Princeton Initiative

Problem for the class exercise on Saturday, September 7, 2024

Professor Yuliy Sannikov

Problem. The goal of this problem is to characterize equilibria in the model of He-Krishnamurthy ("Intermediary Asset Pricing") and to use the iterative method to compute equilibria. There are two agent types: experts and households. Households have log utility, while experts have CRRA utility with relative risk aversion γ, and both with discount rate ρ. New households are born continuously, and the newborn receive labor income at rate lK_t.

Aggregate capital follows the law of motion

$$\frac{dK_t}{K_t} = g\,dt + \sigma\,dZ_t.$$

Capital produces dividend of aK_t. The price of capital per unit is denoted by q_t and follows

$$\frac{dq_t}{q_t} = \mu_t^q\,dt + \sigma_t^q dZ_t.$$

Only experts can hold capital, and they can finance capital by borrowing through risk-free debt and by issuing equity to households, but they must retain fraction of at least $\underline{\chi} = 1/(1+m)$ of risk.

(a) Write down the expression for dr_t^k, for the return on capital.

Experts make optimal consumption and portfolio decisions: they choose how much to borrow and how much outside equity to issue (up to fraction $1-\chi$) to buy capital. Denote the required price of risk of experts by ς_t and recall that $\varsigma_t = \gamma\sigma_t^C$, where σ_t^C is the volatility of aggregate consumption of experts. Denote the value function of a representative expert by

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma}.$$

(b) Write down the law of motion of aggregate net worth of experts N_t as a function of the risk-free rate r_t, the experts' equity share χ_t, the experts' net worth share η_t, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' price of risk ς_t and process v_t. To write the law of motion of N_t, you need to express the experts' consumption rate C_t/N_t as a function of η_t, q_t and v_t.

(c) He and Krishnamurthy assume that inside and outside equity of experts earn the same returns. Thus, the experts' equity held by households earns the risk premium of ς_t times the volatility, even though households' price of risk is higher. Under this assumption, write down the law of motion of world wealth $q_t K_t$, as a function of the risk-free rate r_t, the price of capital q_t, the volatility capital return $\sigma + \sigma_t^q$, the experts' price of risk ς_t and output parameter a.

(d) From your answers to parts (b) and (c), derive the law of motion of the experts' wealth share $\eta_t = N_t/(q_t K_t)$.

(e) Write down the market-clearing condition for output. Hint: Recall that total world output is $(a + l)K_t$, including dividend and labor income of newborn households.

Next, you should determine the size of the "constrained region" where $\chi_t = \underline{\chi}$ and the size of the unconstrained region where $\chi_t = \underline{\chi}$. To do that, you should use the following assumptions of He and Krishnamurthy. Assume that fraction λ of households (i.e. the net worth share of these households is $(1 - \eta_t)\lambda$) are "debt" households who can only hold the risk-free asset. Fraction $1-\lambda$ are "equity" households who can hold outside equity of experts and the risk-free asset. He and Krishnamurthy furthermore assume that equity households cannot use leverage, i.e. the risk of their net worth can be at most equal to the risk of experts' net worth (who hold their own inside equity). Assume (you can verify this later), that it is this constraint that determines the amount of equity that experts can issue.

(f) Derive the value of χ_t as a function of η_t implied by the constraint that equity households cannot use leverage.

The goal of the next questions is to formulate a procedure to compute equilibria using Matlab, using the Iterative Method from the Handbook Chapter.[1] You should use Matlab function payoff_policy_growth to perform the "time step" of the iterative procedure.

(g) Formulate a procedure for the static step. That is, suppose you are given function $v(t, \eta)$ for all η at time t. Find the price of capital $q(t, \eta)$ for all

[1] See Brunnermeier, M. and Y. Sannikov (2016) "Macro, Money, and Finance: A Continuous-Time Approach."

η at time t. Then, given this function, derive the law of motion of η. Derive the backward differential equation that $v(t, \eta)$ must satisfy.

(h) Formulate a procedure for the time step. That is, for the function F = payoff_policy_growth(X, R, MU, S, G, V, lambda), what values of X, R, MU, S, G, V and lambda should you use?

(i) Program the iterative procedure using the terminal condition $v(T, \eta) = a^{-\gamma}\eta^{1-\gamma}$. Use N = 1000. Compute an example for the parameters of He and Krishnamurthy, $\rho = 0.04$, $g = 0.02$, $m = 4$, $a = 1$, $l = 1.84$, $\sigma = 0.09$, $\gamma = 2$ and $\lambda = 0.6$. (See Table 2 of HK - for these parameters I was able to get convergence by setting lambda for payoff policy growth aggressively to 0.9).

Plot, as a function of η, the price of capital q, the risk-free rate r_t, the drift and volatility of η_t (i.e. $\sigma_t^\eta \eta$ and $\mu_t^\eta \eta$), the fraction of equity χ_t held by experts and the experts' consumption rate C_t/N_t.

(j) Replicate Figure 2 from He and Krishnamurthy, where the vertical axis displays the risk premium for capital, i.e. $\varsigma_t(\sigma + \sigma_t^q)$.