Eco529: Modern Macro, Money, and International Finance

Lecture 05: Contrasting Financial Frictions

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
   - Log-utility Model with Fire-sales
   - Contrasting Financial Frictions
   - CRRA-EZ-utility
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
   - Fiscal, Monetary, and Macroprudential Policy

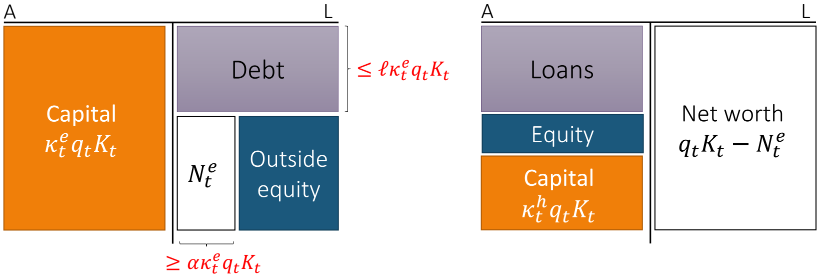
International Macro-Finance Models

1. International Financial Architecture

Digital Money

Two Sectors: Leverage + Skin-in-the-Game Constraint

- Expert sector   Household sector



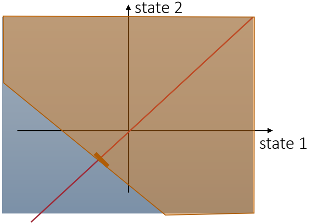
- Households can produce with capital.
  - Productivity $0 < a^h < a^e$
- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- The fraction of aggregate risk held by experts: $\chi_t^e = \frac{\sigma_{N^e,t}}{\sigma_{qK,t}}$
- Experts can issue debt, and outside equity.

  Leverage Constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$.

  Skin in the Game Constraint: $OE_t^e \leqslant (1-\alpha)\kappa_t^e q_t K_t$

Financial Frictions and Distortions

- Belief distortions
  - Match "belief surveys"
- Incomplete markets
  - "natural" leverage constraint (BruSan)
  - Costly state verification (*BGG*)
- $+$ Leverage constraints
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period's price $Rb_t \leqslant q_{t+1} k_t$ (*KM*)
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+1}) k_t$ (*BruPed*)
- Search Friction (*DGP*)



Occasionally binding equity constraint

Two Sector Model Setup: Leverage + Skin-in-the Game

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{dk_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},e}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log(c_t^e) dt\right]$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^h
  $\frac{dk_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},h}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log(c_t^h) dt\right]$

Friction: Can issue

- Risk-free debt,
  Leverage constraint: $-\theta_t^{e,D} \leqslant \ell \theta_t^{e,K}$ (occasionally binding)
- Outside equity,
  Skin-in-the-Game constraint: $-\theta^{e,OE} \leqslant (1-\alpha)\theta_t^{e,K}$ (occasionally binding)

Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**
   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   Fisher separation theorem
   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach

   Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility
4. Numerical model solution
5. KFE: Stationary distribution, fan charts

1b. Overview: Different Approaches

- **Approach 1:** Portfolio Optimization θ
  - Optimization via Stochastic Maximum Principle: most general way, but requires setting up Hamiltonian.
  - Optimization via Martingale Approach: complicated when constraints interact in a non-trivial way (here w/o leverage constraint)
- **Approach 2:** Price-taking Social Planner Approach (κ, χ)

1b. Experts' θ-Choice: Stochastic Maximum Principle

- Experts' problem: (let $r_t^{e,j} := \mathbb{E}[dr_t^{e,j}]/dt$)

$$\max_{c_t^e,\iota_t^e,\theta_t^{e,K},\theta_t^{e,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^e t} u(c_t^e)\mathrm{d}t\right] \; s.t.$$

$$\mathrm{d}n_t^e = \left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]\mathrm{d}t$$
$$+ n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

$$(1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE} \geqslant 0 \text{ (skin in the game)}, (1-\ell)\theta_t^{e,K} + \theta_t^{e,OE} \leqslant 1 \text{ (leverage)}$$

- Denote the multiplier on leverage constraint as λ_t^ℓ, multiplier on skin in the game constraint as λ_t^s. The Hamiltonian can be constructed as $\mathcal{H}_t^e =$

$$e^{-\rho^e t}u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e\left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t) + \theta_t^{e,OE}(r_t^{e,OE} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} - \varsigma_t^e \xi_t^e \overbrace{n_t^e(\theta_t^{e,K} + \theta_t^{e,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e}$$
$$+\xi_t^e n_t^e \lambda_t^\ell \left(1 - (1-\ell)\theta_t^{e,K} - \theta_t^{e,OE}\right) + \xi_t^e n_t^e \lambda_t^s \left((1-\alpha)\theta_t^{e,K} + \theta_t^{e,OE}\right)$$

- Objective function is linear in θ (divide through $\xi_t^e n_t^e$)
  $\Rightarrow$ bang-bang (indifferent or at a constraint)
- FOC w.r.t. c_t is separated/de-coupled from FOC w.r.t. θ_ts as well as ι_t^e
  $\Rightarrow$ Fisher Separation Theorem btw. $c_t^e, \theta_t^e, \iota_t^e$

1b. Households' θ-Choice: Stochastic Maximum Principle

- Households' problem:

$$\max_{c_t^h, \iota_t^h, \theta_t^{h,K}, \theta_t^{h,OE}} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t} u(c_t^h)\mathrm{d}t\right], \ s.t.$$

$$\mathrm{d}n_t^h = \left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K} - r_t) + \theta_t^{h,OE}(r_t^{h,OE}(\iota_t^h) - r_t)\right)\right]\mathrm{d}t + n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

$$\theta_t^{h,K} \geqslant 0 \text{ (household short sale constraint)}$$

- Denote the multiplier on the short selling constraint on capital as λ_t^h. The Hamiltonian can be constructed as:

$$\mathcal{H}_t^h = e^{-\rho^h t} u(c_t^h) + \xi_t^h \overbrace{\left[-c_t^h + n_t^h\left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t) + \theta_t^{h,OE}(r_t^{h,OE} - r_t)\right)\right]}^{\mu_t^{n^h} n_t^h}$$

$$-\varsigma_t^h \xi_t^h \overbrace{n_t^h(\theta_t^{h,K} + \theta_t^{h,OE})(\sigma + \sigma_t^q)}^{\sigma_t^{n^h} n_t^h} + \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Linear in θ_t and Fisher Separation Theorem

1b. θ-Choice: Stochastic Maximum Principle

- Experts' FOC w.r.t. θ:

$$\begin{cases} r_t^{e,K} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi & (1) \\ r_t^{OE} - r_t = \varsigma_t^e(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi & (2) \end{cases}$$

- Households' FOC w.r.t. θ:

$$\begin{cases} r_t^{h,K} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) - \lambda_t^h & (3) \\ r_t^{OE} - r_t = \varsigma_t^h(\sigma + \sigma_t^q) & (4) \end{cases}$$

- Take difference btw (1) and (3) as well as btw (2) and (4)

$$\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^h + (1-\ell)\lambda_t^\ell - (1-\alpha)\lambda_t^\chi,$$

$$0 = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q) + \lambda_t^\ell - \lambda_t^\chi,$$

1b. θ-Portfolio Constraints: Figuring out λs

- Focus on the return gap $r_t^{OE} - r_t^{h,K}$ and $r_t^{e,K} - r_t^{OE}$

$$\begin{cases} r_t^{e,K} - r_t^{OE} = \alpha\lambda_t^{\chi} - \ell\lambda_t^{\ell} \\ r_t^{OE} - r_t^{h,K} = \lambda_t^{h} \end{cases}$$

- Household short selling constraint not binding: $\lambda_t^h = 0$
  - $\lambda_t^{\chi} = 0, \lambda_t^{\ell} > 0$ impossible because $r_t^{e,K} > r_t^{h,K}$
  - $\lambda_t^{\chi} > 0, \lambda_t^{\ell} > 0$ and $\lambda_t^{\chi} > 0, \lambda_t^{\ell} = 0$ both possible
    $\Rightarrow$ Leverage constraint binding
    or Leverage and skin-in-the game both binding
- Household short selling constraint binding: $\lambda_t^h > 0$
  - Define smallest η_t^e such that $\lambda_t^h > 0$
  - $\lambda_t^{\ell} > 0$ impossible because $1/\eta_t^e < 1/\eta^{e,*}$
    $\Rightarrow$ Only skin-in-the-game may bind.
    Intuition: outside equity cannot generate higher return than physical capital

1b. θ-Portfolio to (κ, χ)-Asset/Risk Allocation Constraint

- First order condition (plug in for λs)

$$\frac{a^e - a^h}{q_t} \geqslant \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta-\text{risk premia}}, \quad \text{with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

$$\varsigma_t^e \geqslant \varsigma^h, \quad \text{with equality if } \chi_t^e > \alpha\kappa_t^e$$

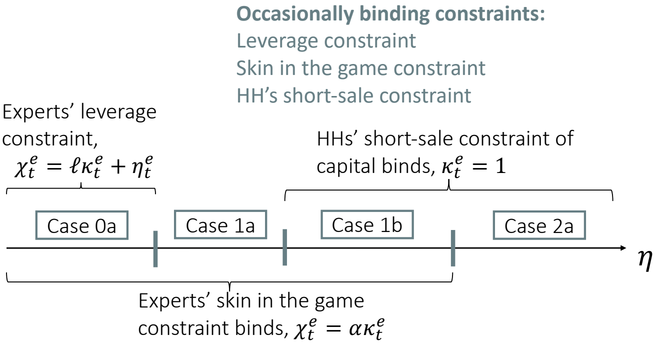
- Constraints were translated from θ space to χ-κ space:
  Skin-in-the-game constraint $\Rightarrow \chi_t^e = \eta_t^{e,K}\theta_t^e + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\geqslant -(1-\alpha)\kappa_t^e} \geqslant \alpha\kappa_t^e$,

  Leverage constraint $\Rightarrow \chi_t^e = \eta_t^e\theta_t^{e,K} + \underbrace{\eta_t^e\theta_t^{e,OE}}_{\leqslant (1-(1-\ell)\theta_t^{e,K})} \leqslant \ell\kappa_t^e + \eta_t^e$

1b. Occasionally Binding Constraints across η

Cases	0a	1a	1b	2a
leverage	$\chi_t^e = \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$
skin in game	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e > \alpha\kappa_t^e$
short-sale	$\kappa_t^e < 1$	$\kappa_t^e < 1$	$\kappa_t^e = 1$	$\kappa_t^e = 1$
Δ-risk premia	$>$	$=$	$>$	$>$
risk-sharing	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$

complementary slackness conditions



1b. θ-Choice: Martingale Approach (aside) (Relaxed Skin-in-the-Game, No Leverage Constraint)

- **Approach 1:** Portfolio Optimization
  - Step 1: Optimization e.g. via Martingale Approach – recall: $\boxed{\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A}$
    - Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q = r_t + \underbrace{[\varsigma_t^e \chi_t^e/\kappa_t^e + \varsigma_t^h (1 - \chi_t^e/\kappa_t^e)]}_{\text{new compared to lecture 04}}(\sigma + \sigma_t^q)$$

    - Of households' capital choice:

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \leqslant r_t + \varsigma_t^h(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1$$

    - Step 2: Capital market clearing to obtain asset/risk allocation κ_t^e, χ_t^e from portfolio weights θs
- **Approach 2:** Price-taking Social Planner Approach

1b. Price Taking Social Planner ⇒ Asset/Risk Allocation

- Maximization within each $\{\}$-term = maximization over weighted sum
- Choose η-weighted sum of expert + HH maximization problem

$$\eta^e\{...\} + \eta^h\{...\}$$

- Why?
  - positive net supply assets become capital and risk shares (of Brownian)
  - zero net supply assets cancel out.

$$\overbrace{\eta_t^e\theta_t^{e,K}}^{\equiv\kappa_t^e}\mathbb{E}[dr_t^{e,K}]/\mathrm{d}t + \overbrace{\eta_t^h\theta_t^{h,K}}^{\equiv\kappa_t^h}\mathbb{E}[dr_t^{h,K}]/dt + \overbrace{(\eta_t^e\theta_t^{e,OE} + \eta_t^h\theta_t^{h,OE})}^{=0}\mathbb{E}[dr_t^{OE}]/dt$$
$$+\underbrace{(\eta_t^e\theta_t^{e,D} + \eta_t^h\theta_t^{h,D})}_{=0}r_t - \varsigma_t^e\underbrace{\eta_t^e(\theta_t^{e,K} + \theta_t^{e,OE})}_{\equiv\chi_t^e}\sigma_t^{r^K} - \varsigma_t^h\underbrace{\eta_t^h(\theta_t^{h,K} + \theta_t^{h,OE})}_{\equiv\chi_t^h}\sigma_t^{r^K}$$

- Translate portfolio constraints in capital and risk share constraints

1b. Portfolio to Asset/Risk Allocation Constraints

- Convert λ-constraints into κ, χ-constraints
  Skin-in-the-game constraint $\Rightarrow \chi_t^e = \eta_t^e \theta_t^e + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\geqslant -(1-\alpha)\kappa_t^e}$,
  Leverage constraint $\Rightarrow \chi_t^e = \eta_t^e \theta_t^e + \underbrace{\eta_t^e \theta_t^{e,OE}}_{\leqslant (1-(1-\ell)\theta_t^{e,K})} \leqslant \ell \kappa_t^e + \eta_t^e$
- **Price-taking social planner's problem:**

$$\max_{\substack{\{\chi_t^e \in [\alpha\kappa_t^e, \kappa_t^e], \chi_t^h = 1-\chi_t^e, \\ \kappa_t^e, \kappa_t^h = 1-\kappa_t^e\}}} \left[\frac{\kappa_t^e a^e + \kappa_t^h a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta\right] - (\varsigma_t^e \chi_t^e + \varsigma_t^h \chi_t^h)\sigma_t^{r^K}$$

End of Proof. Q.E.D

- Linear objective (if frictions take form of constraints)
  - Price of risk adjust such that objective becomes flat *or*
  - Bang-bang solution hitting constraints
- First order condition

$$\frac{a^e - a^h}{q_t} \geqslant \underbrace{\alpha(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)}_{\Delta-\text{risk premia}}, \text{ with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

1b. Price Taking Social Planner $\Rightarrow$ Asset/Risk Allocation

Cases	0a	1a	1b	2a
leverage	$\chi_t^e = \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$	$\chi_t^e < \ell\kappa_t^e + \eta_t^e$
skin in game	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e = \alpha\kappa_t^e$	$\chi_t^e > \alpha\kappa_t^e$
short-sale	$\kappa_t^e < 1$	$\kappa_t^e < 1$	$\kappa_t^e = 1$	$\kappa_t^e = 1$
Δ-risk premia	$>$	$=$	$>$	$>$
risk-sharing	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t > \eta_t$	$\chi_t = \eta_t$

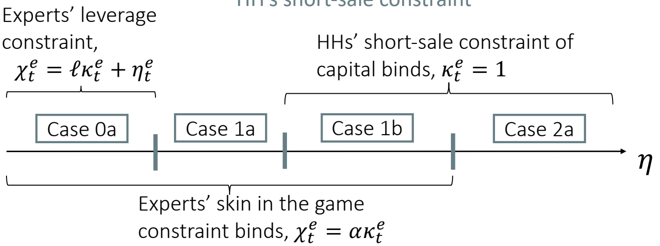
complementary slackness conditions

Occasionally binding constraints:

Leverage constraint

Skin in the game constraint

HH's short-sale constraint



1b. Price Taking Social Planner (General Theorem)

- Price-Taking Planner's Theorem:
  A social planner that takes prices as given chooses a physical asset allocation, κ_t and risk allocation, χ_t that coincides with the choices implied by all individuals' portfolio choices.
- Notation:

$$\boldsymbol{\varsigma}_t = (\varsigma_t^1, ..., \varsigma_t^I)$$
$$\boldsymbol{\chi}_t = (\chi_t^1, ...\chi_t^I)$$
$$\boldsymbol{\sigma}(\boldsymbol{\chi}_t) = (\chi_t^1 \sigma^N, ..., \chi_t^I \sigma^N)$$

- Planner's problem:

$$\max_{\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t} \mathbb{E}_t[\mathrm{d}r_t^N(\boldsymbol{\kappa}_t)]/\mathrm{d}t - \boldsymbol{\varsigma_t} \sigma(\boldsymbol{\chi}_t) \quad (= dr^F/\mathrm{d}t \text{ in equilibrium})$$
$$s.t. \quad F(\boldsymbol{\kappa}_t, \boldsymbol{\chi}_t) \leqslant 0 \qquad (\text{Financial Frictions})$$

Solving Macro Models Step-by-Step

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1. For given C/N-ratio and SDF processes for each i **finance block**
   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   Fisher separation theorem
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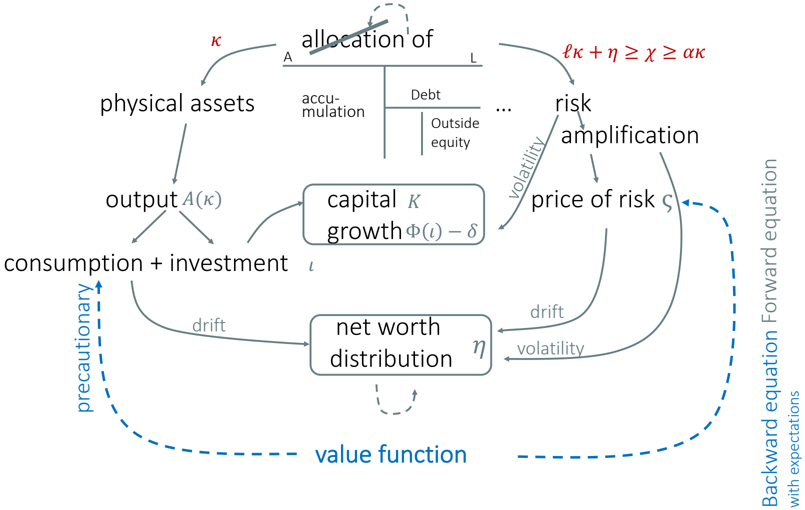
   b. Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach

   Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) $\Rightarrow$ **as in Lecture 04** **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility
4. Numerical model solution
5. KFE: Stationary distribution, fan charts

The Big Pricture



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4a. Obtain κ for Goods Market Clearing (Outside Equity)

- Determination of κ_t
  - Based on difference in risk premia $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$
  - For log utility: $(\sigma_t^{n^e} - \sigma_t^{n^h})(\sigma + \sigma_t^q) = \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$

    Since: $\eta_t^{\eta^e} = \frac{\kappa_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q), \eta_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$, and $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{\eta^e} - \sigma_t^{\eta^h}$
- Hence,

$$\frac{a^e - a^h}{q_t} \geqslant \alpha \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1 \text{ and } \chi_t^e < \ell\kappa_t^e + \eta_t^e.$$

- Determination of χ_t^e:

$$\chi_t^e = \max\{\alpha\kappa_t^e, \eta_t^e\}$$

- Determination of κ_t^e in the leverage constrained region:

$$\kappa_t^e = \frac{\eta_t^e}{\alpha - \ell}$$

4a. Investments and Capital Prices

- Replacing ι_t.
  - Recall from optimal re-investment $\Phi'(\iota) = 1/q_t$:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$$

- Recall from "amplification slide"

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}\frac{\chi_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

4a. Market Clearing

- Output good market:

$$(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t)K_t = C_t$$

$$\Rightarrow \quad \boxed{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t = q_t[\eta_t\rho^e + (1-\eta_t)\rho^h]}$$

- Capital market is taken care off by price-taking social planner approach.
- Risk-free debt market also taken care off by price taking social planner approach (would be cleared by Walras Law anyways)

4b. Algorithm – Static Step

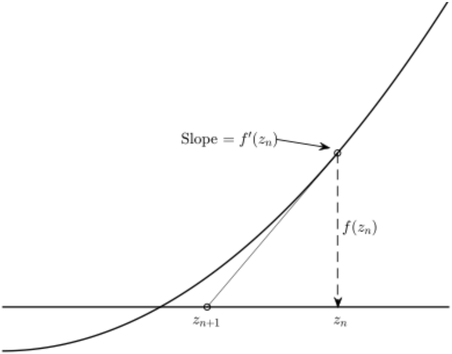
- We have five static conditions

1. $\phi\iota_t = q_t - 1$
2. Planner condition for κ^e_t: $\frac{a^e - a^h}{q_t} \geqslant \alpha \frac{\chi^e_t - \eta^e_t}{(1-\eta^e_t)\eta^e_t}(\sigma + \sigma^q_t)^2$
3. Planner condition for χ^e_t: $\chi^e_t = \max\{\alpha\kappa^e_t, \eta^e_t\}$
4. $\kappa^e_t a^e_t + (1 - \kappa^e_t)a^h - \iota(q_t) - q_t[\eta_t\rho^e + (1 - \eta_t)]\rho^h$
5. $\sigma^q = \frac{q'(\eta^e_t)}{q(\eta^e_t)}(\chi^e_t - \eta^e_t)(\sigma + \sigma^q_t)$
   $\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

- Start at $q(0)$, solve to the right, use different procedure for two η regions depending on κ:

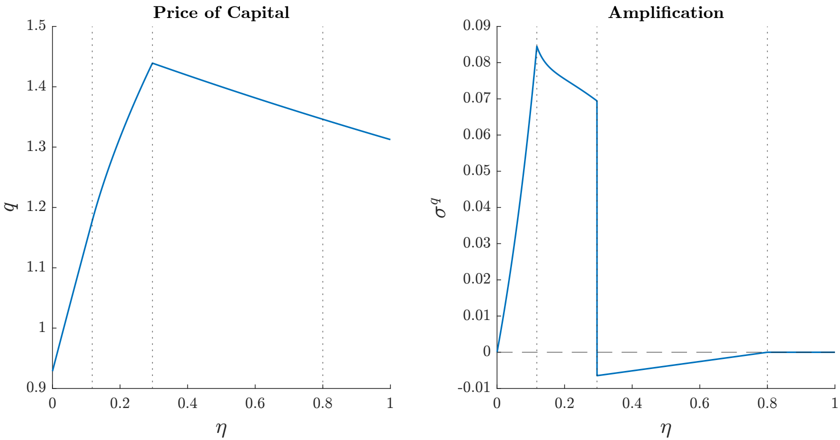
1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (4)
   - Plug in the Planner's condition of χ_t
   - Solve ODE using three equilibrium condition (2),(4) and (5) via Newton's method
   - if $\chi^e_t \geqslant \ell\kappa^e_t + \eta^e_t$, replace κ^e_t by $\frac{\eta^e_t}{\alpha - \ell}$, solve (3) (4) (5) for $\chi(\eta^e)$,$q(\eta^e)$,$\sigma^q(\eta^e)$
2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (4) for $q(\eta^e)$
   (HINT: When constraint binds, we directly substitute in κ^e)

4b. Aside: Newton's Method



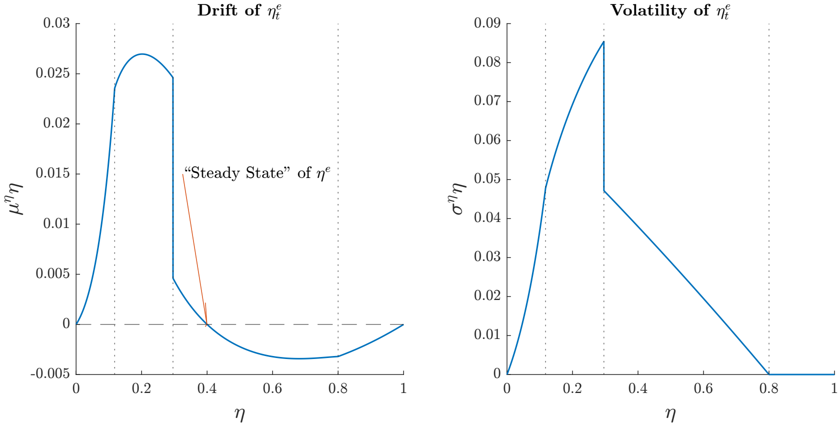
$$\boldsymbol{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, F(\boldsymbol{z}_n) = \begin{bmatrix} \kappa_t^e a_t^e + (1-\kappa_t^e)a^h - \iota(q_t) - q_t[\eta_t \rho^e + (1-\eta_t)\rho^h] \\ q'(\eta_t^e)(\chi_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - \alpha q_t \frac{\chi_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{Planner.} \end{bmatrix}$$

Capital Price and Volatility



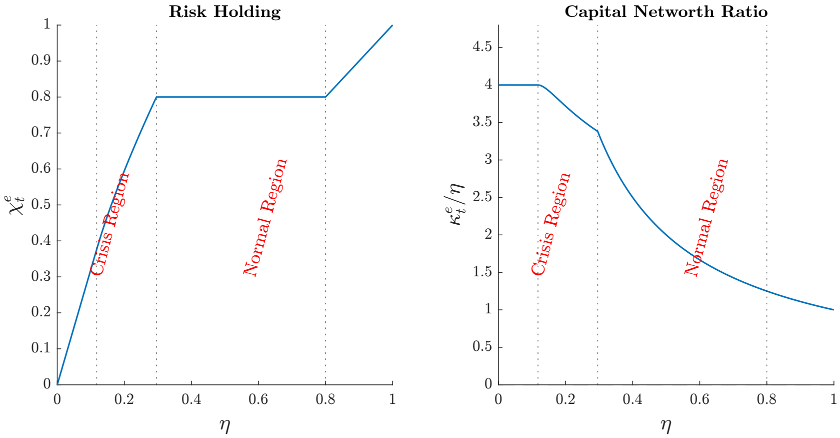
$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta =$0.05, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

Net Worth Evolution: Drift & Volatility



$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

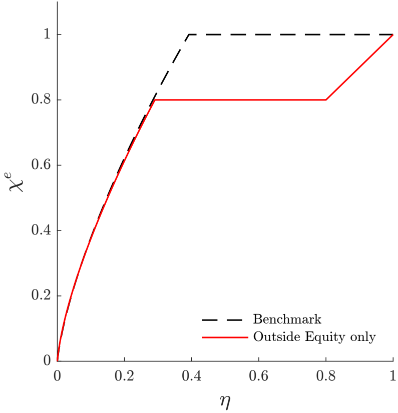
Risk Allocation & Leverage

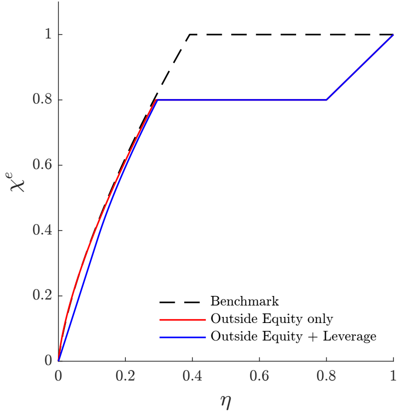


$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$, $\alpha = 0.8$, $\ell = 0.55$.

Risk Allocation: Compare with $\alpha = 1$, $\ell = 1$

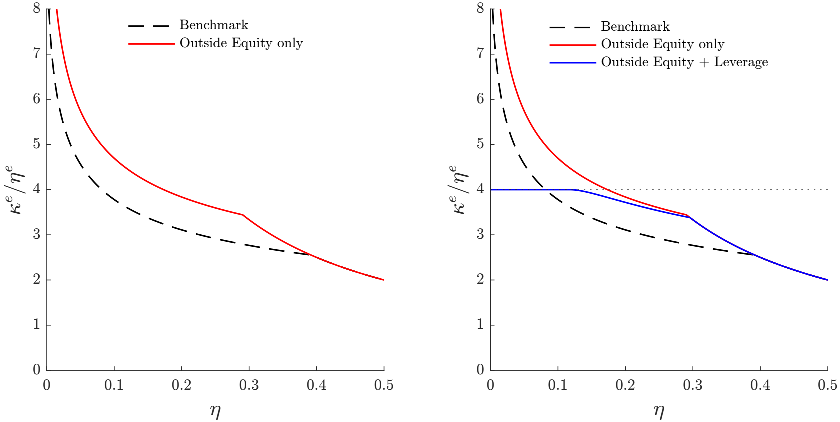
- allow some outside equity $\alpha = .8$ limit leverage $\ell = .55$





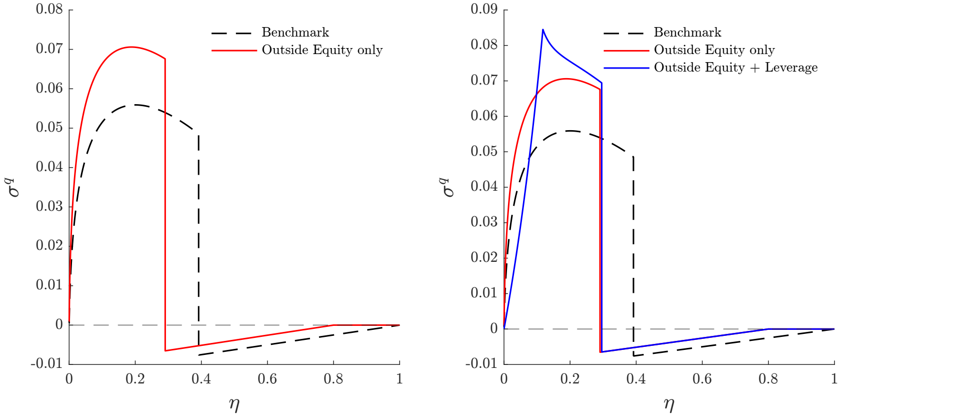
Leverage: Capital Net Worth Ratio

- allow some outside equity $\alpha = .8$ — limit leverage $\ell = .55$



Price Volatility: Compare with $\alpha = 1$, $\ell = 1$

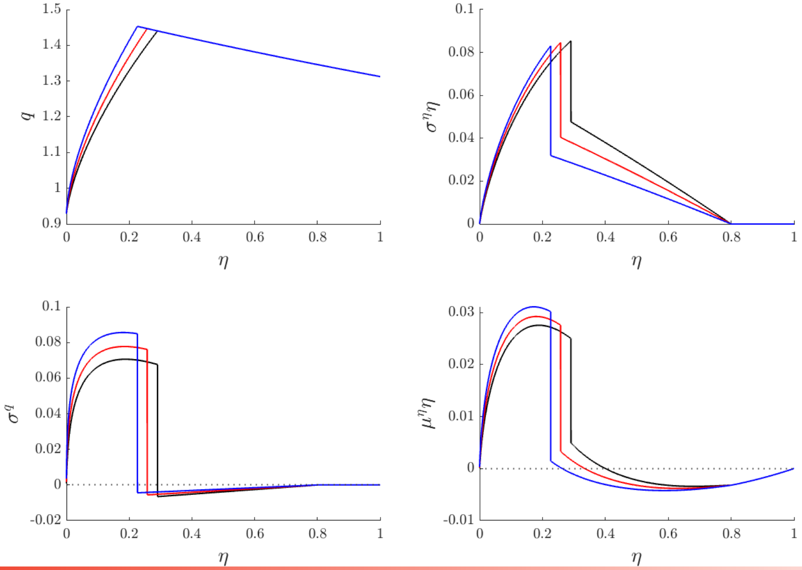
- allow some outside equity $\alpha = .8$ — limit leverage $\ell = .55$



Volatility Paradox $\alpha = 0.8$

- σ^η (as well as $\sigma + \sigma^q$) stays roughly constant as σ varies (even when $\sigma \to 0$)

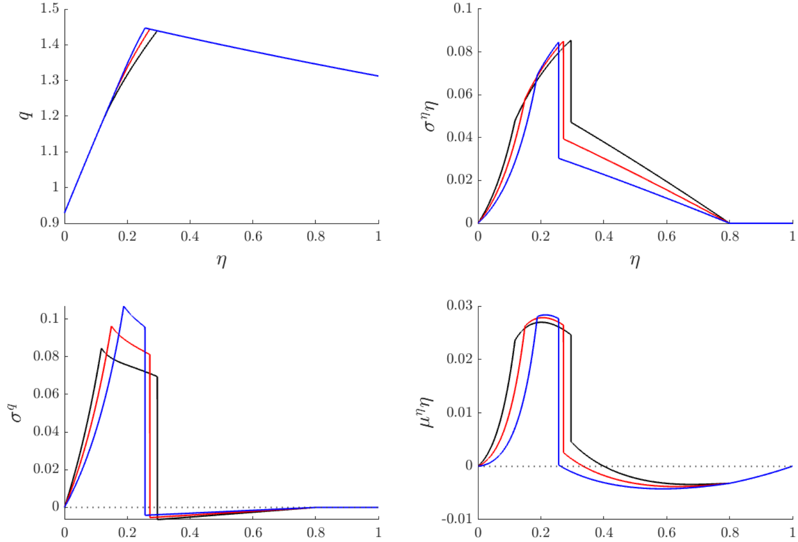
$\sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$



Volatility Paradox $\alpha = 0.8, \ell = 0.55$

- arises in fire-sale region in which leverage constraint does not bind
- leverage constraints lowers volatility and drift

$\sigma = 0.10$, $\sigma = 0.08$, $\sigma = 0.06$



Solving Macro Models Step-by Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**
   Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach
   a. Real investment ι + Goods market clearing *(static)*
   b. Fisher separation theorem
      Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach
      Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) $\Rightarrow$ **as in Lecture 04** **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility
4. Numerical model solution
5. KFE: Stationary distribution, Net worth trap

5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows:

$$\frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

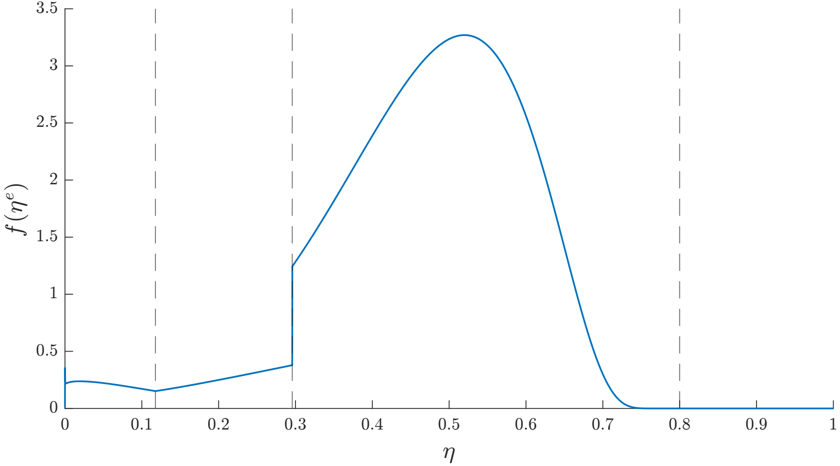
- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"
- Corollary: If stationary distribution $f(\eta)$ exists, it satisfies ODE:

$$0 = -\frac{\mathrm{d}[f(\eta)\mu(\eta)]}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^2[f(\eta)\sigma^2(\eta)]}{\mathrm{d}\eta^2}$$

  - Closed form solution:

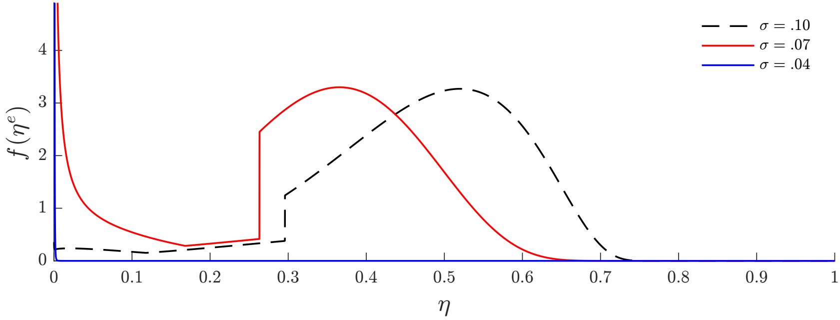
$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)} \mathrm{d}x\right)$$

5. Stationary Distribution



Stationary Distribution for Different σ

- "**Net worth Trap**": Stationary distribution is double-humped shaped
- Lack of Resilience
- Fundamental volatility: $\sigma = .10$, $\sigma = .07$, $\sigma = .04$



- around $\eta = .3$: steady state positive drift but thrown back by skewed shocks
  around $\eta \to 0$: positive drift but thrown back by skewed shocks

Existence of Stationary Distribution

- Observation of comp statics ⇒ stationary dist does not exist for $\sigma = 0.04$
- (Intuition side) When does invariant distribution exist? ⇒ recurrency
  - Forces pull particle out when collapse.
  - "Bounce" back when hitting barrier.
- (Math side) Recall closed form solution:

$$f(\eta) = \frac{\text{Const}}{\sigma^2(\eta)} \exp\left(\int_0^\eta \frac{2\mu(x)}{\sigma^2(x)} \mathrm{d}x\right)$$

  - $f(\eta) \geqslant 0$: probability cannot be negative.
  - $\int f(\eta)\mathrm{d}\eta = 1$: probability distribution is normalizable.

Aside: KFE Analytical Example

- Reflected Geometric Brownian Motion (Reflecting barrier at $x = d$):

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}Z_t - \mathrm{d}U_t, X_t \in (0, d]$$

- KFE:

$$\frac{\partial f}{\partial t} = -\frac{\partial(\mu x f)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2 x^2 f)}{\partial x^2}$$

- Stationary distribution

$$f(x) = \frac{\text{Const}}{\sigma^2 x^2} \exp\left(\int_0^x \frac{2\mu y}{\sigma^2 y^2} \mathrm{d}y\right) = \frac{\frac{2\mu}{\sigma^2} - 1}{d^{\frac{2\mu}{\sigma^2}-1}} x^{\frac{2\mu}{\sigma^2}-2}$$

- Question: when $f(x)$ becomes a density?

Aside: KFE Analytical Example

- Reflected Geometric Brownian Motion (Reflecting barrier at $x = d$):

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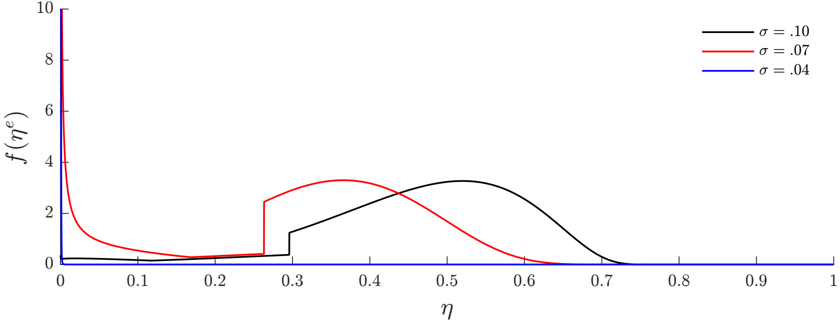
- Stationary distribution

$$f(x) = \frac{\text{Const}}{\sigma^2 x^2} \exp\left(\int_0^x \frac{2\mu y}{\sigma^2 y^2} \mathrm{d}y\right) = \frac{\frac{2\mu}{\sigma^2} - 1}{d^{\frac{2\mu}{\sigma^2}-1}} x^{\frac{2\mu}{\sigma^2}-2}$$

- Question: when $f(x)$ becomes a density?
- "Bouncing back" because of reflecting barrier at $x = d$.
- "Pulled back" by strong enough $\mu(x)$ at $x = 0$.

Stationary Distribution Revisited



- Asymptotic solution ($\eta \to 0$):

$$f(\eta) \sim \left(\frac{2\mu(0)}{\sigma^2(0)} - 1\right) \eta^{\frac{2\mu(0)}{\sigma^2(0)}-2}$$

  - $\frac{2\mu(0)}{\sigma^2(0)} \geqslant 2$: $f(\eta)$ is finite at $\eta = 0$
  - $2 \geqslant \frac{2\mu(0)}{\sigma^2(0)} > 1$: $f(\eta)$ is infinite at $\eta = 0$, but still normalizeable ($\int f \mathrm{d}\eta < \infty$)
  - $1 \geqslant \frac{2\mu(0)}{\sigma^2(0)}$: $f(\eta)$ is infinite at $\eta = 0$, stationary distribution does not exist

Net Worth Trap & Volatility Paradox Interaction

- Net Worth Trap based on volatility paradox interaction with leverage constraint:
  - Leverage constraint depresses μ^η and σ^η
  - High volatility in fire-sale region outside binding leverage constraint
  - As η declines, does μ^η or $(\sigma^\eta)^2$ decline faster?
- Micro- and Macro-Prudential Regulation: Basel I, II, III
  - Basel I: fixed risk-weights and capital requirement
  - Basel II: risk-weights but not time-varying $\Rightarrow$ Net Worth Trap
  - Basel III: Countercyclical capital buffer: (contemporaneous, not past)

Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales
  - liquidity spirals
  - fat tails
- Volatility paradox
- Resilience vs. "Net worth trap" double-humped stationary distribution