Summer School: Macro, Money, and Finance
Problem Set 1: A Continuous Time Kiyotaki-Moore Model

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1 Introducing Physical Investment

In class, we consider a deterministic two sector model with fixed aggregate capital $\bar{K}$. Now we add investment to it. We introduce a concave capital conversion function $\Phi(\iota)$ with assumption $\Phi'(\cdot) > 0, \Phi''(\cdot) < 0$ for investment rate ι and capital depreciation rate δ. For example, consider an agent with capital k_t at time t with investment rate ι_t. His real investment is $\iota_t k_t$, and the capital accumulation is $(\Phi(\iota_t) - \delta)k_t$. This is equivalent to a convex adjustment cost assumption.

1. **Optimal Investment Decision**. Consider the following time line of operating the capitals:
   - At t, agent i purchase capital k_t^i at price q_t.
   - He/she makes the (optimal) investment decision ι_t^i within period $[t, t+dt)$.
   - The capital generates output $a^i(k_t^i)$ at time $t + dt$
   - Finally agent i sell the capital k_{t+dt}^i at market price q_{t+dt}.

   The total gains are:

   $$\pi_t dt = \underbrace{q_{t+dt}k_{t+dt}^i - q_t k_t^i}_{①} + (\underbrace{a^i(k_t^i)}_{②} - \underbrace{\iota_t^i k_t^i}_{③})dt$$

   where ① represents the gain from holding and reselling, ② represents the dividend (output) flow, and ③ represents the investment cost.

   (a) Derive the expression for the return rate, $\frac{\pi_t}{q_t k_t}$. Do the decomposition for ①②③ for π_t.

   (b) Show that optimal investment ι_t^i is the same for $i \in \{e, h\}$, and is determined by:

   $$\frac{1}{q_t} = \Phi'(\iota_t^i).$$

   This is the Tobin's Q condition.

   (c) For conversion function $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, what is the optimal investment ι_t given price q_t?

2. **Amplification via Investment Channel.** Consider the KM-model of the slides. Now we introduce the investment conversion function $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$. We still assume that $a_h(1-\kappa) = \kappa$. Hint: While on the slides the goods market clearing condition stated consumption equals output, now consumption plus investment $\iota_t K_t$ equals output.

   (a) Consider a steady state where collateral constraint is binding. Write down a set of equations such that equilibrium objects (κ^e, η, q) are pinned down. Is there a steady state for K?
   Hint: Consider the scalability (aK) of production technology.

2 Coding Continuous Time Kiyotaki-Moore Model

1. Please replicate the global solution figures (of Lecture 01, slide 23) for the case with ι-investments. You can simply modify the Matlab code provided on the summer school website.
   (If you do not have access to Matlab feel free to convert the Matlab code into a different programming language (Julia, Python, ...) using ChatGPT.