Macrofinance

Lecture 04: Endogenous Risk Dynamics

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Course Overview

Overcoming some Short-comings of Lecture 03

- Basak-Cuoco (1998)
  - $\rho^e > \rho^h$ needed for stationarity
    $\Rightarrow$ capital price q rises after adverse shock $\sigma^q \leqslant 0$ mitigates shock
  - High required risk premium of undercapitalized experts
    is achieved by low risk-free rate (rather than capital price appreciation)
- Kiyotaki-Moore (1997)
  - Fire-sales, since less productive households hold & operate capital
  - (Dynamic) amplification: Capital price drop
    - lowers net worth share of leveraged experts (further)
    - tightens collateral constraint
  - Single shock critique, deterministic "bounce back" to steady state
- **Desired Model Properties:**
  - Self-stabilizing system in normal times (around stochastic SS)
    since adequately capitalized experts can absorb shock (non-linearity)
  - Destabilizing in crises times
  - Endogenous risk and price of risk (cash flows and SDF)
  - Volatility Paradox (Minsky Hypothesis)
  - Endogenous investment and growth of the macroeconomy
  - ("Net worth trap": double-humped distribution conditional on people not died)

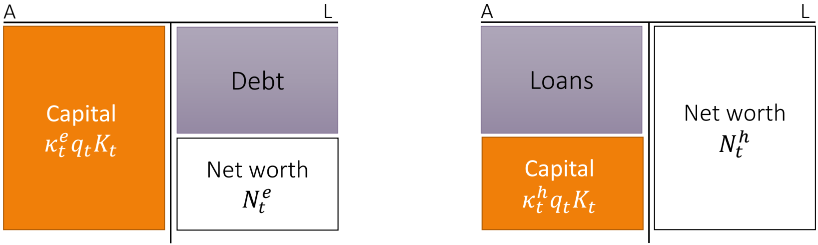
Toolboxes: Technical Innovations

- Introduction of agents' "death" and switching types
- Occasionally binding (short-sale) constraint
  (in addition to natural borrowing limit due to risk aversion)
- Newton Method to solve log-utility numerical example

Two Sector Model: Simple Extension of Basak Cuoco

- Expert sector

Household sector

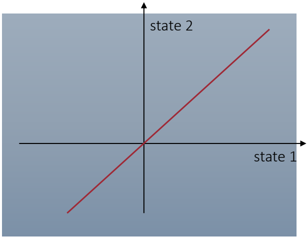


- Households can produce with capital.
  - Productivity $0 < a^h < a^e$
- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- The fraction of aggregate risk held by experts: $\chi_t^e = \frac{\sigma_t^{N^e} N_t^e}{\sigma_t^{qK} q_t K_t}$
- Experts can only issue debt, no outside equity, $\chi_t^e = \kappa_t^e$

Skin-in-the-Game constraint

Financial Frictions and Distortions

- Incomplete markets
  - "natural" leverage constraint (BruSan)
  - Costly state verification (*BGG*)
- $+$ Leverage constraints
  (no "liquidity creation")
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period's price $Rb_t \leqslant q_{t+dt} k_t$ (*KM*)
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+dt}) k_t$ (*BruPed*)



- Search Friction (*DGP*)
- Belief distortions

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$

Household Sector

- Output: $y_t^e = a^h k_t^h$

$$A(\boldsymbol{\kappa}) = \kappa^e a^e + (1 - \kappa^e) a^h$$

Poll 04.01: Why is it important that households can hold capital?

a) to capture fire-sales

b) for households to speculate

c) to obtain stationary distribution

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},e}$
- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^e t}\log(c_t^e)\mathrm{d}t\right]$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h
  $\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$
- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log(c_t^h)\mathrm{d}t\right]$

Friction: Can only issue

- Risk-free debt
  Thus, $\chi_t^e = \kappa_t^e$

Two Sector Model Setup

Expert sector

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- Consumption rate: c_t^e
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Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
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  $\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$
- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\log(c_t^h)\mathrm{d}t\right]$

Poll 04.02: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates, $\rho^e > \rho^h$

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},e}$
- Objective (death):
  $\mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\rho_0^e t}\log(c_t^e)\mathrm{d}t\right]$
  T exponentially distributed with parameter ρ_d^e
  define $\rho^e := \rho_0^e + \rho_d^e$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h
  $\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$
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  T exponentially distributed with parameter ρ_d^h
  define $\rho^h := \rho_0^h + \rho_d^h$

- Upon death of an expert/household, a new agent takes their place, inherits their wealth, and becomes an expert with probability $\zeta^e \in (0, 1)$.

Objective with Death

Lemma

Objective is equivalent to infinite lifetime with higher discount rate $\rho^e = \rho_0^e + \rho_d^e$.

Proof: Conditional on $t < T$, c_t^e is independent of T; since exponential distribution is memoryless, so knowing that T has not occurred does not provide any information. Therefore,

$$
\begin{aligned}
\mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\rho_0^e(t-t_0)}\log(c_t^e)\mathrm{d}t\right] &= \int_{t_0}^{\infty} \rho_d^e e^{-\rho_d^e t_d}\mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\rho_0^e(t-t_0)}\log(c_t^e)\mathrm{d}t \mid T = t_d\right] dt_d \\
&= \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\int_{t_0}^{t_d} \rho_d^e e^{-\rho_d^e t_d} e^{-\rho_0^e(t-t_0)}\log(c_t^e)\mathrm{d}t dt_d\right] \\
&= \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\int_{t}^{\infty} \rho_d^e e^{-\rho_d^e t_d}\mathrm{d}t_d e^{-\rho_0^e(t-t_0)}\log(c_t^e)\mathrm{d}t\right] \\
&= \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} e^{-(\rho_0^e+\rho_d^e)(t-t_0)}\log(c_t^e)\mathrm{d}t\right]
\end{aligned}
$$

Implied Net Worth Dynamics with Death

$$dN_t^e = N_t^e dn_t^e / n_t^e - \rho_d^e (1 - \zeta^e) N_t^e dt + \rho_d^h \zeta^e N_t^h dt$$

$$dN_t^h = N_t^h dn_t^h / n_t^h - \rho_d^h \zeta^e N_t^h dt + \rho_d^e (1 - \zeta^e) N_t^e dt$$

Two Sector Model Setup with Type Switching

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},e}$
- Objective (type switching): $V_{t_0}^e = \mathbb{E}_{t_0}\left[\int_{t_0}^T e^{-\rho^e t}\log(c_t^e)\mathrm{d}t + e^{-\rho^e T} V_T^h\right]$

Household Sector

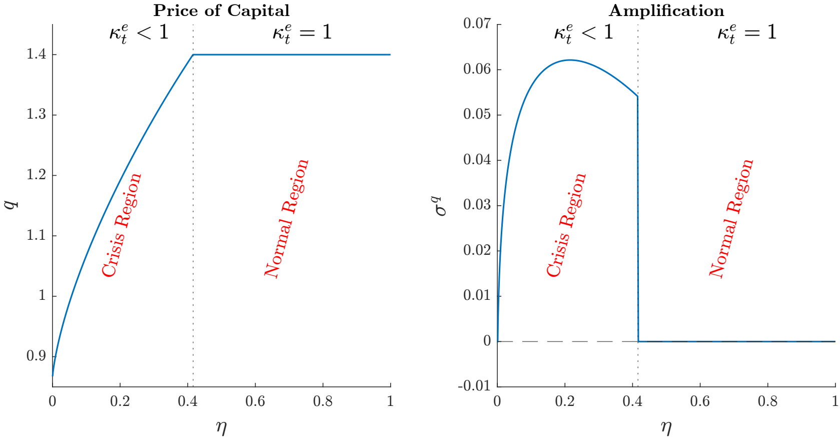
- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h
  $\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$
- Objective (type switching): $V_{t_0}^h = \mathbb{E}_{t_0}\left[\int_{t_0}^T e^{-\rho^h t}\log(c_t^h)\mathrm{d}t + e^{-\rho^h T} V_T^e\right]$

Type Switching Instead of Death

With type switching, we obtain

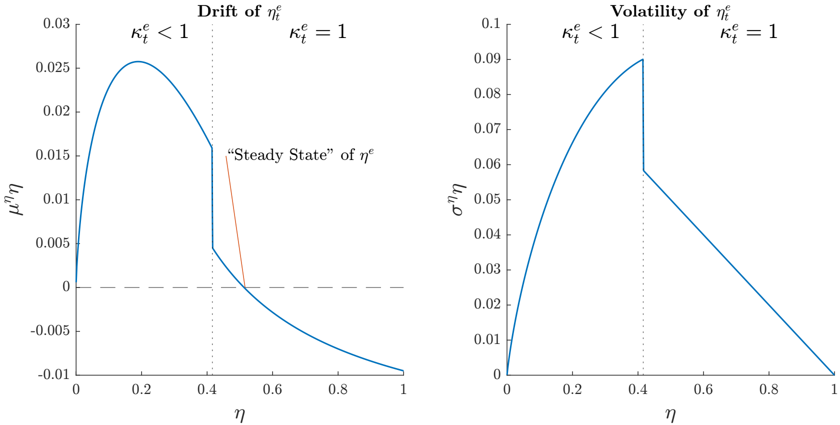
- the effective discount rate is ρ^e (or ρ^h) as in the preferences, not time preference + death rate
- the same $\mathrm{d}N_t^e$ evolution as with death specification
- The optimal consumption choices
  - With log utility: $c_t^e = \rho^e n_t^e$, $c_t^h = \rho^h n_t^h$, so we can solve the problem as before. [Remark: It is more involved to show this formally as co-state/HJB is more complicated.]
  - Beyond log utility: more complicated because behavior changes if agents anticipate that they might be of a different type in the future.

Preview

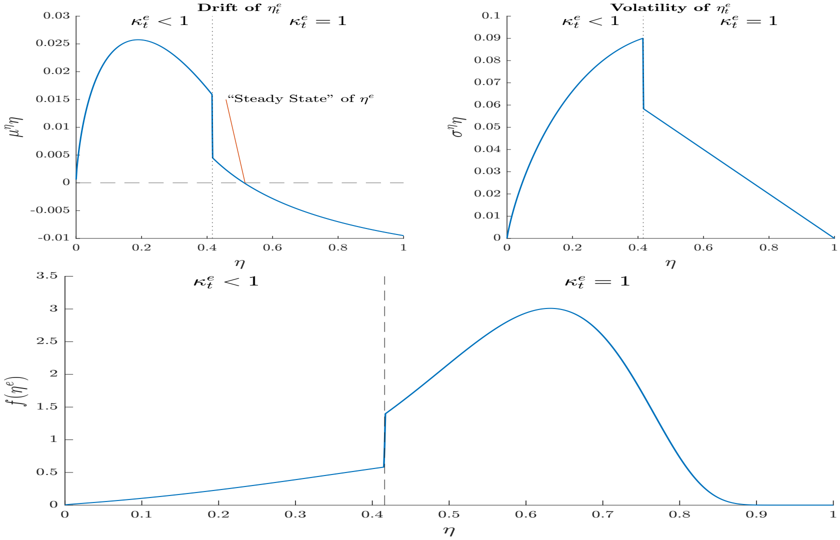


$$\rho^{e,h} = 0.05,\ \rho_0^{e,h} = 0.04,\ \rho_d^{e,h} = 0.01,\ \zeta^e{=}0.05,$$
$$a^e = 0.11,\ a^h = 0.03,\ \sigma = 0.10,\ \delta = 0.05,\ \phi = 10.$$

Preview of μ_η & σ_η



Preview of μ_η & σ_η



Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**
   Toolbox 1: Martingale approach, HJB vs. **Stochastic Maximum Principle** *Approach*
   Fisher separation theorem
   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + Asset market clearing
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility $c = \rho n, \varsigma = \sigma^n$
4. Numerical model solution
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{\mathrm{d}\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i},i}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},i}$$

  - where $\Delta_t^{k,\tilde{i},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i}\mathrm{d}\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$
  - Capital share: $\kappa_t^i = K_t^i/K_t,$ $\qquad \frac{\mathrm{d}K_t}{K_t} = (\Phi(\iota_t) - \delta)\,\mathrm{d}t + \sigma \mathrm{d}Z_t$
- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{i},i}\mathrm{d}\tilde{i}$
  - Across sectors: $N_t = \Sigma_i N_t^i$
  - Net worth share: $\eta_t^i = N_t^i/N_t,$
- Value of capital stock: $q_t K_t,$ $\qquad dq_t/q_t = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t$
  - Idiosyncratic death and type switching does not introduces jump term in q_t-process.
- Postulated SDF-process: $\qquad \frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i}\mathrm{d}t + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i}\mathrm{d}Z_t$

0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Itô)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - $\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i},i}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t$ (without purchases/sales $\mathrm{d}\Delta_t^{k,\tilde{i},i}$)

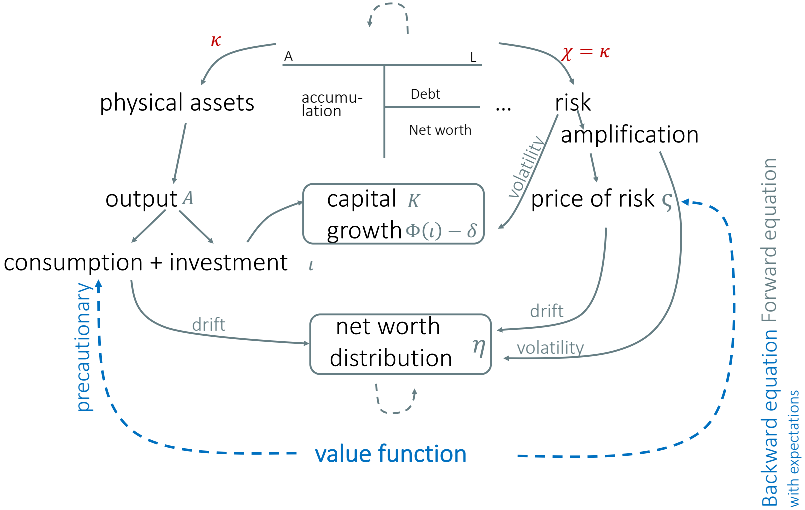
$$\mathrm{d}r_t^k(\iota_t^{\tilde{i},i}) = \left(\overbrace{\frac{a^i - \iota_t^i}{q_t}}^{\text{Dividend yield}} + \overbrace{\Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\mathbb{E}[\text{Capital gain rate}] = \mathbb{E}\ \frac{d(q_t k_t)}{q_t k_t}} \right)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t$$

For aggregate capital return, Replace a^i with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t}V'(n_t^i)$)

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t^i \mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t,\ \varsigma_t^i : \text{price of risk}, \&\ e^{-r_f} = \mathbb{E}[SDF]$$

The Big Picture



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   Fisher separation theorem
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      Special case: log-utility $c = \rho n, \varsigma = \sigma^n$
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1. Stochastic Maximum Principle Approach

- Experts' problem: $\max_{c_t^e, \iota_t^e, \theta_t^{e,K}} \mathbb{E}\left[\int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t\right]$ s.t.

$$\mathrm{d}n_t^e = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t)\right)\right]\mathrm{d}t + n_t^e \theta_t^{e,K}(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

- Households' problem: $\max_{c_t^h, \iota_t^h, \theta_t^h} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t} u(c_t^h) \mathrm{d}t\right]$, s.t. $\theta_t^{h,K} \geqslant 0$,

$$\mathrm{d}n_t^h = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t)\right)\right]\mathrm{d}t + n_t^h \theta_t^{h,K}(\sigma + \sigma_t^q)\mathrm{d}Z_t,$$

- The Hamiltonians can be constructed as

$$\mathcal{H}_t^e = e^{-\rho^e t} u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e \left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} - \varsigma_t^e \xi_t^e \overbrace{n_t^e \theta_t^{e,K}(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e}$$

$$\mathcal{H}_t^h = e^{-\rho^h t} u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t)\right)\right] - \varsigma_t^h \xi_t^h n_t^h \theta_t^{h,K}(\sigma + \sigma_t^q) + \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Lagrange-term, $\lambda^h n_t^h \theta_t^{h,K}$, captures that $\kappa_t^h \geqslant 0$
- FOC w.r.t. c_t is separated/de-coupled from FOC w.r.t. θ_ts as well as ι_t^e
  $\Rightarrow$ Fisher Separation Theorem btw. $c_t^i, \theta_t^i, \iota_t^i$

1a. Individual Agent Choice of ι

- Choice of ι is static problem (and separable) for each t

$$\max_{\iota_t^i} \mathrm{d}r_t^k(\iota_t^i) = \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q \right)$$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents: $\iota_t^i = \iota_t \Rightarrow \frac{\mathrm{d}K_t}{K_t} = (\Phi(\iota_t) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t$
  - Special functional form:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \phi\iota = q - 1$$

- Goods market clearing condition: $(A(\kappa_t) - \iota_t)K_t = \sum_i C_t^i$

1b. θ-Choices: Stochastic Maximum Principle

- The Hamiltonians can be constructed as

$$\mathcal{H}_t^e = e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} - \varsigma_t^e \xi_t^e \overbrace{n_t^e \theta_t^{e,K}(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e}$$

$$\mathcal{H}_t^h = e^{-\rho^e t}u(c_t^h) + \xi_t^h\left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t)\right)\right] - \varsigma_t^h \xi_t^h n_t^e \theta_t^{h,K}(\sigma + \sigma_t^q) + \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Lagrange-term, $\lambda^h n_t^h \theta_t^{h,K}$, captures that $\kappa_t^h \geqslant 0$, i.e. $\kappa_t^e \leqslant 1$
- Objective functions are linear in θ (divide through $\xi_t^i n_t^i$) $\Rightarrow$ bang-bang (or indifferent)
- Take FOC:

$$r_t^{e,K} - r_t - \varsigma_t^e(\sigma + \sigma_t^q) = 0$$
$$r_t^{h,K} - r_t - \varsigma_t^h(\sigma + \sigma_t^q) - \lambda_t^h = 0$$

- Take difference and substitute in for $r_t^{e,K}$, $r_t^{h,K}$

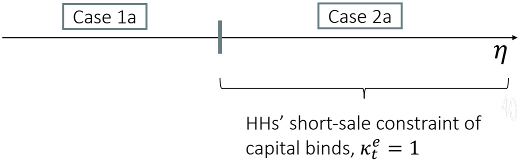
$$\frac{a^e - a^h}{q_t} \geqslant (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) \quad \textit{with equality if} \quad \kappa_t^h > 0$$

1b. Asset, Risk Allocation: Occasionally Binding Constraint

Cases	1a	2a
allocation risk premia	$\kappa_t^e < 1$ $\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$	$\kappa_t^e = 1$ $\frac{a^e - a^h}{q_t} > (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$

complementary slackness conditions

Occasionally binding constraint
(HH's shot-sale constraint of capital)



1b. θ-Choices

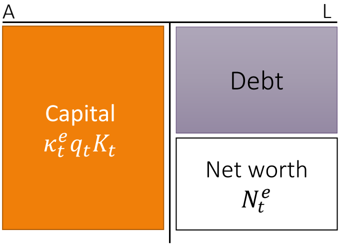
- Experts: $\boldsymbol{\theta}^e = (\theta^{e,K}, \theta^{e,D})$ for capital and debt. $\theta^{e,K} \geqslant 0$. Maximize:

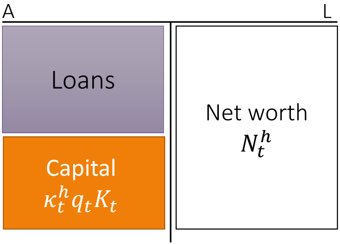
$$\theta_t^{e,K}\mathbb{E}[\mathrm{d}r_t^{e,K}]/\mathrm{d}t + \theta_t^{e,D} r_t - \varsigma_t^e \theta_t^{e,K} \sigma^{r^{e,K}}$$

- Households: $\boldsymbol{\theta}^h = (\theta^{h,K}, \theta^{h,D})$, $\theta^{h,K} \geqslant 0$. Maximize:

$$\theta_t^{h,K}\mathbb{E}[\mathrm{d}r_t^{h,K}]/\mathrm{d}t + \theta_t^{h,D} r_t - \varsigma_t^h \theta_t^{h,K} \sigma^{r^{h,K}}$$

- Expert sector — Household sector





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5. KFE: Stationary distribution, Fan charts

2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks $\longrightarrow$ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

  $\{\mathbf{Z}_{s\in[0,t]}\}$ $\searrow$ $\nearrow$

  net worth distribution

  $\eta_t^e = \frac{N^e}{q_t K_t} \in (0,1)$

- All agents maximize utility
  - Choose: consumption, portfolio, ...
- All markets clear
  - Consumption, capital, debt,

2. Law of Motion of Net Worth Share η_t: Drift μ_t^η

- **Method 1**: Using Itô's quotient rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall ($\chi_t^i = \kappa_t^i$):

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}\mathrm{d}t + r_t\mathrm{d}t + \underbrace{\frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q)}_{\text{risk}}\varsigma_t^i\mathrm{d}t + \frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q)\mathrm{d}Z_t$$

  - For aggregate evolution, include switching terms:

$$\begin{aligned}\frac{\mathrm{d}N_t^i}{N_t^i} &= \frac{\mathrm{d}n_t^i}{n_t^i} - \rho_d^i\zeta^{\neg i}\mathrm{d}t + \rho_d^{\neg i}\zeta^i\frac{N_t^{\neg i}}{N_t^i}\mathrm{d}t \\ &= \left(-\frac{c_t^i}{n_t^i} + r_t + \frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q)\varsigma_t^i - \rho_d^i\zeta^{\neg i} + \rho_d^{\neg i}\zeta^i\frac{N_t^{\neg i}}{N_t^i}\right)\mathrm{d}t + \frac{\chi_t^i}{\eta_t^i}(\sigma+\sigma_t^q)\mathrm{d}Z_t\end{aligned}$$

  - $q_t K_t$ use Ito product rule; $N_t^i/(q_t K_t)$ use Ito ratio rule, $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

$$\mu_t^{\eta^e} = (1-\eta_t^e)\left[-\underbrace{(C_t^e/N_t^e - C_t^h/N_t^h)}_{\text{consumption difference}} + \underbrace{(\varsigma_t^e-\sigma_t^N)\sigma_t^{\eta^e} - (\varsigma_t^h-\sigma_t^N)\sigma_t^{\eta^h}}_{\text{risk premia difference}} + \underbrace{\frac{\rho_d^h\zeta(1-\eta_t^e) - \rho_d^e(1-\zeta)\eta_t^e}{\eta_t^e(1-\eta_t^e)}}_{\text{reshuffling}}\right]$$

  - Note: For log utility $C_t^i/N_t^i = \rho^i$ & price of risk $\varsigma^i = \sigma^{N^i} \Rightarrow \varsigma_t^i - \sigma_t^N = \sigma_t^{\eta^i}$

- **Method 2**: Change of Numeraire + Martingale Approach

2. Law of Motion of Net Worth Share η_t: Drift μ_t^η

- **Method 1**: Using Itô's quotient rule $\eta_t^i = N_t^i/(q_t K_t)$
  - ...
- **Method 2**: Change of Numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Include extra drift term due to reshuffling (death)
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that change of numeraire does not affect ratio η^i!

Asset A: sector i's portfolio return in terms of total wealth $\eta_t^i = \frac{N_t^i}{N_t}$:

$$\frac{d\eta_t^i + \left(\frac{C_t^i}{N_t}\right) dt}{\eta_t^i} = \left(\mu_t^{\eta^i} + \frac{C_t^i}{N_t} + \rho_d^i \zeta^i - \rho_d^i \zeta^i \frac{N_t^i}{N_t}\right) dt + \sigma_t^{\eta^i} dZ_t.$$

Asset B: A benchmark asset with risk-free return $r_t dt$.
Martingale asset pricing formula implies:

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t} + \rho_d^i \zeta^i - \rho_d^i \zeta^i \frac{N_t^i}{N_t} - r_t = \left(\zeta_t^i - \sigma_t^N\right) \sigma_t^{\eta^i}.$$

Taking the weighted sum of the two sectors yields the aggregate equation. Subtracting the one-sector equation from both sides results in the same $\mu_t^{\eta^e}$.

2. σ^η Volatility of Net Worth Share

- Recall Itô quotient rule (only volatility term)
- Since $\eta^e_t = N^e_t / N_t$,

$$\sigma^{\eta^e}_t = \sigma^{N^e}_t - \sigma^{N}_t = \sigma^{N^e}_t - \left(\eta^e_t \sigma^{N^e}_t + (1 - \eta^e_t) \sigma^{N^h}_t \right)$$

- ...

$$\sigma^{\eta^e}_t = (1 - \eta^e_t)(\sigma^{n^e}_t - \sigma^{n^h}_t), \text{ where } \begin{cases} \sigma^{n^e}_t = \frac{\chi^e_t}{\eta^e_t}(\sigma + \sigma^q_t) \\ \sigma^{n^h}_t = \frac{\chi^h_t}{\eta^h_t}(\sigma + \sigma^q_t) = \frac{1-\chi^e_t}{1-\eta^e_t}(\sigma + \sigma^q_t) \end{cases}$$

$$\Rightarrow \qquad \sigma^{\eta^e}_t = \frac{\chi^e_t - \eta^e_t}{\eta^e_t}(\sigma + \sigma^q_t)$$

- Note also: $\eta^e_t \sigma^{\eta^e}_t + \eta^h_t \sigma^{\eta^h}_t = 0 \Rightarrow \sigma^{\eta^h}_t = -\frac{\eta^e_t}{\eta^h_t} \sigma^{\eta^e}_t = -\frac{\eta^e_t}{1-\eta^e_t} \sigma^{\eta^e}_t$

2. Amplification Formula: Loss Spiral

$$\left.\begin{array}{ll} \text{Recall} & \sigma_t^{\eta^e} = \dfrac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q) \\ \text{By Itô's Lemma} & \sigma_t^q = \dfrac{q'(\eta_t^e)}{q(\eta_t^e)}\eta_t^e \sigma_t^{\eta^e} \end{array}\right\} \Rightarrow \sigma_t^q = \underbrace{\frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}}_{\text{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

- Total Volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

- Loss spiral
  - Market illiquidity
    (price impact elasticity)

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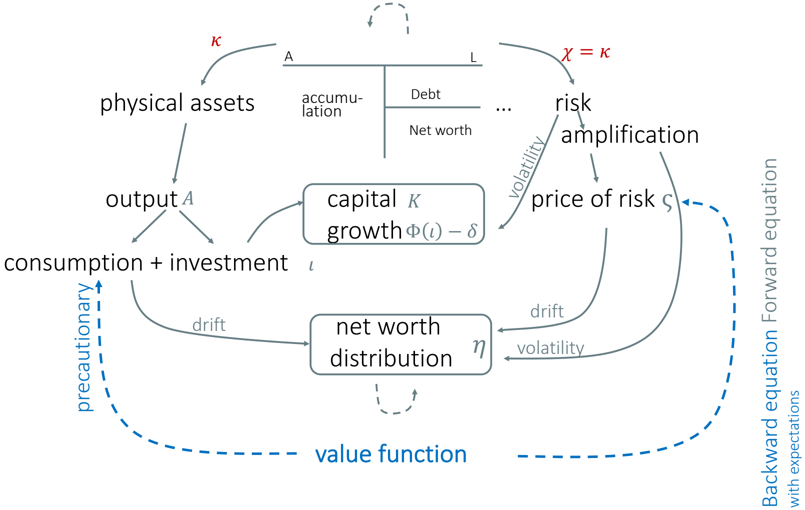
Poll 04.08: Where is the spiral?
a) Sum of infinite geometric series (denominator)
b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity
    (price impact elasticity)

The Big Picture



Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**

   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach

   Fisher separation theorem

   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + Asset market clearing
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω

      Special case: log-utility $c_t^i = \rho^i n_t^i$, $\varsigma_t^i = \sigma_t^{n^i}$
4. Numerical model solution
5. KFE: Stationary distribution, Fan charts

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4a. Obtain κ for Goods Market Clearing

- Determination of κ_t (part of ςs)
  - Based on difference in risk premia: $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$
  - For log utility: $(\sigma_t^{n^e} - \sigma_t^{n^h})(\sigma + \sigma_t^q) = \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$

    Since: $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{\eta^e} - \sigma_t^{\eta^h}$ and $\sigma_t^{\eta^e} = \frac{\kappa_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q), \sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$
- Hence,

$$\boxed{(a^e - a^h)/q_t \geqslant \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2, \text{with equality if } \kappa_t^e < 1}$$

4a. Investments and Capital Prices q

- Replacing ι_t.
  - Recall from optimal re-investment $\Phi'(\iota) = 1/q_t$:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$$

- Recall from "amplification slide"

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}\frac{\kappa_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

4a. Market Clearing

- Output good market:

$$(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t)K_t = C_t$$
$$\Rightarrow \boxed{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t = [\eta_t \rho^e + (1-\eta_t)\rho^h]q_t}$$

- Capital/asset market is taken care
- Risk-free debt market also taken care of by Walras Law

4b. Algorithm – Static Step

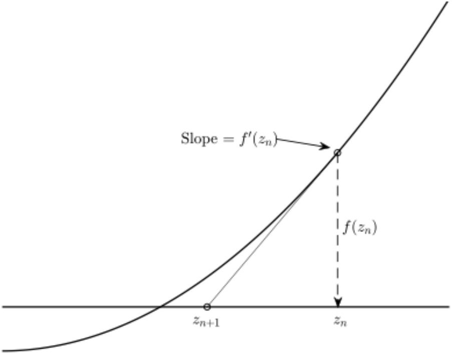
- We have four static conditions

1. Tobin's q: $\phi\iota_t = q_t - 1$
2. Portfolio/asset market clearing, κ_t^e: $\frac{a^e - a^h}{q_t} \geqslant \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Goods market clearing: $\kappa_t^e a_t^e + (1-\kappa_t^e)a^h - \iota(q_t) = [\eta_t^e \rho^e + (1-\eta_t^e)\rho^h]q_t$
4. Amplification: $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q)$
   $\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

- Start at $q(0)$, solve to the right,
  use different procedure for two η^e regions depending on κ^e:

1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (3)
   - Solve ODE using three equilibrium condition (2),(3) and (4) via Newton's method
2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (3) for $q(\eta^e)$

4b. Aside: Newton's Method

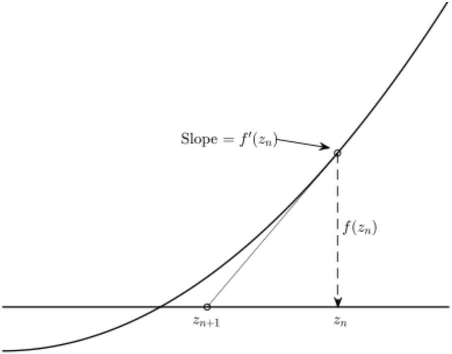


- Find the root of equation system $F(\boldsymbol{z}_n) = 0$ via iterative method:

$$\boldsymbol{z}_{n+1} = \boldsymbol{z}_n - \mathbf{J}_n^{-1}(\boldsymbol{z}_n)F(\boldsymbol{z}_n)$$

  where $\mathbf{J}_n$ is the Jacobian matrix, i.e., $\mathbf{J}_{i,j} = \partial f_i(\boldsymbol{z})/\partial z_j$
- Newton's method does not guarantee global convergence.
- Commonly take several-step iteration.

4b. Aside: Newton's Method



$$\boldsymbol{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, F(\boldsymbol{z}_n) = \begin{bmatrix} \kappa_t^e a_t^e + (1-\kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t^e \rho^e + (1-\eta_t^e)\rho^h] \\ q'(\eta_t^e)(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - q_t \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{portfolio.} \end{bmatrix}$$

Replace red terms from Tobin's Q ι and κ^e-condition.

Code: Getting started

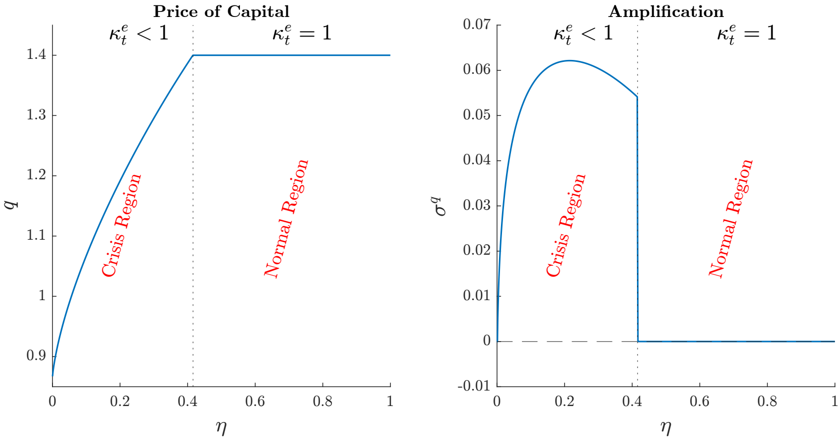
```
%% Parameters and grid
a_e = 0.11; a_h = 0.03;        % production rates
rho_0 = 0.04;                  % time preference
rho_e_d = 0.01;rho_h_d = 0.01;% death rates
rho_e = rho_0 + rho_e_d;       % expert's discount rate
rho_h = rho_0 + rho_h_d;       % household's discount rate
zeta = 0.05;                   % probability of becoming an expert
delta = 0.05; sigma = 0.1;     % decay rate/volatility
phi = 10; alpha = 0.5;         % adjustment cost/equity constraint

N = 501;                          % grid size
eta = linspace(0.0001,0.999,N)'; % grid for \eta

%% Solution
% Solve for q(0)
q0 = (1 + a_h*phi)/(1 + rho_h*phi);

% Inner loop
[Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h,
    sigma, phi, alpha);

S = (Chi - eta).*SSQ; % \sigma_{\eta^e} -- arithmetic volatility of \eta^e
Sg_e = S./eta;        % \sigma^{\eta^e} -- geometric volatility of \eta^e
Sg_h = -S./(1-eta);   % \sigma^{\eta^h} -- geometric volatility of \eta^h

VarS_e = Chi./eta.*SSQ;          % \varsigma^e -- experts' price of risk
VarS_h = (1-Chi)./(1-eta).*SSQ;  % \varsigma^h -- households' price of risk
```

Code: Inner Loop

See lecture notes. inner_loop_log.m:

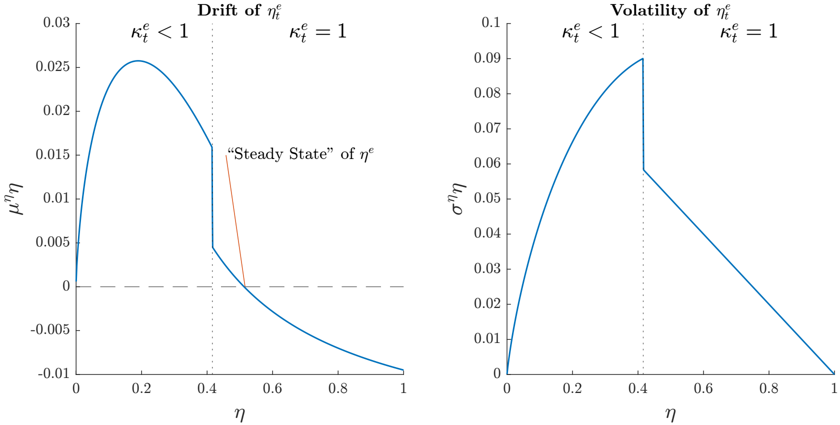
```
function [Q, SSQ, Kappa, Chi, Iota] = inner_loop_log(eta, q0, a_e, a_h, rho_e, rho_h, sigma, phi, alpha)

N = length(eta);
deta = [eta(1); diff(eta)]; % imposes the correct grid step for numerical derivative at \eta^e = 0

% variables
Q = ones(N,1);             % price of capital q
SSQ = zeros(N,1);          % \sigma + \sigma^q
Kappa = zeros(N,1); % capital fraction of experts \kappa

Rho = eta*rho_e + (1-eta)*rho_h; % auxiliary variable: average consumption-to-networth ratio

% Initiate the loop
kappa = 0; q_old = q0; q = q0; ssq = sigma;

% Iterte over eta
% At each step apply Newton's method to F(z) = 0 where z = [q, kappa, ssq]'
% Use chi = alpha*kappa
for i = 1:N
    % Compute F(z_{n-1})
    F = [kappa*(a_e - a_h) + a_h - (q-1)/phi - q*Rho(i);
         ssq*(q - (q - q_old)/deta(i) * (alpha*kappa - eta(i))) - sigma*q;
         a_e - a_h -  q*alpha*(alpha*kappa - eta(i))/(eta(i)*(1-eta(i)))*ssq
```

Solution for $q(\eta)$ and Volatility of q



$\rho^{e,h} = 0.05$, $\rho_0^{e,h} = 0.04$, $\rho_d^{e,h} = 0.01$, ζ^e=0.05,
$a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, δ =0.05, $\phi = 10$.

Solutions: Drift and Volatility of η^e



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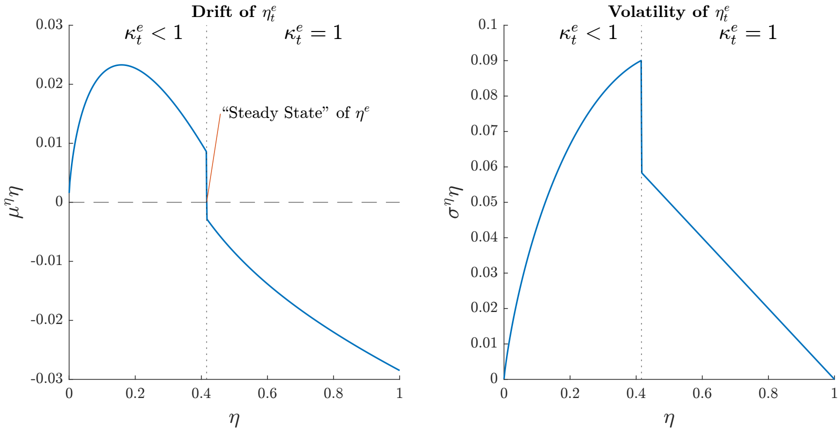
$a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, δ =0.05, $\phi = 10$.

Solutions: η^e-Drift/Volatility with SS at Region Boundary



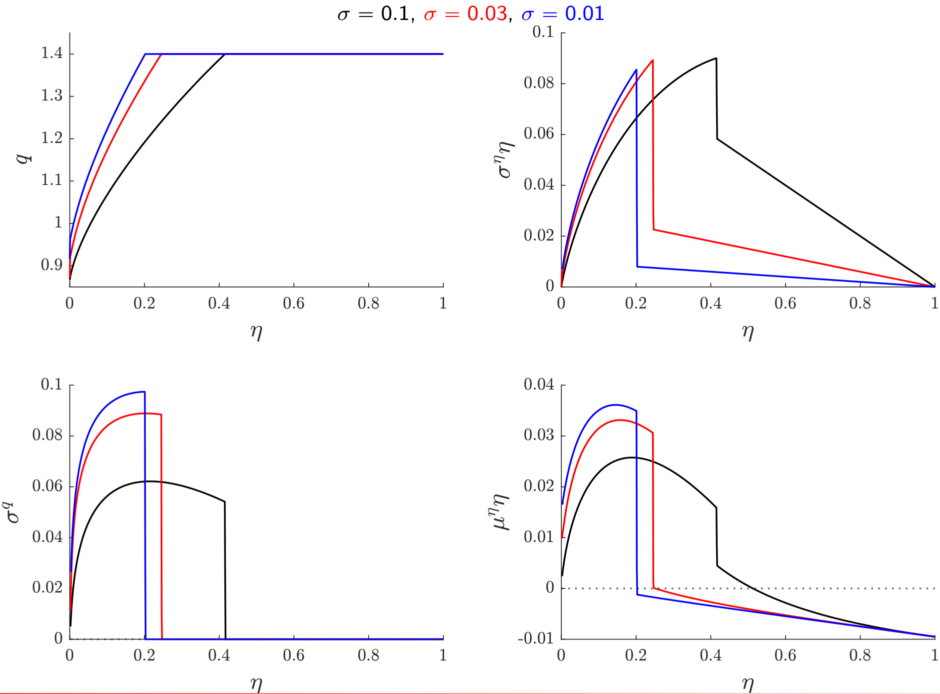
$$\rho^{e,h} = 0.05,\ \rho_0^{e,h} = 0.02,\ \rho_d^{e,h} = 0.03,\ \zeta^e{=}0.05,$$
$$a^e = 0.11,\ a^h = 0.03,\ \sigma = 0.10,\ \delta = 0.05,\ \phi = 10.$$

Poll 04.08: Is it possible for "steady state" lie in $\kappa_t^e < 1$?

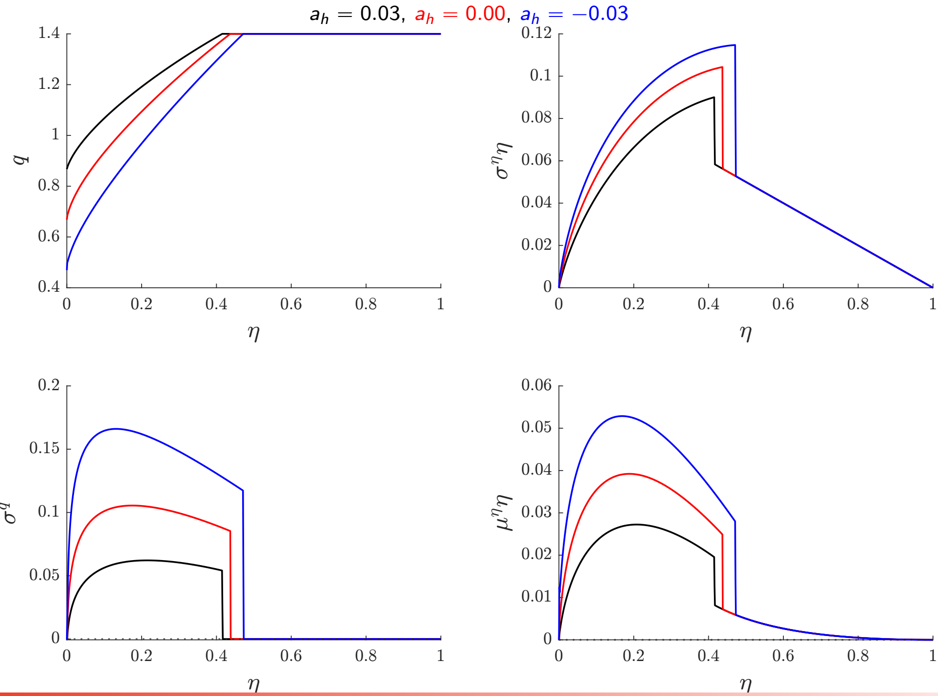
a) yes

b) no

Volatility Paradox



Market Liquidity



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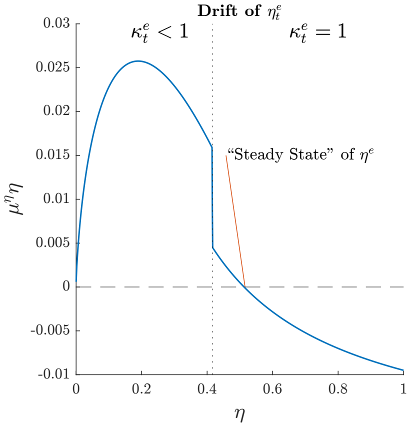
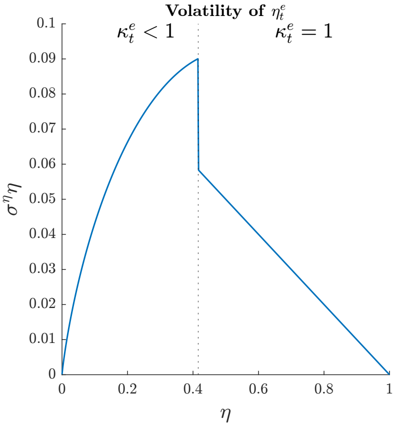
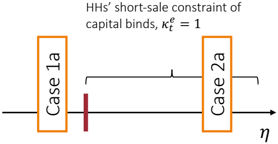
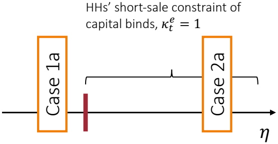
From μ_η, σ_η to Stationary Distribution









5. Kolmogorov Forward Equation

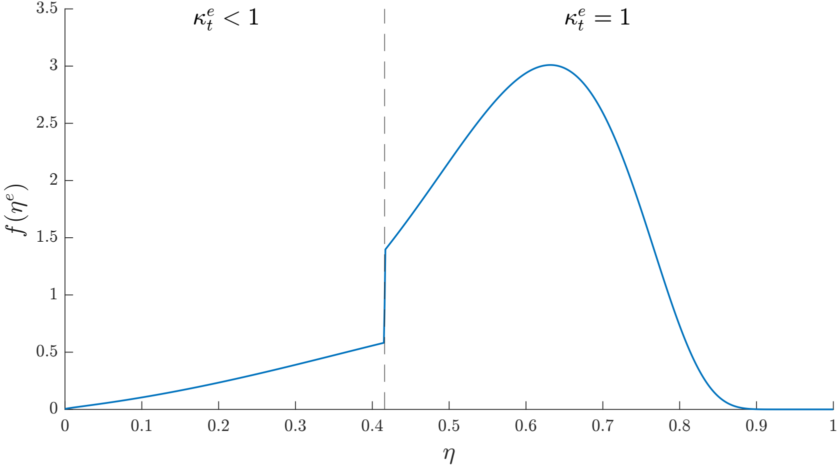
- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows:

$$\frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial[f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"
- Corollary: If stationary distribution $f(\eta)$ exists, it satisfies ODE:

$$0 = -\frac{\partial[f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

5. Stationary Distribution

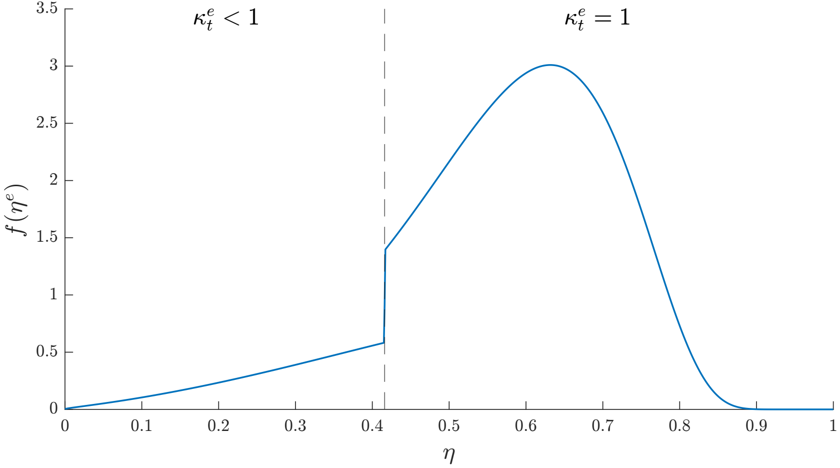


Poll 04.09: Is the constraint always (not just occasionally) binding

a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution



Poll 04.10: What happens for $\rho^e = \rho^h$

a) experts take over the economy $\eta \to 1$

b) there is a steady state

Done ... and Still to Do

- Basak-Cuoco (1998)
  - $\rho^e > \rho^h$ needed for stationarity
    $\Rightarrow$ capital price q rises after adverse shock $\sigma^q \leqslant 0$ mitigates shock
  - High required risk premium of undercapitalized experts
    is achieved by low risk-free rate (rather than capital price appreciation)
- Kiyotaki-Moore (1997)
  - Single shock critique, deterministic "bounce back" to steady state
- **Desired Model Properties:**
  - Self-stabilizing system in normal times (around stochastic SS)
    since adequately capitalized experts can absorb shock (non-linearity)
  - Destabilizing in crises times
  - Endogenous risk and price of risk (cash flows and SDF)
  - Volatility Paradox (Minsky Hypothesis)
  - Endogenous investment and growth of the macroeconomy
- **Still to do**
  - Explicit debt issuance constraint
  - "Net worth trap": double-humped distribution conditional on people not died