, i.e. similar to Basel III, regulation which imposes a counter cyclical capital buffer. Solve the case for
  1. for the case in which the skin-in-the game constraint does not allow the issuance of any outside equity.
  2. for the case in which experts can issue outside equity and debt up to the collateral constraint $D_t^e \leqslant \ell(\eta_t)\kappa_t q_t K_t$
- Generalize the analysis for the case in which
  the leverage parameter ℓ is a function of the volatility $\sigma + \sigma_t^q$
  1. for the case in which the skin-in-the game constraint does not allow the issuance of any outside equity.
  2. for the case in which experts can issue outside equity and debt up to to the collateral constraint $D_t^e \leqslant \ell(\sigma + \sigma_t^q)\kappa_t q_t K_t$