Online Summer School
Macro, Money, Finance
Problem Set 4

July 10, 2024

Please submit your solutions to the dropbox link by 7/16/2024 8:30am (EST).

1 Money Model with Stochastic Volatility

Consider the model of Lecture 10 with log utility and without government policy ($\mu^{\mathcal{B}} = i = \mathscr{G} = \tau = 0$).[1] In this problem, we add stochastic volatility to the model. Suppose idiosyncratic risk $\tilde{\sigma}_t$ evolves according to the exogenous stochastic process

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t,$$

where $\tilde{\sigma}^{ss}$, b and ν are positive constants.

1. Use goods market clearing and optimal investment to express q^K, $q^{\mathcal{B}}$ and ι in terms of $\vartheta := \frac{q^{\mathcal{B}}}{q^K + q^{\mathcal{B}}}$
2. Derive the "money valuation equation" using martingale method:
   (a) First, write down:
      - Returns on capital ($dr_t^{K,\tilde{i}}$) and bonds ($dr_t^{\mathcal{B}}$), and the law of motion for individual wealth $n_t^{\tilde{i}}$. Keep in mind that q_t^K and $q_t^{\mathcal{B}}$ are now stochastic processes.
      - Prices of idiosyncratic and aggregate risks, $\tilde{\varsigma}_t$ and ς_t respectively. You may take as given that prices of risk are volatility loadings of net worth.

   (b) Use the martinagale pricing condition:

   $$\frac{\mathbb{E}[dr_t^{K,\tilde{i}}]}{dt} - \frac{\mathbb{E}[dr_t^{\mathcal{B}}]}{dt} = \varsigma_t(\sigma_t^{r^{K,\tilde{i}}} - \sigma_t^{r^{\mathcal{B}}}) + \tilde{\varsigma}_t(\tilde{\sigma}_t^{r^{K,\tilde{i}}} - \tilde{\sigma}_t^{r^{\mathcal{B}}})$$

   and market clearing conditions to derive an expression of the form $\mu_t^{\vartheta} = f(\vartheta_t, \tilde{\sigma}_t)$, where function f only depends on model parameters (the "money valuation equation").[2]

[1] There can still be a (constant) supply of bonds $\mathcal{B}_t \neq 0$.

[2] You should derive an expression for μ_t^{ϑ} using Ito's Lemma and the definition of $\vartheta_t := \frac{q_t^{\mathcal{B}}}{q_t^K + q_t^{\mathcal{B}}}$ to substitute $\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}$ in the martingale condition.

3. Suppose that $\nu = 0$ and the economy is at the steady state with $\tilde{\sigma}_t = \tilde{\sigma}^{ss}$.
   - Derive expressions for $q^{\mathcal{B}}$, q^K and ϑ in the monetary and non-monetary equilibria.
   - What is the smallest value of $\tilde{\sigma}^{ss}$ that allows for a monetary equilibrium? Denote this value by $\tilde{\sigma}^{ss}_{min}$.
   - Suppose that $\tilde{\sigma}^{ss} > \tilde{\sigma}^{ss}_{min}$, what happens to $q^{\mathcal{B}}$, q^K and ϑ as $\tilde{\sigma}^{ss}$ rises?
   - Suppose that $0 < \tilde{\sigma}^{ss} < \tilde{\sigma}^{ss}_{min}$, what happens to $q^{\mathcal{B}}$, q^K and ϑ as $\tilde{\sigma}^{ss}$ falls?

4. Suppose that $\nu > 0$ and solve the model numerically:
   (a) Set $a = 0.2$, $\phi = 1$, $\delta = 0.05$, $\rho = 0.01$, $\tilde{\sigma}^{ss} = 0.2$, $b = 0.05$, $\nu = 0.02$.
   (b) Since $\tilde{\sigma}_t$ follows a Cox–Ingersoll–Ross process, it is distributed according to Gamma distribution with parameters $\alpha = 2b\tilde{\sigma}^{ss}/\nu^2$ and $\beta = 2b/\nu^2$:
   $$f(\tilde{\sigma}) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\tilde{\sigma}^{\alpha-1}e^{-\beta\tilde{\sigma}}$$
   Based on this, suggest a grid for $\tilde{\sigma}$ and construct the M matrix using `build_M.m`.
   (c) Apply Ito's lemma to $\vartheta_t = \vartheta(\tilde{\sigma}_t)$, and equate the drift term with $\vartheta_t \mu_t^{\vartheta}$, using the expression for μ_t^{ϑ} from question 2. This gives you an HJB-looking equation for $\vartheta(\tilde{\sigma})$.
   (d) Solve the model using value function iteration:
      i. Rewrite the money valuation equation such that in the discretized form you get:
      $$\rho\vartheta = \mathbf{u}(\vartheta) + \mathbf{M}\vartheta$$
      ii. Write a loop that updates $\vartheta(\tilde{\sigma})$ with the implicit method:
      $$\vartheta_{t-\Delta t} = \Big((1 + \rho\Delta t)\mathbf{I} - \Delta t\mathbf{M}\Big)^{-1}\Big(\Delta t\mathbf{u}(\vartheta_t) + \vartheta_t\Big)$$
      iii. Iterate over $\vartheta(\tilde{\sigma})$ until convergence.
   (e) Plot $\vartheta, q^{\mathcal{B}}, q^K, r^f, \varsigma, \tilde{\varsigma}$ as functions of $\tilde{\sigma}$.[3] Explain the dependence of the variables on $\tilde{\sigma}$.

[3] To compute r^f you would be using Ito's formula and the martingale pricing formula for $dr^{K,\tilde{i}}$ or $dr^{\mathcal{B}}$.