Summer School: Macro, Money, and Finance
Problem Set 2

June 26, 2024

Please submit your solutions to the dropbox link by 7/2/2024 8:30am (EST).

1 Portfolio Choice Problem with Log Utility

Consider an infinitely-lived household with logarithmic preferences over consumption $\{c_t\}_{t\geq 0}$,

$$U_0 := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right].$$

The household has initial wealth $n_0 > 0$ and does not receive any endowment or labor income. Wealth can be invested into two assets. A risk-free bond with (instantaneous) return $r^b dt$ and a risky stock with return $r^s dt + \sigma dZ_t$, where Z_t is a Brownian motion. Here, r^b, r^s, and σ are constant parameters.

The household's net worth evolution is

$$dn_t = -c_t dt + n_t\left(\left(1-\theta_t^s\right)r^b dt + \theta_t^s\left(r^s dt + \sigma dZ_t\right)\right),$$

where θ_t^s denotes the fraction of wealth invested into the stock. The household chooses consumption $\{c_t\}_{t\geq 0}$ and portfolio shares $\{\theta_t^s\}_{t\geq 0}$ to maximize utility U_0 subject to the net worth evolution (and a solvency constraint $n_t \geq 0$).

1. **Solving the problem using the HJB Equation**:

   In this part, you will solve the consumption-portfolio choice problem using the Hamilton-Jacobi-Bellman (HJB) equation. The state space of this decision problem is one-dimensional with state variable n_t, so you can denote the household's value function by $V(n)$.

   (a) Write down the (deterministic) HJB equation for the value function $V(n)$.

   (b) Take first-order conditions with respect to all choice variables.

   (c) Let's make a guess that optimal consumption is proportional to net worth, $c(n) = an$ with some constant $a > 0$ (to be determined below). Use the first-order condition for consumption derived in part (b) to turn this into a guess for the value function $V(n)$.

   Hint: Don't forget to add an integration constant (call it b) when moving from $V'(n)$ to $V(n)$; $V(n)$ is the sum of two terms.

(d) Use your guess for $V(n)$ to simplify the first-order condition for θ^s and solve the resulting equation for $\theta^s(n)$.

(e) Substitute the optimal choices and the guess for $V(n)$ into the HJB equation to eliminate $V(n)$, $V'(n)$, $V''(n)$, c, θ^s and the max operator.

(f) The resulting equation in step (e) has to hold for all $n > 0$ (if it does not, the previous guess was incorrect). Show that this is indeed possible if we choose a and b appropriately. What are the required values for a and b?

(g) (**Not required**) For $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ with $\gamma \neq 1$, redo (c) – (f).

2. **Solving the problem using the Stochastic Maximum Principle**:

Now consider the same decision problem as before but approach it with the stochastic maximum principle instead of the HJB equation.

(a) Denote by ξ_t the costate for net worth n_t and by $\sigma_{\xi,t}$ its (arithmetic) volatility loading (that is $d\xi_t = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t$ with some drift $\mu_{\xi,t}$). Write down the Hamiltonian of the problem.

(b) The choice variables have to maximize the Hamiltonian at all times. Take the first-order conditions in this maximization problem.

(c) Let's again make the guess $c_t = an_t$ with an unknown constant $a > 0$. Use the first-order condition for consumption derived in part (b) to turn this into a guess for the costate ξ_t. Also determine the implied costate volatility $\sigma_{\xi,t}$.

(d) Determine the optimal solution for θ_t^s.

(e) Write down the costate equation for ξ_t and substitute in your guess for c_t, the implied guesses for ξ_t and $\sigma_{\xi,t}$, and the implied optimal solution for θ_t^s. Show that the costate equation is indeed satisfied (and hence the guess was correct) if you choose a suitably. Which value(s) for a work?

(f) Verify that the optimal solution coincides with the one you obtained from the HJB approach. Also show that $\xi_t = e^{-\rho t}V'(n_t)$, where V is the value function determined previously.