Eco529: Modern Macro, Money, and International Finance

Lecture 10: One Sector Monetary Model with Heterogenous Agents

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Fall, 2023

Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model: FTPL and Monetarism Elements
2. Multi-sector Model, Real vs. Nominal Bonds, Banks, "The I Theory of Money"
3. Price Rigidity - New Keynesian Elements
4. Welfare Analysis & Optimal Policy

International Macro-Finance Models

1. International Financial Architecture

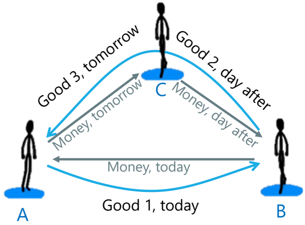
Digital Money

Overview

- Intuition for different "Monetary Theories"
- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs
- Monetary model with time-varying idiosyncratic risk
  - Safe Asset with CAPM-$\beta < 0$,
  - Exorbitant Privilege, Laffer Curve
  - Safe Asset-Bubble Complementarity
  - Modern Debt Sustainability Analysis
- Medium of Exchange Role

The 3 Roles of Money

- Store of value
  - Bond is less risky than other "capital" – no idiosyncratic risk
  - Govt bond is a special safe asset
- Medium of exchange
  - Overcome double-coincidence of wants problem
  - (Narrow) money is special gov. bond
- Unit of account
  - Intratemporal: Numeraire bounded rationality
  - Intertemporal: Debt contracts incomplete markets
    Wage contracts wage/price stickiness



- Record keeping device – money is memory
  - *Virtual ledger*

The 3 Roles of Money

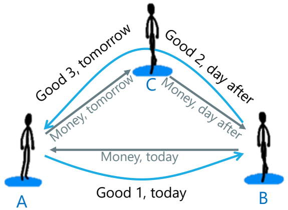
- Store of value
  - Bond is less risky than other "capital" – no idiosyncratic risk
  - Govt bond is a special safe asset
    – helps to partially overcome incomplete markets/OLG frictions
    (- helps to relax colleteral constraints)
  - Fiscal Theory of Price Level (FTPL):

$$\frac{\mathcal{B}_t+\mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t}(\text{primarysurpluses})_s ds + \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds + \frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T+\mathcal{M}_T}{\mathcal{P}_T}$$

    - Monetary vs. fiscal dominance

- Medium of exchange
  - Overcome double-coincidence of wants problem
  - (Narrow) money is special gov. bond
    - helps to overcome double-coincidence of wants friction
    (cash-in-advance, money in utility, shopping time models)
    - lower interest rate Δi_s
  - **Monetarisms**: Quantity Equation
    $\nu_t \mathcal{M}_t = \mathcal{P}_t T_t$ (or $\mathcal{P}_t Y_t$)

- Unit of account
  - Intratemporal: Numeraire bounded rationality
  - Intertemporal: Debt contracts incomplete markets
    New Keynesian wage/price stickiness



Credit, Money, Reserves, and Government Debt

- Credit vs. Money
  - Credit — zero net supply
  - Money (Gov. bond) — positive net supply

    - Perfect credit renders money useless
- Gov. Debt vs. Money in form of Cash and Reserves
  - Gov. debt: convenience yield as it relaxes collateral constraint
  - Money $\mathcal{M}_t$ has lower interest rate Δi if it offers medium of exchange role in addition
    - Reserves: Interest bearing

      - Special form of government debt:

        - Infinite maturity — more like equity (no rollover risk)
        - Zero duration — more like overnight debt
        - Banking system can't offload it – **Financial Repression**
    - Cash: extra convenience yield and zero interest $\Rightarrow$ lower return by Δi
    - Fintech revolution erodes extra convenience yield

Simplify to One Sector Model

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \mathrm{d}\Delta_t^{k,\tilde{i},e} + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

- Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^e t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$

Friction: Can issue

- Risk-free debt only

~~Household Sector~~

- ~~Output: $y_t^h = a^h k_t^h$~~
- ~~Consumption rate: c_t^h~~
- ~~Investment rate: ι_t^h~~

~~$$\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \mathrm{d}\Delta_t^{k,\tilde{i},h} + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$~~

- ~~Objective: $\mathbb{E}_0\left[\int_0^\infty e^{-\rho^h t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$~~

Model Overview

- Continuous time, infinite horizon, one consumption good
- Continuum of agents
  - Operate capital with constant idiosyncratic risk, AK production technology
  - Can trade capital and government bond, Extension: add diversified equity claims
- Government
  - Exogenous spending
  - Taxes output
  - Issues (nominal) bonds = money (since no medium of exchange friction)
- Financial Frictions: incomplete markets
  - Agents cannot trade idiosyncratic risk
- Aggregate risk: fluctuations in volatility of idio risk (& capital productivity)

Model with Capital + Safe Asset

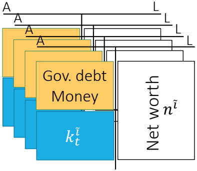
- Each heterogenous citizen $\tilde{i} \in [0,1]$:

$$\mathbb{E}_t \left[\int_0^\infty e^{-\rho s} \left(\frac{(c_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} \right) \mathrm{d}s \right] + f(\mathscr{G} K_s), \text{ where } K_t := \int k_t^{\tilde{i}} \mathrm{d}\tilde{i}$$

$$s.t. \frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}} \mathrm{d}t + \mathrm{d}r_t^{\mathcal{B}} + (1-\theta_t^{\tilde{i}})(\mathrm{d}r_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - \mathrm{d}r_t^{\mathcal{B}}) \& \text{ No Ponzi}$$

- Each citizen operates physical capital $k_t^{\tilde{i}}$
  - Output (net investment): $y_t^{\tilde{i}} \mathrm{d}t = (a k_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}}) \mathrm{d}t$
  - $\frac{\mathrm{d}k_t^{\tilde{i}}}{k_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i}}) - \delta \right) \mathrm{d}t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^{\tilde{i}} + \mathrm{d}\Delta_t^{k,\tilde{i}}$,

    ($\mathrm{d}\tilde{Z}_t^{\tilde{i}}$ idiosyncratic Brownian)
  - Output tax $\tau a k_t^{\tilde{i}} \mathrm{d}t$
- No aggregate risk $\mathrm{d}Z_t$
- Financial Friction: Incomplete markets: no $\mathrm{d}\tilde{Z}_t^{\tilde{i}}$-claims



Government: Taxes, Bond/Money Supply, Gov. Budget

- Policy Instruments ($K_t := \int k_t^{\tilde{i}} \mathrm{d}\tilde{i}$)
  - Government spending $\mathscr{G} K_t$ (with exogenous $\mathscr{G}$)
  - Proportional output tax $\tau a K_t$
  - Nominal government debt supply $\frac{\mathrm{d}\mathcal{B}_t}{\mathcal{B}_t} = \mu_t^{\mathcal{B}} \mathrm{d}t$
  - Floating nominal interest rate i_t on outstanding bonds
- Government budget constraint (BC)

$$\underbrace{(\mu_t^{\mathcal{B}} - i_t)}_{\check{\mu}_t^{\mathcal{B}} :=} \mathcal{B}_t + \mathcal{P}_t K_t \quad \underbrace{(\tau a - \mathscr{G})}_{\substack{s := \\ \text{primary surplus per K}}} \quad = 0$$

i_t is not for market clearing, payment/redistribution to bond holders, $q^{\mathcal{B}}$ clears bond market

  - No No-Ponzi constraint (Equilibrium selection):

Related Literature on Money as Store of Value

\Friction	OLG	Incomplete Markets + idiosyncratic risk	
Risk	deterministic	endowment risk borrowing constraint	return risk Risk tied up with Individual capital
Only money	Samuelson	Bewley	“I Theory without I”
With capital	Diamond	Aiyagari	Brunnermeier-Sannikov (AER PP 2016)

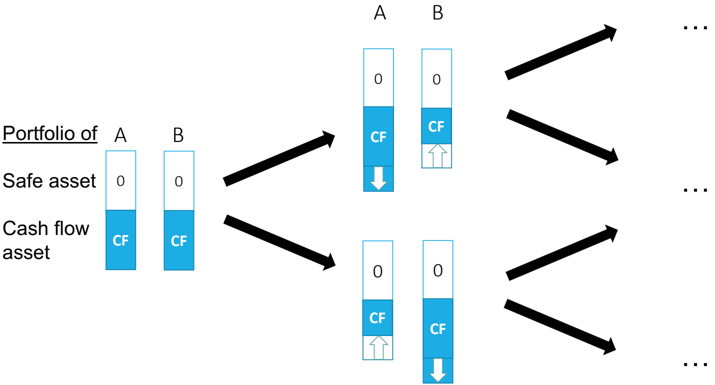
Aside 2: BruSan meets Bewley-Huggett-Aiyagari

- General setup that encompasses BHA and BruSan:
- $\max_{c,\theta} \mathbb{E} \int_0^\infty e^{-\rho t} u(c_t) \mathrm{d}t$
  - $\mathrm{d}n_t^{\tilde{i}} = -c_t^{\tilde{i}} \mathrm{d}t + \mathrm{d}y_t^{\tilde{i}} + n_t^{\tilde{i}} \left(r\mathrm{d}t + (1-\theta)(\mathrm{d}r_t^{k,\tilde{i}} - r\mathrm{d}t \right)$
  - $\mathrm{d}r_t^{k,\tilde{i}} = r^k \mathrm{d}t + \tilde{\sigma}^k \mathrm{d}\tilde{Z}_t^{k,\tilde{i}}$
  - $\mathrm{d}y_t^{\tilde{i}} = -y_t^{\tilde{i}} \mathrm{d}t + \tilde{\sigma}^y \mathrm{d}\tilde{Z}_t^{y,\tilde{i}}$
- BruSan: $\tilde{\sigma}^y = 0$ ... to rebuild capital stock after shock and
  to smooth consumption(by reducing risk)
  Precautionary savings with safe asset
- Bewley-Huggett-Aiyagari:
  $\tilde{\sigma}^k = 0$ ... to smooth consumption (capital is also a safe asset)
  - Risk does not scale with net worth $\Rightarrow \frac{c_t}{n_t}$ and portfolio θ_t depends on net worth

Incomplete Market Friction and Safe Assets

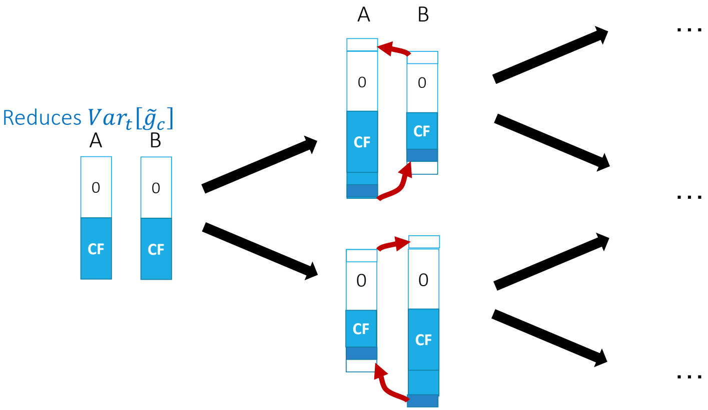
$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t[PV_{\xi^{**}}(\text{cash flow}^j)] + \mathbb{E}_t[PV_{\xi^{**}}(\text{service flow}^j)] \qquad (\xi^{**} = \text{new SDF})$$

Example: $= 0$



Incomplete Market Friction and Safe Assets

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t[PV_{\xi^{**}}(\text{cash flow}^j)] + \mathbb{E}_t[PV_{\xi^{**}}(\text{service flow}^j)]$$

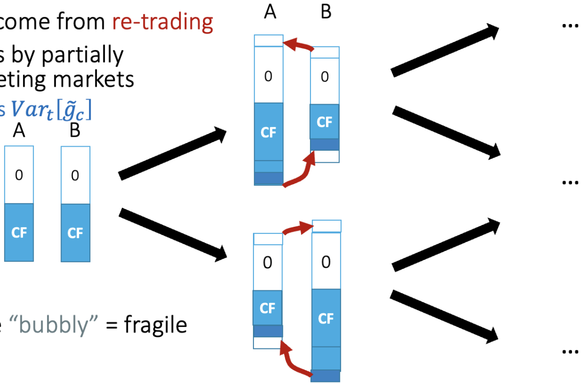


Incomplete Market Friction and Safe Assets

$$\frac{\mathcal{B}_t}{\mathcal{P}_t} = \mathbb{E}_t[PV_{\xi**}(\text{cash flow}^j)] + \mathbb{E}_t[PV_{\xi**}(\text{service flow}^j)]$$

- Value come from re-trading
- Insures by partially completing markets

  Reduces $Var_t[\tilde{g}_c]$



- Can be "bubbly" = fragile

Solving Macro Models Step-by-Step

0 Postulate aggregates, price processes and obtain return processes

1 For given C/N-ratio and SDF processes for each i **finance block**

- a Real investment ι + Goods market clearing *(static)*

  Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach

- b Portfolio choice θ (idio shock) + Asset market clearing or

  Asset allocation κ & risk allocation χ

  Toolbox 2: "price-taking social planner approach" – Fisher separation theorem

  Toolbox 3: Change in numeraire to total wealth (including SDF)

  - "money evaluation/FTPL equation" ϑ

2 Evolution of state variable ~~η (and~~ K) **forward equation**

3 Value functions **backward equation**

- a Value fcn. as fcn. of investment opportunities ω

  Special case: log-utility, constant investment opportunities

- b Separating value fcn. $V^i(n^{\tilde{i}}; \eta, K)$ into $v^i(\eta)(\tilde{\eta}^{\tilde{i}})^{1-\gamma}\nu(K)$
- c Derive $\check{\rho} = C/N$-ratio and ς price of risk

4 Numerical model solution

- a Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
- b Solve PDE via value function iteration

5 KFE: Stationary distribution, fan charts

Assets, Aggregate Resource Constraint, and Markets

- Assets: capital and bonds
  - q_t^K Capital price
  - $q_t^{\mathcal{B}} := \frac{\mathcal{B}_t}{\mathcal{P}_t}/K_t$ value of the bonds per unit of capital
  - $\vartheta_t := \frac{\frac{\mathcal{B}_t}{\mathcal{P}_t}}{q_t^K K_t + \frac{\mathcal{B}_t}{\mathcal{P}_t}} = \frac{q_t^{\mathcal{B}}}{q_t^K + q_t^{\mathcal{B}}}$ share of bond wealth
  - Postulate Ito price processes $\mathrm{d}q_t^K/q_t^K = \mu_t^{q,K}\mathrm{d}t, \mathrm{d}q_t^{\mathcal{B}}/q_t^{\mathcal{B}} = \mu_t^{q,\mathcal{B}}\mathrm{d}t, \mathrm{d}\vartheta_t/\vartheta_t = \mu_t^{\vartheta}\mathrm{d}t$
  - SDF for each $\tilde{i}$ agent: $\mathrm{d}\xi_t^{\tilde{i}}/\xi_t^{\tilde{i}} = -r_t^{f,\tilde{i}}dt - \varsigma_t^{\tilde{i}}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$
- Aggregate resource constraints:
  - Output: $C_t + \iota_t K_t + \mathscr{G} K_t = aK_t$
  - Capital: $\int k_t^{\tilde{i}}\mathrm{d}\Delta k_t^{k,\tilde{i}}\mathrm{d}\tilde{i} = 0$
- Markets: Walrasian goods, bonds, and capital markets

Poll: Why do risk-free rate and price of risk not depend on individual $\tilde{i}$?

a) risk-free bond can be traded

b) aggregate risk can be traded

c) CRRA utility for all agents with same γ

0. Return on Gov. Bond/Money

- Number of Bonds/coins follows:

$$\frac{\mathrm{d}\mathcal{B}_t}{\mathcal{B}_t} = (\check{\mu}_t^{\mathcal{B}} + i_t)\mathrm{d}t$$

  - Where i_t is interest paid on government bonds/outside money (reserves)
- Return on Gov. Bond/Money: in output numeraire:

$$\mathrm{d}r_t^{\mathcal{B}} = i_t\mathrm{d}t + \underbrace{\frac{\mathrm{d}(1/\mathcal{P}_t)}{1/\mathcal{P}_t}}_{-\textit{inflation}} = i_t\mathrm{d}t + \underbrace{\frac{\mathrm{d}(q_t^{\mathcal{B}}K_t/\mathcal{B}_t)}{q_t^{\mathcal{B}}K_t/\mathcal{B}_t}}_{-\textit{inflation}} \quad \text{Fisher Equation}$$

$$= \frac{\mathrm{d}(q_t^{\mathcal{B}}K_t)}{q_t^{\mathcal{B}}K_t} - \check{\mu}_t^{\mathcal{B}}\mathrm{d}t$$

- Seigniorage (excluding interest paid to money holders)

0. Distribution of "Seigniorage"

1. Proportionally to bond/money holdings
   - No real effects, only nominal
2. Proportionally to capital holdings
   - Bond/Money return decreases with $\mathrm{d}\mathcal{B}_t$ (change in debt level/money supply)
   - Capital return increases $\Longleftarrow$
   - Pushes citizens to hold more capital
3. Proportionally to net worth
   - Fraction of seigniorage goes to capital - same as 2.
   - Rest of seigniorage goes to money holders - same as 1.
4. Per capita
   - No real effects: Ricardian Equivalence
     people simply borrow against the transfers they expect to receive
   - **Remark**: In a setting with borrowing constraint like in Bewley, Hugget, Aiyagari, etc. per capita seigniorage transfers are not neutral

0. Return on Capital (with seigniorage rebate terms)

$$\mathrm{d}r_t^{K,\tilde{i}} = \frac{a(1-\tau) - \iota_t^{\tilde{i}}}{q_t^K}\mathrm{d}t + \frac{\mathrm{d}(q_t^K k_t^{\tilde{i}})}{q_t^K k_t^{\tilde{i}}}$$

$$= \left(\frac{a(1-\tau) - \iota_t^{\tilde{i}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \mu_t^{q^K} \right) \mathrm{d} + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

- Use government budget constraint to substitute out τ (and $\mathcal{B}_t/\mathcal{P}_t = q_t^{\mathcal{B}} K_t$)

$$\underbrace{(\mu_t^{\mathcal{B}} - i_t)}_{\check{\mu}_t^{\mathcal{B}} :=} q_t^{\mathcal{B}} + \underbrace{(\tau a - \mathscr{G})}_{s=} = 0$$

$$\mathrm{d}r_t^{K,\tilde{i}} = \left(\frac{a - \mathscr{G} - \iota_t^{\tilde{i}}}{q_t^K} + \Phi(\iota_t^{\tilde{i}}) - \delta + \frac{q_t^{\mathcal{B}}}{q_t^K}\check{\mu}_t^{\mathcal{B}} \right) \mathrm{d} + \tilde{\sigma}\mathrm{d}\tilde{Z}_t^{\tilde{i}}$$

1. Optimal Choices

- Optimal investment rate (for $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$):

$$\phi\iota_t = q_t^K - 1$$

- Consumption:

$$\frac{c_t}{n_t} =: \check{\rho}_t \Rightarrow C_t = \check{\rho}_t \left(q_t^{\mathcal{B}} + q_t^K\right) K_t$$

  - Looking ahead to Step 3:
    When is $\frac{c}{n}$ constant? Recall $\frac{c}{n} = \rho^{1/\gamma}\omega^{1-1/\gamma}$
    - Log-utility, $\gamma = 1$: $\check{\rho} = \rho$
    - In steady state:
      ω investment opportunity/networth multiplier is constant

Optimal Choices & Market Clearing

- Optimal investment rate:

$$\phi\iota_t = q_t^K - 1 \qquad \iota_t = \frac{1}{\phi}(q_t^K - 1)$$

- Consumption: Goods market

$$\frac{c_t}{n_t} =: \check{\rho}_t \Rightarrow C_t = \check{\rho}_t \left(q_t^{\mathcal{B}} + q_t^K\right) K_t = (a - \iota_t - \mathscr{G})K_t$$

- Portfolio Capital market
  - Solve for θ_t later $\quad 1 - \theta_t = \frac{q_t^K}{q_t^K + q_t^B} =: 1 - \vartheta_t$

Bond market clears by Walras law

$\boxed{\vartheta = \text{fraction of wealth in nominal claims}}$

Equilibrium (before solving for portfolio choice)

Equilibrium:

$$
\boxed{
\begin{aligned}
q_t^{\mathcal{B}} &= \vartheta_t \frac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t} \\
q_t^{K} &= (1-\vartheta_t)\frac{1+\phi\check{a}}{(1-\vartheta_t)+\phi\check{\rho}_t} \\
\iota_t &= \frac{(1-\vartheta_t)\check{a}-\check{\rho}_t}{(1-\vartheta_t)+\phi\check{\rho}_t}
\end{aligned}
}
$$

$\check{a} = a - \mathscr{G}$

For log utility

$\check{\rho}_t = \rho$

- Moneyless equilibrium with $q_t^{\mathcal{B}} = 0 \Rightarrow \vartheta_t = 0$
- Next, determine portfolio choice.

Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**
   a. Real investment ι + Goods market clearing *(static)*
      Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice θ (idio shock) + Asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "price-taking social planner approach" – Fisher separation theorem
      Toolbox 3: Change in numeraire to total wealth (including SDF)
      - "money evaluation/FTPL equation" ϑ
2. Evolution of state variable ~~η (and~~ K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of investment opportunities ω
      Special case: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^{\tilde{i}}; \eta, K)$ into $v^i(\eta)(\tilde{\eta}^{\tilde{i}})^{1-\gamma}\nu(K)$
   c. Derive $\check{\rho} = C/N$-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, fan charts

1.b Portfolio choice θ: Bond/Money Evaluation/FTPL Equation

- Recall martingale method
  - Excess expected return of risky asset A to risky asset B:

$$\boxed{\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B) + \tilde{\varsigma}_t^i(\tilde{\sigma}_t^A - \tilde{\sigma}_t^B)}$$

- 4 alternative derivations:
  - In consumption numeraire
    i. Expected excess return of capital w.r.t. bond return
    ii. Expected excess return of net worth (portfolio) w.r.t. bond return
  - In total net worth numeraire
    iii. Expected excess return of capital w.r.t. bond return
    iv. Expected excess return of individual net worth (=net worth share) w.r.t. bond return (per bond)

1. Portfolio choice θ: Bond Evaluation/FTPL Equation

Price capital relative to money in consumption numeraire

- Asset pricing equation (martingale method): $\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} = \tilde{\varsigma}_t^i \tilde{\sigma}^{\tilde{i}}$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}]}{\mathrm{d}t} = \frac{\check{a}-\iota_t}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^K} = r_t^f + \tilde{\varsigma}_t \tilde{\sigma}$$

$$\Uparrow$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} = \Uparrow \quad -\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{B}}} = r_t^f \quad \Uparrow$$

$$\frac{\mathbb{E}_t[\mathrm{d}r_t^{K,\tilde{i}}]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}r_t^{\mathcal{B}}]}{\mathrm{d}t} = \frac{\check{a}-\iota_t}{q_t^K} + \underbrace{\frac{1}{1-\vartheta_t}\check{\mu}^{\mathcal{B}}}_{\text{Seigniorage redistribution}} + \underbrace{\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}}_{=-\mu_t^{\vartheta}/(1-\vartheta_t)} = \tilde{\varsigma}\tilde{\sigma}$$

- Goods market clearing: $\check{\rho}(q_t^{\mathcal{B}} + q_t^K)K_t = (\check{a} - \iota_t)K_t \Rightarrow \frac{\check{a}-\iota_t}{q_t^K} = \frac{\check{\rho}}{1-\vartheta_t}$
- Price of idiosyncratic risk: $\tilde{\varsigma}_t = \ldots$

  Hint: $\mathrm{d}(q^K + q^{\mathcal{B}}) = \mathrm{d}q^K + \mathrm{d}q^{\mathcal{B}}$

$$\mu^{q^K+q^{\mathcal{B}}} = \frac{\mu_{q^K+q^{\mathcal{B}}}}{q^K+q^{\mathcal{B}}} = \frac{\mu^{q^K} q^K + \mu^{q^{\mathcal{B}}} q^{\mathcal{B}}}{q^K+q^{\mathcal{B}}} = (1-\vartheta)\mu^{q^K} + \vartheta\mu^{q^{\mathcal{B}}}$$

$$\sigma^{q^K+q^{\mathcal{B}}} = \frac{\sigma_{q^K+q^{\mathcal{B}}}}{q^K+q^{\mathcal{B}}} = \frac{\sigma^{q^K} q^K + \sigma^{q^{\mathcal{B}}} q^{\mathcal{B}}}{q^K+q^{\mathcal{B}}} = (1-\vartheta)\sigma^{q^K} + \vartheta\sigma^{q^{\mathcal{B}}}$$

3. Deriving price of idiosyncratic risk $\tilde{\varsigma}^{\tilde{i}}$ and C/N-ratio $\check{\rho}$

- Recall and plug in:
  - $\tilde{\varsigma}_t^{\tilde{i}} = -\tilde{\sigma}_t^{\xi}$, where $\xi_t^{\tilde{i}} = e^{-\rho t}(c_t^{\tilde{i}})^{-\gamma} = e^{-\rho t}(\check{\rho} n_t^{\tilde{i}})^{-\gamma}$
  - Since $\check{\rho} = c_t/n_t$ is constant for CRRA
  - $\tilde{\varsigma}^{\tilde{i}} = \gamma \tilde{\sigma}^{n^{\tilde{i}}} = \gamma(1-\vartheta)\tilde{\sigma}$
- For log utility $\gamma = 1$:
  - $\tilde{\varsigma}^{\tilde{i}} = \tilde{\sigma}^{n^{\tilde{i}}}$
  - $\check{\rho} = \rho$

4. Portfolio choice θ: Bond/Money Evaluation/FTPL Equation

Price capital relative to money in consumption numeraire

- Asset pricing equation (martingale method): $\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} = \tilde{\varsigma}_t^{i}\tilde{\sigma}^{\tilde{i}}$

$$\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}]}{dt} = \frac{\check{a}-\iota_t}{q_t^K} + \frac{q_t^{\mathcal{B}}}{q_t^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^K} = r_t^f + \tilde{\varsigma}_t\tilde{\sigma}$$

$$\Uparrow$$

$$\frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} = \quad \Uparrow \quad -\check{\mu}^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q^{\mathcal{B}}} = r_t^f \quad \Uparrow$$

$$\frac{\mathbb{E}_t[dr_t^{K,\tilde{i}}]}{dt} - \frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} = \frac{\check{a}-\iota_t}{q_t^K} + \underbrace{\frac{1}{1-\vartheta_t}}_{\text{Seigniorage redistribution}}\check{\mu}^{\mathcal{B}} + \underbrace{\mu_t^{q^K} - \mu_t^{q^{\mathcal{B}}}}_{=-\mu_t^{\vartheta}/(1-\vartheta_t)} = \tilde{\varsigma}\tilde{\sigma}$$

- Goods market clearing: $\check{\rho}(q_t^{\mathcal{B}} + q_t^K)K_t = (\check{a}-\iota_t)K_t \Rightarrow \frac{\check{a}-\iota_t}{q_t^K} = \frac{\check{\rho}}{1-\vartheta_t}$
- Price of idiosyncratic risk: $\tilde{\varsigma}_t = \gamma\tilde{\sigma}_t^n = \gamma(1-\theta_t)\tilde{\sigma}$
- Capital market clearing: $1-\theta_t = 1-\vartheta_t$
- FTPL/Money Valuation Equation: $\mu_t^{\vartheta} = \check{\rho} + \check{\mu}_t^{\mathcal{B}} - \gamma(1-\vartheta_t)^2\tilde{\sigma}^2$

In steady state $\mu_t^{\vartheta} = 0$: $(1-\vartheta) = \sqrt{\check{\rho} + \check{\mu}^{\mathcal{B}}}/(\sqrt{\gamma}\tilde{\sigma})$

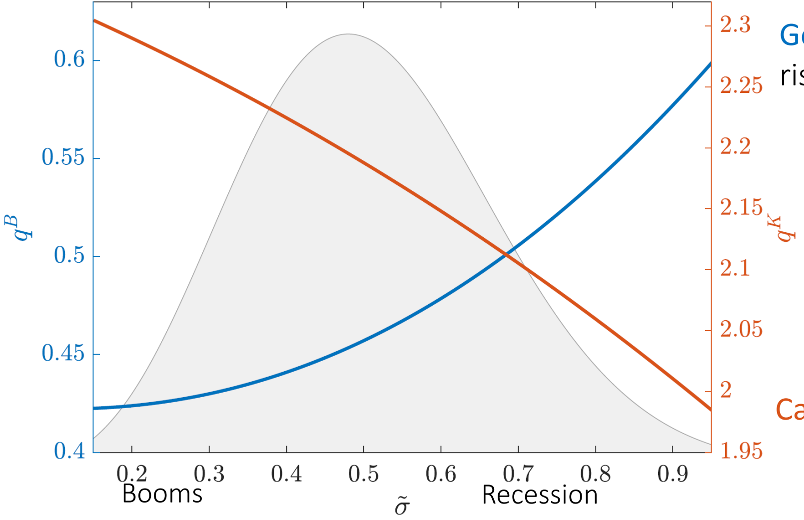
Two Stationary Equilibria – in closed form for log-utility

Non-Monetary	Monetary
$q_t^{\mathcal{B}} = 0$	$\frac{\mathcal{B}_0}{\mathcal{P}_0}/K_t = q^{\mathcal{B}} = \frac{(\tilde{\sigma}-\sqrt{\rho+\breve{\mu}^{\mathcal{B}}})(1+\phi\check{a})}{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}+\phi\tilde{\sigma}\rho}$
$q_t^K = \frac{1+\phi\check{a}}{1+\phi\rho}$	$q^K = \frac{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}(1+\phi\check{a})}{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}+\phi\tilde{\sigma}\rho}$
$\iota = \frac{\check{a}-\rho}{1+\phi\rho}$	$\iota = \frac{\check{a}\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}-\tilde{\sigma}\rho}{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}+\phi\tilde{\sigma}\rho}$

- Note: For log utility $\check{\rho} = \rho$ time preference rate
- ϕ adjustment cost for investment rate
- $\breve{\mu}_t^{\mathcal{B}} = \mu_t^{\mathcal{B}} - i_t$ bond issuance rate beyond interest rate
- $\check{a} = a - \mathscr{G}$ part of TFP not spend on gov.

Flight-to-Safety: Comparative static w.r.t. $\tilde{\sigma}$

- Comparative static w.r.t. idiosyncratic risk $\tilde{\sigma}$



Gov. debt value rises in recessions

Capital price

Comparative static w.r.t. $\breve{\mu}^B$ - Fiscal Policy s

- Recall $\breve{\mu}^{\mathcal{B}}_t \mathcal{B}_t + \mathcal{P}_t K_t \underbrace{(\tau_t a_t - \mathscr{G}_t)}_{s_t :=} = 0$

- $\breve{\mu}^{\mathcal{B}} = 0 \Rightarrow s = 0$ no primary surplus (no cash payoff for bond)
  - $q^{\mathcal{B}} K = \frac{\mathcal{B}}{\mathcal{P}} > 0$ bond trades at a **bubble** due to service flow

- $\breve{\mu}^{\mathcal{B}} > 0 \Rightarrow s < 0$ primary deficit (constant fraction of GDP)
  - As long as $q^{\mathcal{B}} > 0$ "mine the bubble"

- $\breve{\mu}^{\mathcal{B}} < 0 \Rightarrow s > 0$ and $r > g$ primary surplus (constant fraction of GDP)
  - $q^{\mathcal{B}} K_t = \mathbb{E}_t[PV_{r^f}(sK_t)]$ no bubble, but service flow
  - $\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty e^{-r^f t} s K_t \mathrm{d}t\right]$

- Real risk-free rate
  - $r^f = \underbrace{\left(\Phi(\iota(\breve{\mu}^{\mathcal{B}})) - \delta\right)}_{=g} - \breve{\mu}^{\mathcal{B}}$

Recall: Ponzi Scheme vs. Bubble

- Ponzi Scheme:
  - Rollover existing *short-term debt* with growing new (short-term) debt
    Which needs to be rolled over again.
  - Growth: Ponzi scheme grows is new debt exceeds amount needed to pay off old one
- Bubble:
  - Issue *long-term asset*, whose value exceeds fundamental value (of e.g. zero)
  - Growth: issue each period additional bubble assets

Aside: Two Stationary Equilibria for CRRA γ utility

Non-Monetary	Monetary
$q^{\mathcal{B}} = 0$	$q^{\mathcal{B}} = \frac{\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}\right)(1+\phi\check{a})}{\sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}} = \frac{\left(\sqrt{\gamma}\tilde{\sigma} - \sqrt{\check{\rho}-s/q^{\mathcal{B}}}\right)(1+\phi\check{a})}{\sqrt{\check{\rho}-s/q^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$
$q^K = \frac{1+\phi\check{a}}{1+\phi\check{\rho}_0}$	$q^K = \frac{\sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}(1+\phi\check{a})}{\sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$
$\iota = \frac{\check{a}-\check{\rho}_0}{1+\phi\check{\rho}_0}$	$\iota = \frac{\check{a}\sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}-\sqrt{\gamma}\tilde{\sigma}\check{\rho}}{\sqrt{\check{\rho}+\breve{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$

- For log utility:
  $\check{\rho} = \check{\rho}_0 = \rho$
  $\gamma = 1$

ρ time preference rate
ϕ adjustment cost for investment rate
$\breve{\mu}^{\mathcal{B}}_t = \mu^{\mathcal{B}}_t - i_t$bond issuance rate beyond interest rate
$\check{a} = a - \mathscr{G}$ part of TFP not spend on gov.

3. Descaling by Idiosyncratic Risk for CRRA utility

For CRRA utility

- Rephrase the conjecture value function as

$$V_t^{\tilde{i}} = \frac{(\omega_t^{\tilde{i}} n_t^{\tilde{i}})^{1-\gamma}}{1-\gamma} = \underbrace{\left(\omega_t \frac{N_t^i}{K_t}\right)^{1-\gamma}}_{:=v_t} \underbrace{\left(\frac{n_t^{\tilde{i}}}{N_t^i}\right)^{1-\gamma}}_{:=\left(\tilde{\eta}_t^{\tilde{i}}\right)^{1-\gamma}} \frac{K_t^{1-\gamma}}{1-\gamma}$$

- v_t^i depend only on aggregate state η_t
- Itô's quotient rule:

$$\frac{d\tilde{\eta}_t^{\tilde{i}}}{\tilde{\eta}_t^{\tilde{i}}} = \frac{d(n_t^{\tilde{i}}/N_t^i)}{n_t^{\tilde{i}}/N_t^i} = \left(\mu_t^{n^{\tilde{i}}} - \mu_t^N + (\sigma_t^N)^2 - \sigma^N \sigma_t^{n^{\tilde{i}}}\right) dt + \left(\sigma_t^{n^{\tilde{i}}} - \sigma^N\right) dZ_t + \tilde{\sigma}_t^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}} = \tilde{\sigma}_t^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}}$$

- Itô's Lemma:

$$\frac{d(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}}{(\tilde{\eta}_t^{\tilde{i}})^{1-\gamma}} = -\frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2 dt + (1-\gamma)\tilde{\sigma}^{n^{\tilde{i}}} d\tilde{Z}_t^{\tilde{i}}$$

4. BSDE for v_t^i

$$\frac{\mathrm{d}V_t^{\tilde{i}}}{V_t^{\tilde{i}}} = \frac{\mathrm{d}\left(v_t^i (\tilde{\eta}_t^{\tilde{i}})^{1-\gamma} K_t^{1-\gamma}\right)}{v_t^i (\tilde{\eta}_t^{\tilde{i}})^{1-\gamma} K_t^{1-\gamma}}$$

- By Itô's product rule:

$$= \left[\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\cancel{\sigma}^2 + (\tilde{\sigma}^{n^{\tilde{i}}})^2\right) + \cancel{(1-\gamma)\sigma\sigma_t^v}\right]\mathrm{d}t + \text{volatility terms}$$

- Recall by consumption optimality $\frac{\mathrm{d}V_t^{\tilde{i}}}{V_t^{\tilde{i}}} - \rho\mathrm{d}t + \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$ follows a martingale
  - Hence, drift above $= \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$
- BSDE:

$$\mu_t^v + (1-\gamma)(\Phi(\iota_t) - \delta) - \frac{1}{2}\gamma(1-\gamma)\left(\tilde{\sigma}^{n^{\tilde{i}}}\right)^2 = \rho - \frac{c_t^{\tilde{i}}}{n_t^{\tilde{i}}}$$

  - In steady state $\mu_t^v = 0$, $\frac{c^{\tilde{i}}}{n^{\tilde{i}}} =: \check{\rho} = \rho - (1-\gamma)(\Phi(\iota_t) - \delta) + \frac{1}{2}\gamma(1-\gamma)((1-\vartheta)\tilde{\sigma})^2$

4. Numerical Solution for CRRA $\gamma \neq 1$

- Solve numerically

$$\check{\rho} = \rho - (1-\gamma)(\Phi(\iota) - \delta) + \frac{1}{2}\gamma(1-\gamma)((1-\vartheta)\tilde{\sigma})^2$$

  - After plugging in
  - $\iota = \frac{\check{a}\sqrt{\check{\rho}+\check{\mu}^{\mathcal{B}}} - \sqrt{\gamma}\tilde{\sigma}\check{\rho}}{\sqrt{\check{\rho}+\check{\mu}^{\mathcal{B}}} + \phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}}$
  - $1 - \vartheta = \sqrt{\check{\rho} + \check{\mu}^{\mathcal{B}}}/(\sqrt{\gamma}\tilde{\sigma})$

Remark on r_t^f in Consumption Numeraire

- From portfolio choice using $\varsigma = \gamma\sigma$ (here: $\sigma = 0$)

$$
\begin{aligned}
r^f &= \Phi(\iota) - \delta - \mu^{\mathcal{B}} - \gamma\sigma^2 \\
r^f &= \frac{1}{\phi}\log\frac{\sqrt{\check{\rho}+\check{\mu}^{\mathcal{B}}}(1+\phi a)}{\sqrt{\check{\rho}+\check{\mu}^{\mathcal{B}}}+\phi\sqrt{\gamma}\tilde{\sigma}\check{\rho}} - \delta - \mu^{\mathcal{B}} - \gamma\sigma^2
\end{aligned}
$$

  - For $\phi = 0 : r^f = a - \frac{\sqrt{\gamma}\tilde{\sigma}\check{\rho}}{\sqrt{\check{\rho}+\check{\mu}^{\mathcal{B}}}} - \delta - \mu^{\mathcal{B}} - \gamma\sigma^2$
  - For $\phi = \infty : r^f = -\delta - \mu^{\mathcal{B}} - \gamma\sigma^2$
- Remark: money supply growth
  - Increases ι as portfolio choice is tilted towards capital
  - Depresses real r^f one-to-one because ...

$$
\begin{aligned}
r^f &= \rho + \gamma\mu^c - \gamma(\gamma+1)((\sigma^c)^2 + (\tilde{\sigma}^c)^2)/2 \\
r^f &= \rho + \gamma[\Phi(\iota(\check{\mu}^{\mathcal{B}})) - \delta] - \gamma(\gamma+1)((\sigma^c)^2 + (1-\vartheta(\mu^{\mathcal{B}}))^2(\tilde{\sigma})^2)/2
\end{aligned}
$$

  - ... agents hold more idiosyncratic risk

2 Asset Pricing Perspectives

Agent i's SDF, ξ_t^i: $\mathrm{d}\xi_t^i/\xi_t^i = -r_t^f \mathrm{d}t - \varsigma_t^i \mathrm{d}\tilde{Z}_t^i$

- Buy and Hold Perspective: *First aggregate and then iterate (overtime)*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = q_t^{\mathcal{B}} K_t = \lim_{T\to\infty} \left(\mathbb{E}\Big[\underbrace{\int_0^T \xi_t^i \text{AssetCashflow}_t \mathrm{d}t}_{\text{of whole issuance}}\Big] + \underbrace{\mathbb{E}[\xi_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T}]}_{\text{Bubble term}} \right)$$

  - If all agents i are marginal investors of aggregate risk asset
  - Note: if primary surplus is negative & grows with economy at r^f and $r^f < g$ first term is $+\infty$ and second term is $-\infty$

- Dynamic Trading Perspective: *First iterate (over time) and then aggregate*

- Dynamic trading strategy leads to cashflows conditional on idiosyncratic risks
- Denote η^i the *share of asset* held by agent i (Note: individual transversality condition holds)

$$\eta_t^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \xi_t^i (\eta_t^i \text{AssetCashflow}_t + \eta_t^i \text{TradingCashflow}_t) \mathrm{d}t \right] + \ldots \right)$$

2 Asset pricing perspectives

Agent i's SDF, ξ_t^i: $\mathrm{d}\xi_t^i/\xi_t^i = -r_t^f \mathrm{d}t - \tilde{\varsigma}_t^i \mathrm{d}\tilde{Z}_t^i$

- Buy and Hold Perspective: *First aggregate and then iterate (overtime)*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = q_t^{\mathcal{B}} K_t = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \xi_t^i \text{AssetCashflow}_t \mathrm{d}t \right] + \underbrace{\mathbb{E}[\xi_T^i P_T]}_{\textit{Bubble term}} \right)$$

of whole issuance

  - If all agents i are marginal investors of aggregate risk asset
  - Note: if primary surplus is negative & grows with economy at r^f and $r^f < g$ first term is $+\infty$ and second term is $-\infty$

- Dynamic Trading Perspective: *First iterate (over time) and then aggregate*

  - Dynamic trading strategy leads to cashflows conditional on idiosyncratic risks
  - Denote η^i the *share of asset* held by agent i (Note: individual transversality conditions hold)

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\int \mathbb{E} \left[\int_0^T \xi_t^i (\eta_t^i \text{AssetCashflow}_t + \eta_t^i \text{TradingCashflow}_t) \mathrm{d}t \right] + \dots \mathrm{d}i \right)$$

$$= \mathbb{E}\left[\int_0^T \overbrace{\int \xi_t^i \eta_t^i \mathrm{d}i}^{\xi_t^{**} :=} \text{AssetCashflow}_t \mathrm{d}t \right] + \mathbb{E}\left[\int_0^T \overbrace{\int \xi_t^i \eta_t^i \mathrm{d}i}^{\xi_t^{**} :=} \text{TradingCashflow}_t \mathrm{d}t \right]$$

service flow term

  - Discount rate $\mathbb{E}[\mathrm{d}r^\eta]/\mathrm{d}t = r^f + \tilde{\varsigma}\tilde{\sigma}$
  - ξ^i and η^i are negatively correlated $\Rightarrow$ depresses weighted "Quasi-SDF" (higher discount rate)

Quasi-SDF $\xi_t^{**} = \int \xi_t^i \eta_t^i di$?

- Total networth (incl. bubble wealth) $= \mathbb{E}_t\left[\int_t^\infty \underbrace{\frac{\int \xi_s^i \eta_s^i \mathrm{d}i}{\int \xi_t^i \eta_t^i \mathrm{d}i}}_{\xi_s^{**}/\xi_t^{**}} C_s \mathrm{d}s\right]$
  - Net worth share weighted SDF
  - "Representative agent SDF"
  - Complete markets: $\xi_t^{**} = \xi_t$
- Quasi risk-free rate r^{f**} (in cts. Ito world)

$$\overbrace{\underbrace{\underset{\rho}{\text{Preference rate}} + \underset{\gamma\mu_c}{\text{Ramsey term}} - \underset{\frac{1}{2}\gamma(\gamma+1)[\underbrace{(\sigma_t^c)^2}_{\text{agg risk}}}{\text{Precautionary savings/self-insurance}}}_{r^{f**}} + \underbrace{(\tilde{\sigma}_t^c)^2}_{\text{idio risk}}]}^{r^f}$$

Quasi-SDF $\xi_t^{**} = \int \xi_t^i \eta_t^i di$?

$$
\begin{aligned}
\xi_t^{**} &= \int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i} \\
&= \int e^{-\rho t} \frac{u'(c_t^{\tilde{i}})}{u'(c_0^{\tilde{i}})} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i} = \int e^{-\rho t} \left(\frac{c_t^{\tilde{i}}}{c_0^{\tilde{i}}} \right)^{-\gamma} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i} = \int e^{-\rho t} \left(\frac{\not{\rho} n_t^{\tilde{i}}}{\not{\rho} n_0^{\tilde{i}}} \right)^{-\gamma} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i}
\end{aligned}
$$

- For log utilty $\gamma = 1$: $\xi_t^{**} = \int e^{-\rho t} \left(\frac{n_t^{\tilde{i}}}{n_0^{\tilde{i}}} \right) \eta_t^{\tilde{i}} \mathrm{d}\tilde{i} = e^{-\rho t} \frac{N_0}{N_t}$
- Total net worth (incl. bubble wealth) $= N_t = \mathbb{E}_t \left[\int_t^{\infty} \underbrace{\frac{\int \xi_s^i \eta_s^i \mathrm{d}i}{\int \xi_t^i \eta_t^i \mathrm{d}i}}_{\xi_s^{**}/\xi_t^{**}} C_s \mathrm{d}s \right]$
    - Net worth share weighted SDF
    - "Representative agent SDF"
    - Complete markets: $\xi_t^{**} = \xi_t$

Deriving FTPL - traditional

- Money valuation equation for log utility $\gamma = 1$:

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t (\underbrace{\rho + \overbrace{g}^{\Phi(\iota)-\delta} - g - (1-\vartheta_t)^2 \tilde{\sigma}^2}_{=r^f} + \check{\mu}_t^B)$$

- Integrate forward:

$$\vartheta_0 = \mathbb{E} \int_0^T e^{-r^f t} e^{gt} (-\check{\mu}_t^{\mathcal{B}}) \vartheta_t \mathrm{d}t + \mathbb{E} e^{-r^f T} e^{gT} \vartheta_T \text{recall gov. budget constraint: } \check{\mu}_t^{\mathcal{B}} = -s/q_t^{\mathcal{B}}$$

$$= \mathbb{E}\left[\int_0^\infty e^{-r^f t} e^{gt} \frac{s}{q_t^{\mathcal{B}}} \vartheta_t \mathrm{d}t\right] + \mathbb{E} e^{-(r^f - g)T} \vartheta_T$$

$$= \mathbb{E}\left[\int_0^\infty e^{-(r^f-g)t} \frac{sK_t}{N_t} \mathrm{d}t\right] + \mathbb{E} e^{-(r^f-g)T} \frac{\mathcal{B}_T}{\mathcal{P}_T N_T}$$

- Multiply by N_0 : $\vartheta_0 N_0 = \mathbb{E}\left[\int_0^T e^{-(r^f-g)t} \underbrace{\frac{N_0}{N_t}}_{e^{-gt}} sK_t \mathrm{d}t\right] + \mathbb{E}\left[e^{-(r^f-g)T} \underbrace{\frac{N_0}{N_t}}_{e^{-gT}} \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]$

- FTPL equaiton: $\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T e^{-r^f t} sK_t \mathrm{d}t\right] + \mathbb{E}\left[e^{-(r^f-g)T} \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]$

(for $r^f > g$ take limit $T \to \infty$, $\mathbb{E}\left[e^{-(r^f-g)T} \frac{\mathcal{B}_T}{\mathcal{P}_T}\right] \to 0$)

Deriving FTPL – separating service flow with SDF ξ_t^{**}

- Money valuation equation for log utility $\gamma = 1$:

$$\vartheta_t \mu_t^{\vartheta} = \vartheta_t(\rho - (1-\vartheta_t)^2 \tilde{\sigma}^2 + \check{\mu}_t^{B})$$

- Integrate forward:

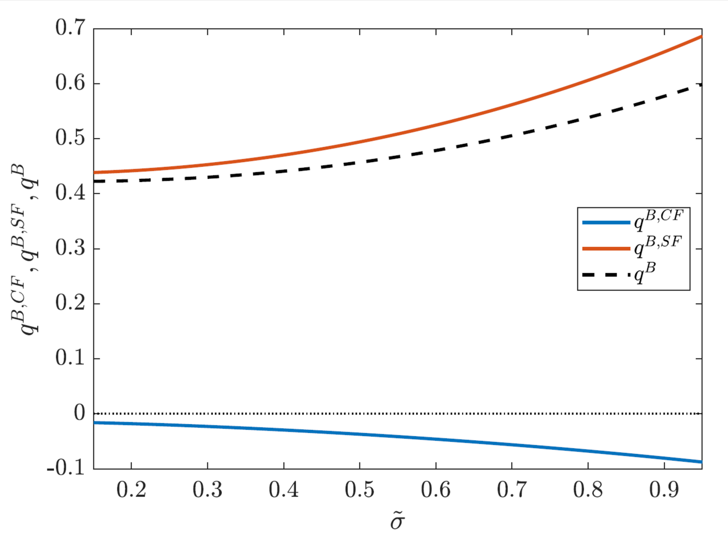
$$\begin{aligned}
\vartheta_0 &= \mathbb{E} \int_0^\infty e^{-\rho t}(-\check{\mu}_t^{\mathcal{B}} + (1-\vartheta_t)^2 \tilde{\sigma}^2)\vartheta_t \mathrm{d}t \\
&= \mathbb{E} \int_0^\infty e^{-\rho t} \frac{s}{q_t^{\mathcal{B}}} \vartheta_t \mathrm{d}t + \mathbb{E} \int_0^\infty e^{-\rho t} (1-\vartheta_t)^2 \tilde{\sigma}^2 \vartheta_t \mathrm{d}t \\
&= \mathbb{E} \int_0^\infty e^{-\rho t} \frac{sK_t}{N_t} \vartheta_t \mathrm{d}t + \mathbb{E} \int_0^\infty e^{-\rho t} (1-\vartheta_t)^2 \tilde{\sigma}^2 \frac{\mathcal{B}_t}{\mathcal{P}_t N_t} \mathrm{d}t
\end{aligned}$$

- Multiply by N_0

$$\vartheta_0 N_0 = \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**} := \int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i}} sK_t \mathrm{d}t\right] + \mathbb{E}\left[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**} := \int \xi_t^{\tilde{i}} \eta_t^{\tilde{i}} \mathrm{d}\tilde{i}} \underbrace{(1-\vartheta_t)^2 \tilde{\sigma}^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}}_{\textit{service flow}} \mathrm{d}t\right]$$

Safe Asset: Service flow ≫ Cash flow

- $$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**}:=\int \xi_t^{\tilde{i}}\eta_t^{\tilde{i}}d\tilde{i}} sK_t dt\right] + \mathbb{E}\left[\int_0^\infty \underbrace{e^{-\rho t}\frac{N_0}{N_t}}_{\xi_t^{**}:=\int \xi_t^{\tilde{i}}\eta_t^{\tilde{i}}d\tilde{i}} \underbrace{(1-\vartheta_t)^2\tilde{\sigma}^2\frac{\mathcal{B}_t}{\mathcal{P}_t}}_{\text{service flow}} dt\right]$$

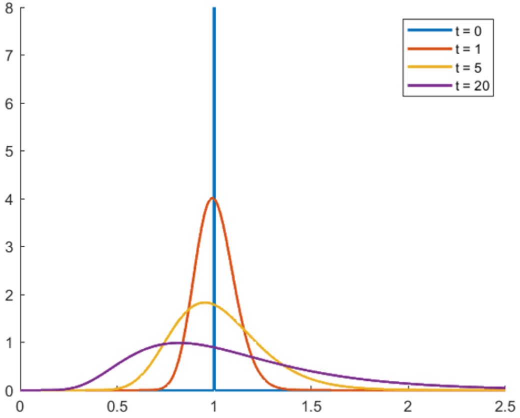


Service Flow Term vs. Bubble Term

- Service flow
  - Partial insurance via retrading – (partially undo incomplete markets)
    - Bewley . . . smooth consumption
    - BruSan . . . retrade capital and safe asset + smooth consumption
  - Remaining (idiosyncratic) risk depresses cash flow return
- Bubble
  - $\lim_{T\to\infty} \mathbb{E}[\bar{\xi}_t P_t] > 0$, if $r^f \leqslant g_t$
    r^f is depressed by precautionary savings (incl. uninsurable idiosyncratic risk)
  - Transversality condition holds for each individual, but not in aggregate
    $\neq$ complete markets
- Ponzi scheme is not feasibly for everyone, No Ponzi constraint may be binding
  - Who can run a Ponzi scheme? exorbitant privilege
    ... assigned by equilibrium selection
  - Likely to government, private entities are subject to solvency constraint

Remark: Cross-sectional Net Worth Distribution

- $\tilde{\eta}_t^{\tilde{i}} = \frac{\tilde{n}_t^{\tilde{i}}}{N_t}$ is non-stationary ... and log-normally distributed



- Next: Extend model with net worth reset jumps to η^*

Aside 1: Model Extension with Idiosyncratic Resilience

- With Poisson intensity λ net worth $\tilde{\eta}_t^{\tilde{i}-}$ jumps from to SS-η
- Log-utility $\Rightarrow$ same returns, no impact on equilibrium
  (because $\mathrm{d}J$ shock impact is independent of portfolio choice θ)

$$\frac{\mathrm{d}n_t^{\tilde{i}}}{n_t^{\tilde{i}}} = \Bigg(-\overbrace{\rho}^{c_t^{\tilde{i}}/n_t^{\tilde{i}}} + \overbrace{g}^{r_t^B=\Phi(\iota)-\delta} + (1-\theta_t)\overbrace{\frac{a-\iota}{q_t^K}}^{\mathbb{E}[\mathrm{d}r_t^{k,\tilde{i}}]/\mathrm{d}t - r_t^B}\Bigg)\mathrm{d}t + (1-\theta_t)\tilde{\sigma}\mathrm{d}Z_t^{\tilde{i}} + j_t^{n,\tilde{i}}\mathrm{d}J_t^{n,\tilde{i}}, \frac{\mathrm{d}N_t}{N_t} = g\mathrm{d}t$$

$$\frac{\mathrm{d}\eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}} = \underbrace{\left(-\rho + (1-\vartheta_t)\frac{a-\iota}{q^K}\right)}_{=0}\mathrm{d}t + (1-\vartheta_t)\tilde{\sigma}\mathrm{d}Z_t^{\tilde{i}} + j_t^{n,\tilde{i}}\mathrm{d}J_t^{n,\tilde{i}} = (1-\vartheta_t)\tilde{\sigma}\mathrm{d}Z_t^{\tilde{i}} + j_t^{n,\tilde{i}}\mathrm{d}J_t^{n,\tilde{i}}$$

- Set $j_t^{n,\tilde{i}} = \frac{\eta^* - \eta_t^{\tilde{i}}}{\eta_t^{\tilde{i}}}$
- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = \frac{(1-\vartheta)^2\tilde{\sigma}^2}{2}\frac{\partial^2(\eta^2 g(\eta))}{\partial\eta} - \lambda g(\eta)$$

- There is a kink at η^*

Aside 1: Model Extension with Idiosyncratic Resilience

- KFE (for all $\eta \neq \eta^*$) is given by:

$$0 = g''(\eta)\eta^2 + 4g'(\eta)\eta + \left(2 - \frac{2\lambda}{(1-\vartheta)^2\tilde{\sigma}^2}\right) g(\eta)$$

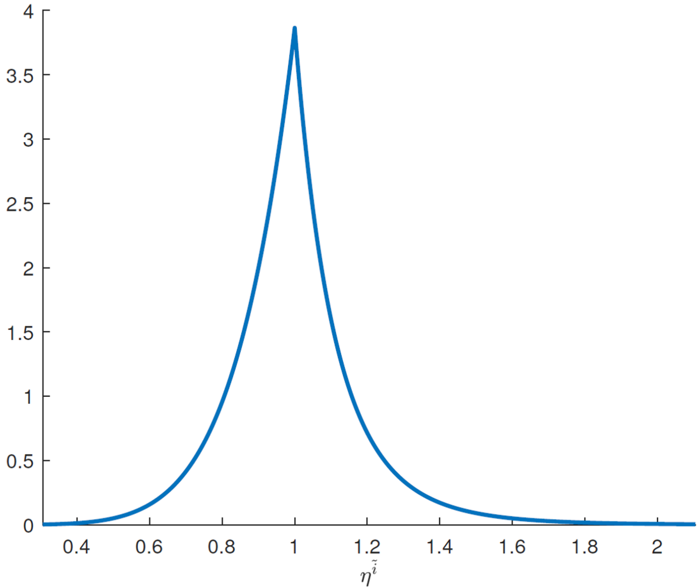
- Euler's equation – has closed-form solutions:

$$\begin{aligned} g(\eta) &= C_1\eta^{\alpha_1} + C_2\eta^{\alpha_2} \text{ for } \eta < \eta^* \\ g(\eta) &= C_3\eta^{\alpha_3} + C_4\eta^{\alpha_4} \text{ for } \eta \geqslant \eta^* \end{aligned}$$

$$\int_0^\infty g(\eta)\mathrm{d}\eta = 1, \quad \lim_{\eta\to 0} g(\eta) = \lim_{\eta\to\infty} g(\eta) = 0$$

- $+$ continuity at η^*: $\alpha_1 = \frac{\alpha-3}{2}.\alpha_2 = -\frac{\alpha+3}{2}, \alpha = \sqrt{\frac{8\lambda}{(1-\vartheta)^2\tilde{\sigma}^2} + 1}$
- Solution under $\eta^* = 1$: $C_1 = C_4 = \frac{2\lambda}{(1-\vartheta^2)\tilde{\sigma}^2\alpha}, C_2 = C_3 = 0$

Aside 1: Model Extension with Idiosyncratic Resilience



Aside 1: Model Extension with Idiosyncratic Resilience

- Homework: Generalize setting with CRRA $\gamma \neq 1$.
- Portfolio choice is independent of jump risk.
- Consumption-net worth ratio c_t/n_t is affected $\Rightarrow$ impacts q^B (and ϑ) [see good market clearing condition]
- How generalize numerical procedure of slide 37 (Step 4 with CRRA)?
- extreme redistribution $\lambda \to \infty$
  - Does this impact the bubble mining?

Overview

- Intuition for different "Monetary Theories"
- Monetary Model with one sector with constant idiosyncratic risk
  - Safe Asset and Service Flows
  - Bubble (mining) or not
  - 2 Different Asset Pricing Perspectives/SDFs
- Monetary model with time-varying idiosyncratic risk
  - Safe Asset with CAPM-$\beta < 0$,
  - Exorbitant Privilege, Laffer Curve
  - Safe Asset-Bubble Complementarity
  - Modern Debt Sustainability Analysis
- Medium of Exchange Role