Macrofinance

Lecture 02: Stochastic Optimization in Continuous Time

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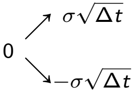
Overview of Lecture 02

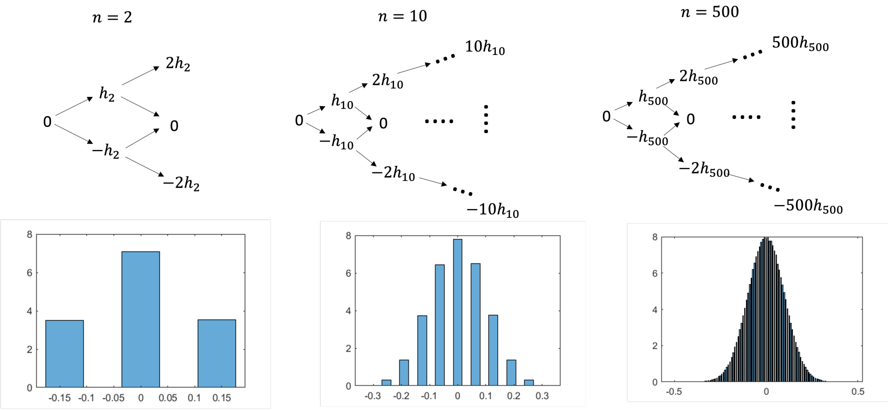
- Basic Itô Calculus
- Single-Agent Consumption-Portfolio Choice
- Stochastic Control Methods in Continuous Time
  - Hamilton-Jacobi-Bellman (HJB) Equation
  - Stochastic Maximum Principle (Pontryagin)
  - Martingale Method

Brownian Motion $\mathrm{d}Z$

- Brownian Motion as a binomial tree over Δt.



- More steps with shrinking step size: $h_n = \sigma\sqrt{\Delta t/n}$



Notations for Itô's Process

- Arithmetic Itô's Process: $dX_t = \mu_{X,t}\mathrm{d}t + \sigma_{X,t}\mathrm{d}Z_t$
  - X in the subscript of μ and σ
  - $\mu_{X,t}$ and $\sigma_{X,t}$ (can be) time varying
- Geometric Itô's Process: $dX_t = \mu_t^X X_t\mathrm{d}t + \sigma_t^X X_t\mathrm{d}Z_t$
  - X in the superscript of μ and σ.
  - Example: Stock goes up 32% or down 32% over a year (256 trading days):

$$\sigma^X = \frac{32\%}{\sqrt{256}} = 2\%$$

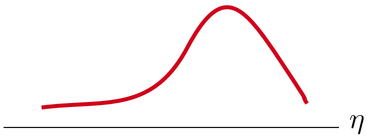
- Note: This is not a general convention, but used during this course.

Itô Processes: Characterization, Skewness over Δt

- Itô processes ... fully characterized by drift and volatility

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}Z_t$$

  - Arithmetic Itô's Process: $\mathrm{d}X_t = \mu_{X,t}\mathrm{d}t + \sigma_{X,t}\mathrm{d}Z_t$
  - Geometric Itô's Process: $\mathrm{d}X_t = \mu_t^X X_t\mathrm{d}t + \sigma_t^X X_t\mathrm{d}Z_t$
- Characterization for full volatility dynamics on Prob.-space
  - Discrete time: Probability loading on states
    conditional expectations $\mathbb{E}[X|Y]$ difficult to handle
  - Cts. time Loading on a Brownian Motion $\mathrm{d}Z_t$ captured by σ
- Normal distribution for *dt*, yet with skewed distribution for $\Delta t > 0$



  - If σ_t is time-varying
  - E.g. from normal-*dt* to log-normal-Δt and vice versa (geometric $\mathrm{d}X_t$.)

Basics of Itô's Calculus

- Itô's Lemma in geometric notation:

$$df(X_t) = \left[f'(X_t)\mu_t^X X_t + \frac{1}{2} f''(x) \left(\sigma_t^X X_t \right)^2 \right] \mathrm{d}t + f'(X_t)\sigma_t^X X_t \mathrm{d}Z_t$$

  - Example: SDF's volatility for CRRA utility: $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, u'(c) = c^{-\gamma}$

$$\xi_t = e^{-\rho t} \frac{c_t^{-\gamma}}{c_0^{-\gamma}} \Rightarrow \sigma_t^{\xi} = -\gamma \sigma_t^c$$

- Itô product rule: (stock price * exchange rate)

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)\mathrm{d}t + (\sigma_t^X + \sigma_t^Y)\mathrm{d}Z_t$$

- Itô ratio rule:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = [\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X)]\mathrm{d}t + (\sigma_t^X - \sigma_t^Y)\mathrm{d}Z_t$$

Overview

- Basic Itô Calculus
- Single-Agent Consumption-Portfolio Choice
- Stochastic Control Methods in Continuous Time
  - Hamilton-Jacobi-Bellman (HJB) Equation
  - Stochastic Maximum Principle (Pontryagin)
  - Martingale Method

Single-Agent Consumption-Portfolio Choice

- Choose consumption $\{c_t\}_{t=0}^{\infty}$ and portfolio weights to $\{\theta_t\}_{t=0}^{\infty}$ to maximize:

$$\mathbb{E}\left[\int_0^{\infty} e^{-\rho t} u(c_t) dt\right], \quad \text{with } u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

- Subject to:
  - Net worth evolution

$$\forall t > 0 : \mathrm{d}n_t = -c_t \mathrm{d}t + n_t[\theta_t r_t \mathrm{d}t + (1-\theta_t)\mathrm{d}r_t^a]$$

  - A solvency constraint: $\forall t > 0, n_t \geqslant 0$.

    alternatively, a "no Ponzi condition" leads to identical solution
- Beliefs about:
  - r_t risk-free rate
  - $\mathrm{d}r_t^a$ risky asset return process with risk premium δ_t^a: $\mathrm{d}r_t^a = (r_t + \delta_t^a)\mathrm{d}t + \sigma_t^a \mathrm{d}Z_t$
  - Take prices/returns as given

State Space

- Suppose returns are a function of state variable η_t:

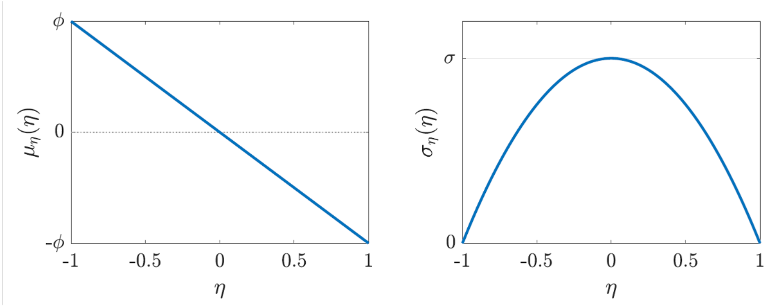
$$r_t = r(\eta_t), \quad \delta_t^a = \delta^a(\eta_t), \quad \sigma_t^a = \sigma^a(\eta_t)$$

- η_t evolves according to a diffusion process:

$$\mathrm{d}\eta_t = \mu_t^\eta(\eta_t)\eta_t \mathrm{d}t + \sigma_t^\eta(\eta_t)\eta_t \mathrm{d}Z_t$$

  - with initial state η_0 given
- Then decision problem has two state variables:
  - n_t controlled state
  - η_t external state
- For each initial state (n_0, η_0) we have a separate decision problem

Example: Functional Forms



- η-evolution (implies $\eta_t \in (-1, 1)$)

$$\mu^\eta \eta = \mu_\eta = -\phi\eta, \qquad \sigma_\eta(\eta) = \sigma(1 - \eta^2)$$

- Asset returns:

$$r(\eta) = r^0 + r^1\eta, \quad \delta^a(\eta) = \delta^0 - \delta^1\eta, \quad \sigma^a(\eta) = \sigma^0 - \sigma^1\eta$$

- With parameters: $r^0, r^1, \delta^0, \delta^1, \sigma^0, \sigma^1 \geqslant 0$

Stochastic Control Methods in Continuous Time

- **Hamilton-Jacobi-Bellman (HJB) Equation**
  - **Continuous-time version of Bellman Equation**
  - **Requires Markovian formulation w explicit state space defin.:** $V(\cdot)$ **vs** $V_t(\cdot)$
  - **Solve (Postulate) value function** $V(n, \eta)$
- Stochastic Maximum Principle
  - Conditions that characterize path of optimal solution (as opposed to whole value function)
  - Closer to discrete-time Euler equations than Bellman equation
  - Does not require Markovian problem structure
  - Solve (Postulate) co-state variable ξ_t^i
- Martingale Method
  - (Very general) shortcut for portfolio choice problem
  - Yields interpretable equations (effectively linear factor pricing equations)
  - But: tailored to specific problems (portfolio choice), non-trivial to apply elsewhere
  - Postulate SDF process: $\mathrm{d}\xi_t^i/\xi_t^i$

Method 1: Hamilton-Jacobi-Bellman (HJB) Equation

- Stochastic version of single-agent consumption-portfolio choice
- HJB differential equation
- Special cases:
  - Constant returns
  - Time-varying returns

Value Function and Principle of Optimality

- Notation:
  - $\mathcal{A}(n,\eta)$: set of admissible choices $\{c_t,\theta_t\}_{t=0}^{\infty}$ given the initial conditions: $n_0 = n, \eta_0 = \eta$
  - $\mathcal{A}_T(n,\eta)$: set of policies $\{c_t,\theta_t\}_{t=0}^{T}$ over $[0,T]$ that have admissible extensions to $[0,\infty)$, $\{c_t,\theta_t\}_{t=0}^{\infty} \subset \mathcal{A}(n,\eta)$
- Define the value function of the decision problem:

$$V(n,\eta) := \max_{\{\theta_t,c_t\}_{t=0}^{\infty}\in\mathcal{A}(n,\eta)} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho t}u(c_t)dt\right]$$

- It is easy to see that V satisfies the Bellman principle of optimality: for all $T>0$

$$V(n,\eta) := \max_{\{\theta_t,c_t\}_{t=0}^{T}\subset\mathcal{A}_T(n,\eta)} \mathbb{E}_t\left[\int_0^{T} e^{-\rho t}u(c_t)\mathrm{d}t + e^{-\rho T}V(n_T,\eta_T)\right]$$

(where n_T depends on the choice $\{\theta_t,c_t\}_{t=0}^{T}$ over $[0,T]$.)

A Stochastic Version of the HJB Equation: Derivation

- With $V_t := V(n_t, \eta_t)$, can write the principle of optimality as:

$$0 = \max_{\{\theta_t, c_t\}_{t=0}^T \subset \mathcal{A}_T(n_0,\eta_0)} \mathbb{E}_t \left[\int_0^T e^{-\rho t} u(c_t) \mathrm{d}t + e^{-\rho T} V_T - V_0 \right]$$

- By integrating by part:

$$e^{-\rho T} V_T - V_0 = -\rho \int_0^T e^{-\rho t} V_t \mathrm{d}t + \int_0^T e^{-\rho t} dV_t$$

- Combine with previous equation:

$$0 = \max_{\{\theta_t, c_t\}_{t=0}^T \subset \mathcal{A}_T(n_0,\eta_0)} \mathbb{E}_t \left[\int_0^T e^{-\rho t} (u(c_t) - \rho V_t) \mathrm{d}t + \int_0^T e^{-\rho t} dV_t \right]$$

- Divide by T, and take limit $T \downarrow 0$:

Literally this yields the following equation only for $t = 0$, but we can shift time to any intitial time due to Markovian

$$\boxed{\rho V_t \mathrm{d}t = \max_{c_t, \theta_t} \{ u(c_t) \mathrm{d}t + \mathbb{E}[dV_t] \}}$$

A Stochastic Version of the HJB Equation: Interpretation

- Stochastic version of HJB:

$$\rho V_t \mathrm{d}t = \max_{c_t, \theta_t} \{u(c_t)\mathrm{d}t + \mathbb{E}[dV_t]\}$$

- $=$ implicit backward stochastic differential equation (BSDE) for value process V_t
- What does it mean?
  - Stochastic: equation for the stochastic process V_t is not a deterministic function
  - Differential equation: relates time differential dV_t to process value V_t (& other variables)
  - Backward: forward-looking equation that must be solved backward in time, determines only expected time differential $\mathbb{E}[dV_t]$, volatility process is part of the solution
  - Implicit: $\mathbb{E}[dV_t]$ is not explicitly solved for, instead part of non-linear expression on right-hand side (due to max operator)

Digression: Alternative Derivation: Time Approximation

- Usual way of writing discrete time Bellman Equation ($\beta := e^{-\rho}$)

$$V(n_t, \eta_t) = \max_{c_t, \theta_t} \{u(c_t) + \beta \mathbb{E}_t[V(n_{t+1}, \eta_{t+1})]\}$$

- More generally, with generic period length $\Delta t > 0$ ($\beta = e^{-\rho \Delta t}$):

$$V(n_t, \eta_t) = \max_{c_t, \theta_t} \{u(c_t)\Delta t + \beta \mathbb{E}_t[V(n_{t+\Delta t}, \eta_{t+\Delta t})]\}$$

Subtract $\beta V(n_t, \eta_t)$ from both sides:

$$\frac{1-\beta}{\Delta t} V(n_t, \eta_t)\Delta t = \max_{c_t, \theta_t} \{u(c_t)\Delta t + \beta \mathbb{E}_t[V(n_{t+\Delta t}, \eta_{t+\Delta t}) - V(n_t, \eta_t)]\}$$

Taking the limit $\Delta t \to 0$ yields again:

$$\rho V(n_t, \eta_t)\mathrm{d}t = \max_{c_t, \theta_t} \{u(c_t)dt + \mathbb{E}_t[dV(n_t, \eta_t)]\}$$

Method 1: Hamilton-Jacobi-Bellman (HJB) Equation

- Stochastic version of single-agent consumption-portfolio choice
- HJB differential equation
- Special cases:
  - Constant returns
  - Time-varying returns

The (Deterministic) HJB Equation

- Next step: transform stochastic version of HJB into a (non-stochastic) differential equation
- General idea: use Itô's lemma to express $\mathbb{E}[dV_t]$ in terms of derivatives of value function V_t

Poll: The (Deterministic) HJB Equation

- Which of the following is the correct one? [Recall the definition $V_t = V(n_t, \eta_t)$]

[**a**] $$\mathbb{E}[dV_t] = \Big(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t}\Big)dt$$

[**b**] $$\mathbb{E}[dV_t] = \Big(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma^2_{n,t} + \partial_{\eta\eta} V(n_t,\eta_t)\sigma^2_{\eta,t}\right)\Big)dt$$

[**c**] $$\mathbb{E}[dV_t] = \Big(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\Big(\partial_{nn} V(n_t,\eta_t)\sigma^2_{n,t} + \partial_{\eta\eta} V(n_t,\eta_t)\sigma^2_{\eta,t} + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\Big)\Big)dt$$

[**d**] $$\mathbb{E}[dV_t] = \Big(\partial_n V(n_t,\eta_t)\mu_{n,t} + \partial_\eta V(n_t,\eta_t)\mu_{\eta,t} + \frac{1}{2}\left(\partial_{nn} V(n_t,\eta_t)\sigma^2_{n,t} + \partial_{\eta\eta} V(n_t,\eta_t)\sigma^2_{\eta,t}\right) + \partial_{\eta n} V(n_t,\eta_t)\sigma_{\eta,t}\sigma_{n,t}\Big)dt$$

The (Deterministic) HJB Equation

- Next step: transform stochastic version of HJB into a (non-stochastic) differential equation
- General idea: use Itô's lemma to express $\mathbb{E}[dV_t]$ in terms of derivatives of value function V_t
  Here, $V_t = V(n_t, \eta_t)$, so we can write:

$$\rho V_t \mathrm{d}t = \max_{c_t, \theta_t} \Bigg(u(c_t) + \partial_n V(n_t, \eta_t)\mu_{n,t} + \partial_\eta V(n_t, \eta_t)\mu_{\eta,t} \\ + \frac{1}{2}\left(\partial_{nn} V(n_t, \eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t, \eta_t)\sigma_{\eta,t}^2\right) + \partial_{\eta n} V(n_t, \eta_t)\sigma_{\eta,t}\sigma_{n,t} \Bigg) dt$$

The (Deterministic) HJB Equation

- Next step: transform stochastic version of HJB into a (non-stochastic) differential equation
- General idea: use Itô's lemma to express $\mathbb{E}[dV_t]$ in terms of derivatives of value function V_t
  Here, $V_t = V(n_t, \eta_t)$, so we can write:

$$\rho V_t \mathrm{d}t = \max_{c_t, \theta_t} \Big(u(c_t) + \partial_n V(n_t, \eta_t)\mu_{n,t} + \partial_\eta V(n_t, \eta_t)\mu_{\eta,t} \\ + \frac{1}{2}\left(\partial_{nn} V(n_t, \eta_t)\sigma_{n,t}^2 + \partial_{\eta\eta} V(n_t, \eta_t)\sigma_{\eta,t}^2\right) + \partial_{\eta n} V(n_t, \eta_t)\sigma_{\eta,t}\sigma_{n,t} \Big) dt$$

- For this problem, drifts and volatilities are:

$$\mu_{n,t} = -c_t + n_t\left[r(\eta_t) + (1-\theta_t)\delta^a(\eta_t)\right] \qquad \mu_{\eta,t} = \mu_\eta(\eta_t)$$
$$\sigma_{n,t} = n_t(1-\theta_t)\sigma^a(\eta_t) \qquad \sigma_{\eta,t} = \sigma_\eta(\eta_t)$$

The (Deterministic) HJB Equation

- Combining the previous equation and dropping dt and time subscripts:

$$\begin{aligned}\rho V(n,\eta) = &\max_c \left(u(c) - \partial_n V(n,\eta)c\right) \\ &+ \max_\theta \Big\{ \partial_n V(n,\eta) n(r(\eta) + (1-\theta)\delta^a(\eta)) \\ &\qquad + \left(\frac{1}{2}\partial_{nn} V(n,\eta) n(1-\theta)\sigma^a(\eta) + \partial_{\eta n} V(n,\eta)\sigma_\eta(\eta)\right) n(1-\theta)\sigma^a(\eta) \Big\} \\ &+ \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta} V(n,\eta)(\sigma_\eta(\eta))^2\end{aligned}$$

This is a nonlinear partial differential equation (PDE) for $V(n,\eta)$
Note: nonlinearity enters through the max operator

Method 1: Hamilton-Jacobi-Bellman (HJB) Equation

- Stochastic version of single-agent consumption-portfolio choice
- HJB differential equation
- Special cases:
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Special Case: Constant Returns

Lets first assume that returns are constant: $r_t = r, \delta_t^a = \delta^a, \sigma_t^a = \sigma^a$

Can then drop η from the problem and write the HJB as:

$$\rho V(n) = \max_c \left(u(c) - V'(n)c\right) + \max_\theta \left(V'(n)n(r + (1-\theta)\delta^a) + \frac{1}{2}V''(n)n^2((1-\theta)\sigma^a)^2\right)$$

To solve this equation, first solve optimizations:

- optimal consumption choice: marginal utility of consumption = marginal value of wealth

$$u'(c) = V'(n)$$

- optimal portfolio choice: Merton portfolio weight

$$1-\theta = \left(-\frac{V''(n)n}{V'(n)}\right)^{-1} \frac{\delta^a}{(\sigma^a)^2}$$

Remarks:

- this has a flavor of mean-variance portfolio choices: $-\frac{V''(n)n}{V'(n)}$ is the relative risk aversion, δ^a is the excess return and $(\sigma^a)^2$ is the risky asset's variance

Solving HJB for Constant Return Case

- We could now plug optimal choices and solve the resulting ODE numerically
- Instead for this problem: guess functional form and solve analytically
- Guess: $V(n) = \frac{u(\omega n)}{\rho}$ with some constant $\omega > 0$. Plugging into HJB equaiton
  - $\gamma = 1$ (log utility) $\Rightarrow V(n) = \frac{1}{\rho}(\log \omega + \log n)$

$$\log \omega + \log n = \log \rho + \log n - 1 + \frac{1}{\rho}\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)$$

  - $\gamma \neq 1$:

$$\rho \frac{(\omega n)^{1-\gamma}}{\rho} = \gamma \rho^{1/\gamma} \omega^{1-1/\gamma} \frac{(\omega n)^{1-\gamma}}{\rho} + (1-\gamma)\left(r + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)\right)\frac{(\omega n)^{1-\gamma}}{\rho}$$

In both cases, n cancels out, thus verifying our guess (we can then solve for ω)

Full Solution for Constant Return Case

- Value function:

$$V(n) = \frac{u(\omega n)}{\rho}$$

- Optimal choices:

$$\begin{cases} c(n) = \rho^{1/\gamma}\omega^{1-1/\gamma} n \\ 1-\theta(n) = \dfrac{1}{\gamma}\dfrac{\delta^a}{(\sigma^a)^2} \end{cases}$$

- Constant ω in the value function (for $\gamma \neq 1$):

$$\omega = \rho\left(1 + \frac{\gamma-1}{\gamma}\frac{1}{\rho}\left(r - \rho + \frac{1}{2\gamma}\left(\frac{\delta^a}{\sigma^a}\right)^2\right)\right)^{\frac{\gamma}{\gamma-1}}$$

Discussion of Optimal Consumption Choice

$$c_t/n_t = \rho^{1/\gamma}\omega_t^{1-1/\gamma}$$

- Reaction of c/n to investment opportunities ω depends on EIS $\psi := 1/\gamma$:
  - i $\psi < 1$ better investment opportunities $\Rightarrow$ consumption $\uparrow$, savings $\downarrow$
  - ii $\psi > 1$ better investment opportunities $\Rightarrow$ consumption $\downarrow$, savings $\uparrow$
  - iii $\psi = 1$ consumption-wealth ratio independent of investment opportunities
- Why this ambiguous relationship? Two effects:
  1. income effect:
     - improved investment opportunities ω make investor effectively richer
     - investor responds by increasing consumption in all periods
  2. substitution effect:
     - improved investment opportunities ω makes saving more attractive
     - to benefit from them, investor reduces consumption now to get more consumption later

  $\psi < 1$ substitution effect weak (consumption smoothing desire), income effect dominates

  $\psi > 1$ investor less averse against variation in consumption, substitution effect dominates

Discussion of Optimal Consumption Choice

- Combining the previous equation and dropping dt and time subscripts:

$$\begin{aligned}\rho V(n,\eta) = & \max_c \left(u(c) - \partial_n V(n,\eta)c\right) \\ & + \max_\theta \Big\{ \partial_n V(n,\eta) n(r(\eta) + (1-\theta)\delta^a(\eta)) \\ & \qquad + \left(\frac{1}{2}\partial_{nn} V(n,\eta) n(1-\theta)\sigma^a(\eta) + \partial_{\eta n} V(n,\eta)\sigma_\eta(\eta)\right) n(1-\theta)\sigma^a(\eta) \Big\} \\ & + \partial_\eta V(n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{\eta\eta} V(n,\eta)(\sigma_\eta(\eta))^2\end{aligned}$$

Solution method 1: solve this two-dimensional PDE for V numerically
Solution method 2: guess $V(n,\eta) = \frac{u(\omega(\eta)n)}{\rho}$ and reduce to one-dimensional ODE for $\omega(\eta)$

Time-Varying Returns: Optimal Consumption and Portfolio

- Optimal consumption choice (after using guess from previous slide)

$$c(n,\eta) = \rho^{1/\gamma}(\omega(\eta))^{1-1/\gamma} n$$

  - as for constant returns, but now investment opportunities $\omega(\eta)$ are state-dependent

- Optimal portfolio choice (after using guess from previous slide)

$$1-\theta(n,\eta) = \underbrace{\frac{1}{\gamma}\frac{\delta^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\frac{\omega'(\eta)}{\omega(\eta)}\sigma_\eta(\eta)\sigma^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{hedging demand}}$$

  - additional hedging demand term that depends on covariance $\sigma^\omega\sigma^a$ of investment opportunities with asset return

Time-Varying Returns: Hedging Demand

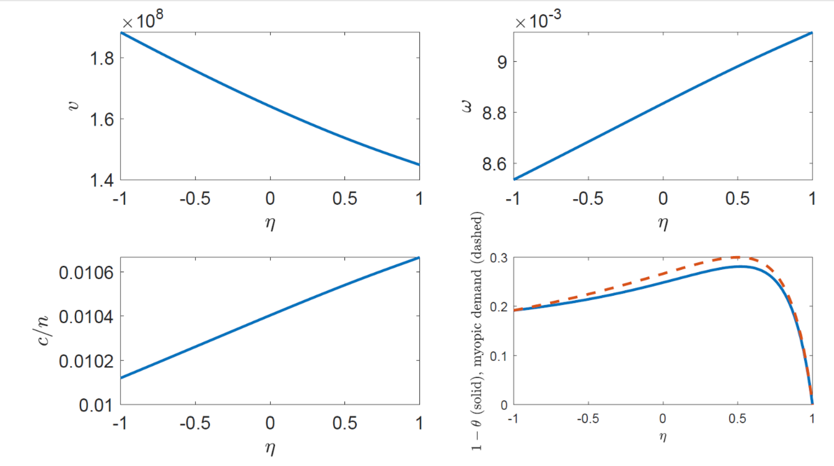
$$1-\theta(n,\eta) = \underbrace{\frac{1}{\gamma}\frac{\delta^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\frac{\omega'(\eta)}{\omega(\eta)}\sigma_\eta(\eta)\sigma^a(\eta)}{(\sigma^a(\eta))^2}}_{\text{hedging demand}}$$

- Why should variation in future investment opportunities be relevant for portfolio choice? Two opposing motives:
  1. If investment opportunities are good, it is valuable to have any resources available
     - → invest in assets that pay off in states in which investment opportunities are good
  2. If investment opportunities are bad, that's bad time for investor and additional wealth is valuable
     - → invest in assets that pay off in states in which investment opportunities are bad
- Which of the two dominates depends on γ:
  a. $\gamma < 1$, investor not very risk averse, prefer to have resources when it is profitable to invest
  b. $\gamma > 1$, investor sufficiently risk averse to want to hedge against bad times
  c. $\gamma = 1$, the two forces cancel out, investor acts myopically
- Remark: a very conservative investor ($\gamma \to \infty$) only cares about the hedging component

Determining Investment Opportunities

- When substituting optimal choices into HJB, n cancels out, and we get ODE for $\omega(\eta)$
- One can solve this numerically for the function $\omega(\eta)$
- Details will be provided in Lecture 06 (later)
  - (E.g., solve equivalently for $v(\eta) := (\omega(\eta))^{1-\gamma}$ which is a "more linear" (less kinky) ODE.)

Example Solution



Parameters:

$\rho = 0.02, \gamma = 5, \phi = 0.2, \sigma = 0.1, r^0 = 0.02, r^1 = 0.01, \delta^0 = 0.3, \delta^1 = 0.03, \sigma^0 = 0.15$

Stochastic Control Methods in Continuous Time

- Hamilton-Jacobi-Bellman (HJB) Equation
  - Continuous-time version of Bellman Equation
  - Requires Markovian formulation with explicit definition of state space: $V(\cdot)$ vs $V_t(\cdot)$
  - Solve (Postulate) value function $V(n, \eta)$
- **Stochastic Maximum Principle**
  - **Conditions that characterize path of optimal solution (as opposed to whole value function)**
  - **Closer to discrete-time Euler equations than Bellman equation**
  - **Does not require Markovian problem structure**
  - **Solve (Postulate) co-state variable ξ_t^i**
- Martingale Method
  - (Very general) shortcut for portfolio choice problem
  - Yields interpretable equations (effectively linear factor pricing equations)
  - But: tailored to specific problems (portfolio choice), non-trivial to apply elsewhere
  - Postulate SDF process: $\mathrm{d}\xi_t^i/\xi_t^i$

Method 2: Stochastic Maximum Principle

- Define the Hamiltonian

$$H_t^i = e^{-\rho t}\frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i n_t^i \mu_t^{n^i} + \sigma_{\xi^i,t} n_t^i \sigma_t^{n^i}$$
$$= \underbrace{e^{-\rho t}\frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i\left[-c_t^i + n_t^i(1-\theta_t^i)(r_t+\delta_t^a) + n_t^i\theta_t^i r_t + \frac{\sigma_{\xi^i,t}}{\xi_t^i} n_t^i(1-\theta_t^i)\sigma_t^{r^a}\right]}_{=:H(t,n_t^i,c_t^i,\theta_t^i,\xi_t^i,\sigma_{\xi^i,t})}$$

- ξ_t^i is called the costate of the system
  - it plays the role of a dynamic Lagrange multiplier on the state evolution for n_t^i
  - $\sigma_{\xi^i,t}$ is the costate volatility (dZ_t-loading of ξ_t^i)
- Stochastic Maximum Principle: under certain (convexity) conditions, the optimal choices c_t^i, θ_t^i maximize the Hamiltonian and the costate ξ_t^i satisfies the BSDE

$$\mathrm{d}\xi_t^i = -\partial_n H(t, n_t^i, c_t^i, \theta_t^i, \xi_t^i, \sigma_{\xi^i,t})\mathrm{d}t + \sigma_{\xi^i,t}\mathrm{d}Z_t$$

Note: $\sigma_{\xi^i,t}$ is part of the solution

Maximum Principle – Step 1: Maximizing the Hamiltonian

- Hamiltonian (from last slide):

$$H_t^i = e^{-\rho t}\frac{(c_t^i)^{1-\gamma}}{1-\gamma} + \xi_t^i \left[-c_t^i + n_t^i(1-\theta_t^i)(r_t + \delta_t^a) + n_t^i \theta_t^i r_t + \frac{\sigma_{\xi^i,t}}{\xi_t^i} n_t^i (1-\theta_t^i)\sigma_t^{r^a} \right]$$

- FOC w.r.t c_t^i, θ_t^i

$$e^{-\rho t}(c_t^i)^{-\gamma} = \xi_t^i$$

$$\delta_t^a = -\frac{\sigma_{\xi^i,t}}{\xi_t^i}\sigma_t^{r^a}$$

- Let's define $\varsigma_t^i := -\frac{\sigma_{\xi^i,t}}{\xi_t^i}$, then second condition becomes

$$\delta_t^a = \varsigma_t^i \sigma_t^{r^a}$$

  - interpretation: ς_t^i is the (shadow) price of risk
  - will become clearer in martingale method below: costate ξ_t^i coincides with the SDF

Maximum Principle – Step 2: Costate Equation

- Costate equation (additional FOC)

$$\mathrm{d}\xi_t^i = -\frac{\partial H_t^i}{\partial n_t^i}\mathrm{d}t - \varsigma_t^i \xi_t^i \mathrm{d}Z_t$$

- The drift of ξ_t^i is given by:

$$\mu_{\xi^i,t} = -\frac{\partial H_t}{\partial n_t^i} = -\xi_t^i\left[(1-\theta_t^i)(r_t+\delta_t^a) + \theta_t^i r_t - \varsigma_t^i(1-\theta_t^i)\sigma_t^{r^a}\right]$$

$$= -\xi_t^i\Big[r_t + (1-\theta_t^i)\underbrace{(\delta_t^a - \varsigma_t^i \sigma_t^{r^a})}_{=0 \text{ by portf. choice}}\Big]$$

- Hence,

$$\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = -r_t \mathrm{d}t - \varsigma_t^i \mathrm{d}Z_t$$

- Equivalent but shorter:

$$\boxed{\frac{\mathbb{E}_t[\mathrm{d}\xi_t^i]}{\xi_t^i} = -r_t \mathrm{d}t}$$

Remark:

- this is essentially the consumption Euler equation
- using consumption FOC: $\frac{\mathbb{E}_t[d((c_t^i)^{-\gamma})]}{(c_t^i)^{-\gamma}dt} = \rho - r_t$

Solving the Costate Equation I: Transform into PDE

- Costate equation plays similar role as stochastic version of HJB
- We can transform it into a PDE by applying Ito to $\xi_t^i = \xi(t, n_t^i, \eta_t)$:

$$\begin{aligned}
0 = {} & r(\eta)\xi(t,n,\eta) + \partial_t \xi(t,n,\eta) \\
& + \partial_n \xi(t,n,\eta)\left(-c^*(t,n,\eta;\xi) + n\left(r(\eta) + (1-\theta^*(t,n,\eta;\xi)\delta^a(\eta)\right)\right) \\
& + \partial_\eta \xi(t,n,\eta)\mu_\eta(\eta) + \frac{1}{2}\partial_{nn}\xi(t,n,\eta)n^2(1-\theta^*(t,n,\eta;\xi))^2(\sigma^a(\eta))^2 \\
& + \partial_{\eta n}\xi(t,n,\eta)\sigma_\eta(\eta)n(1-\theta^*(t,n,\eta;\xi))\sigma^a(\eta) + \frac{1}{2}\partial_{\eta\eta}\xi(t,n,\eta)(\sigma_\eta(\eta))^2
\end{aligned}$$

  where $c^*(t,n,\eta;\xi)$, $\theta^*(t,n,\eta;\xi)$ are the optimal choices that maximize H_t^i (which depend on ξ_t^i)
- **Link to HJB**: we obtain same PDE if we take derivative in HJB with respect to n and substitute $\xi(t,n,\eta) = e^{-\rho t}V(n,\eta)$
  - reason: ξ_t^i acts like Lagrange multiplier on the net worth evolution
  - envelope theorem: ξ_t^i is marginal (time-*zero*) utility benefit of giving agent i an additional unit of (time t) wealth

Example: HJB-Costate Eq. Connection without η-State

- Suppose returns are constant, value function $V(n)$ satisfies HJB

$$\rho V(n) = u(c^*) + V'(n)\left(-c + n(r + (1-\theta^*)\delta^a)\right) + \frac{1}{2}V''(n)n^2((1-\theta^*)\sigma^a)^2$$

- Take derivative w.r.t. n and multiply by $e^{-\rho t}$
(can ignore dependence of c^* and θ^* on n by envelope theorem)

$$\rho e^{-\rho t}V'(n) = e^{-\rho t}V''(n)\left(-c^* + n(r + (1-\theta^*)\delta^a)\right) + \frac{1}{2}e^{-\rho t}V'''(n)n^2((1-\theta^*)\sigma^a)^2$$
$$+ e^{-\rho t}V'(n)\left(r + (1-\theta^*)\delta^a\right) + e^{-\rho t}V''(n)n((1-\theta^*)\sigma^a)^2$$

- Substitute $\xi(t,n) := e^{-\rho t}V'(n)$ and use the following facts:
  - $\partial_t \xi(t,n) = -\rho\xi(t,n)$
  - $\varsigma(t,n) = -\frac{\partial_n \xi(t,n)}{\xi(t,n}n(1-\theta^*)\sigma^a = -\frac{\partial_n V''(n)}{V'(n)}n(1-\theta^*)\sigma^a$ (by Ito)
  - $\delta^a - \varsigma(t,n)(1-\theta^*)\sigma^a = 0$ (by optimal portfolio choice)

$$0 = r\xi(t,n) + \partial_t\xi(t,n) + \partial_n\xi(t,n)\left(-c^* + n(r + (1-\theta^*)\delta^a)\right) + \frac{1}{2}\partial_{nn}\xi(t,n)n^2((1-\theta^*)\sigma^a)^2$$

- This is the same PDE as on previous slide

Solving the Costate Equation II: Guess and Verify

- We can also keep the stochastic costate equation and verify a guess
- E.g. in constant returns case, $V(n) = \frac{u(\omega n)}{\rho}$ suggests guess $\xi_t = e^{-\rho t} \frac{\omega}{\rho} u'(\omega n_t)$
- This guess implies:
  - from first-order conditions:

$$c_t^i = \rho^{1/\gamma} \omega^{1-1/\gamma} n_t^i$$
$$1 - \theta = \frac{1}{\gamma} \frac{\delta^a}{(\sigma^a)^2}$$

  - by Ito:

$$\frac{d\xi_t}{\xi_t} = -\rho dt + \frac{dn_t^{-\gamma}}{n_t^{-\gamma}} = \left(-\rho - \gamma \mu_t^n + \frac{\gamma(\gamma+1)}{2} (\sigma_t^n)^2\right) dt - \gamma \sigma_t^n dZ_t$$

- Substituting drift into costate equation:
  - case $\gamma = 1$: verifies guess (ω drops out)
  - case $\gamma \neq 1$: yields algebraic equation for ω (same as in HJB approach)

Stochastic Control Methods in Continuous Time

- Hamilton-Jacobi-Bellman (HJB) Equation
  - Continuous-time version of Bellman Equation
  - Requires Markovian formulation with explicit definition of state space: $V(\cdot)$ vs $V_t(\cdot)$
  - Solve (Postulate) value function $V(n, \eta)$
- Stochastic Maximum Principle
  - Conditions that characterize path of optimal solution (as opposed to whole value function)
  - Closer to discrete-time Euler equations than Bellman equation
  - Does not require Markovian problem structure
  - Solve (Postulate) co-state variable ξ_t^i
- **Martingale Method**
  - **(Very general) shortcut for portfolio choice problem**
  - **Yields interpretable equations (effectively linear factor pricing equations)**
  - **But: tailored to specific problems (portfolio choice), non-trivial to apply elsewhere**
  - **Postulate SDF process:** $\mathrm{d}\xi_t^i/\xi_t^i$

Method 3: Martingale Approach – Discrete Time

$$\max_{\{c_t,\boldsymbol{\theta}_t\}_{\tau=t}^{T}} \mathbb{E}_t\left[\sum_{\tau=t}^{T}\frac{1}{(1+\rho)^{\tau-t}}u(c_\tau)\right]$$
$$s.t. \quad \boldsymbol{\theta}_t\boldsymbol{p}_t = \boldsymbol{\theta}_{t-1}(\boldsymbol{p}_t + \boldsymbol{d}_t) - c_t, \quad \text{for all } t$$

- FOC w.r.t θ_t at t
$$\xi_t p_t = \mathbb{E}\left[\xi_{t+1}(p_{t+1} + d_{t+1})\right]$$

  where $\xi_t = \frac{u'(c_t)}{(1+\rho)^t}$ is the (multi-period) stochastic discount factor (SDF)
  - If projected on asset span, then pricing kernel ξ_t^*
  - Note: $MRS_{t,\tau} = \xi_{t+\tau}/\xi_t$
- Consider portfolio, where one reinvests dividend d
  - Portfolio is a self-financing trading strategy, A, with price, p_t^A
$$\xi_t p_t^A = \mathbb{E}_t\left[\xi_{t+1}p_{t+1}^A\right]$$
  - $\xi_t p_t^A$ is a martingale.

Method 3: Martingale Approach – Cts. Time

$$\max_{\{c_t,\boldsymbol{\theta}_t\}_{t=0}^{\infty}} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho t}u(c_t)dt\right]$$

$$s.t. \quad \frac{\mathrm{d}n_t}{n_t} = -\frac{c_t}{n_t}\mathrm{d}t + \sum_j \theta_t^j \mathrm{d}r_t^j + \text{labor income/endowment/taxes}$$

$$n_0 \text{ given}$$

- Portfolio Choice: Martingale Approach
  - Let x_t^A be the value of a "self-financing trading strategy" (reinvest dividends)
- Theorem: $\xi_t x_t^A$ *follows a martingale, i.e., drift = 0*
  - Let $\frac{dx_t^A}{x_t^A} = \mu_t^A \mathrm{d}t + \sigma_t^A \mathrm{d}Z_t$, postulate $\frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i}\mathrm{d}t + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i}\mathrm{d}Z_t$. Then by product rule:

$$\frac{d(\xi_t^i x_t^A)}{\xi_t^i x_t^A} = \underbrace{\left(-r_t^i + \mu_t^A - \varsigma_t^i\sigma_t^A\right)}_{=0}\mathrm{d}t + \text{volatility term} \Rightarrow \boxed{\mu_t^A = r_t^i + \varsigma_t^i\sigma_t^A}$$

    - For risk-free asset, i.e., $\sigma_t^A = 0$, $r_t^f = r_t^i$
    - Excess expected return to risky asset B: $\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$

Remark: What is ξ_t for CRRA utility

- ξ_t is $e^{-\rho t}u'(c_t) = e^{-\rho t}c_t^{-\gamma}$. [Note: $dc_t = \mu_t^c c_t \mathrm{d}t + \sigma_t^c c_t \mathrm{d}Z_t$]
- Apply Itô's Lemma:
  - Note: $u'' = -\gamma c^{-\gamma-1}, u''' = \gamma(\gamma+1)c^{-\gamma-2}$

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -\underbrace{\left(\rho + \gamma\mu_t^c - \frac{1}{2}\gamma(\gamma+1)(\sigma_t^c)^2\right)}_{r_t^f}\mathrm{d}t - \underbrace{\gamma\sigma_t^c}_{\varsigma_t}\mathrm{d}Z_t$$

  - Risk free rate r_t^f
  - Price of risk ς_t
  - Aside: Epstein-Zin(-Duffie) preferences with EIS ψ

$$r^f = \rho + \psi^{-1}\mu_t^c - \frac{1}{2}\gamma(\psi^{-1}+1)(\sigma_t^c)^2$$

Method 3: Martingale Approach - Cts. Time

- Proof 1: Stochastic Maximum Principle (see Handbook chapter)
- Proof 2: Intuition (calculus of variation)
  Remove from the optimum Δ at t_1 and add back at t_2

$$V(n,\omega,t) = \max_{\{\iota_s,\boldsymbol{\theta}_s,c_t\}_{s=t}^{\infty}} \mathbb{E}_t\left[\int_0^{\infty} e^{-\rho(s-t)}u(c_s)ds|\omega_t = \omega\right]$$

  - s.t. $n_t = n$

$$e^{-\rho t_1}\frac{\partial V}{\partial n}(n^*_{t_1}, x_{t_1}, t_1)x^A_{t_1} = \mathbb{E}\left[e^{-\rho t_2}\frac{\partial V}{\partial n}(n^*_{t_2}, x_{t_2}, t_2)x^A_{t_2}\right]$$

  - See Lecture notes and Merkel's handout

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Conclusion

- Basic Itô Calculus
- Single-Agent Consumption-Portfolio Choice
- Stochastic Control Methods in Continuous Time
  - Hamilton-Jacobi-Bellman (HJB) Equation
  - Stochastic Maximum Principle (Pontryagin)
  - Martingale Method