Eco529: Modern Macro, Money, and International Finance

Lecture 04: Endogenous Risk Dynamics

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
   - Log-utility Model with Fire-sales
   - Contrasting Financial Frictions
   - CRRA-EZ-utility
   - Evolution of Distribution, Fan Charts
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
   - Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models

1. International Financial Architecture

Digital Money

Desired Model Properties

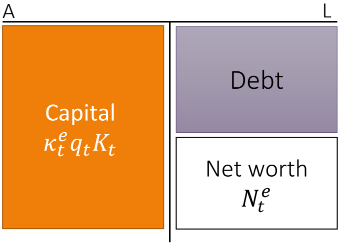
- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- ("Net worth trap" double-humped stationary distribution)

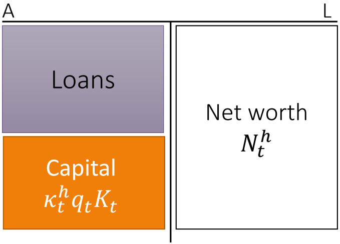
Toolboxes: Technical Innovations

- Occasionally binding (short-sale) constraint
  (in addition to natural borrowing limit due to risk aversion)
- Price setting social planner to find capital and risk allocation
- Change of numeraire
  - Easily incorporate aggregate fluctuations
  - To use martingale methods more broadly
- Newton Method to solve log-utility numerical example

Two Sector Model: Simple Extension of Basak Cuoco

- Expert sector
  Household sector



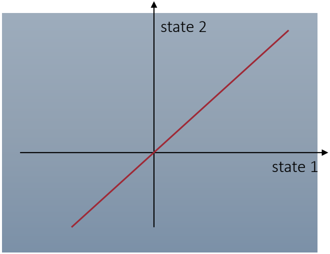


- Households can produce with capital.
  - Productivity $0 < a^h < a^e$
- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- The fraction of aggregate risk held by experts: $\chi_t^e = \frac{\sigma_t^{N^e}}{\sigma_t^{qK}}$
- Experts can only issue debt, no outside equity, $\chi_t^e = \kappa_t^e$

Skin-in-the-Game constraint

Financial Frictions and Distortions

- Belief distortions
  - Match "belief surveys"
- **Incomplete markets**
  - **"natural" leverage constraint (BruSan)**
  - Costly state verification (*BGG*)
- + Leverage constraints (no "liquidity creation")
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period's price $Rb_t \leqslant q_{t+1} k_t$ (*KM*)
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+1}) k_t$ (*BruPed*)
- Search Friction (*DGP*)



Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$

Household Sector

- Output: $y_t^e = a^h k_t^h$

$$A(\boldsymbol{\kappa}) = \kappa^e a^e + (1 - \kappa^e) a^h$$

Poll 04.01: Why is it important that households can hold capital?

a) to capture fire-sales

b) for households to speculate

c) to obtain stationary distribution

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{\mathrm{d}k_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},e}$$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{\mathrm{d}k_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},h}$$

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Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{dk_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},e}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log(c_t^e) dt\right]$

Household Sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h
  $\frac{dk_t^{h,\tilde{i}}}{k_t^{h,\tilde{i}}} = \left(\Phi(\iota_t^{h,\tilde{i}}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},h}$
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log(c_t^h) dt\right]$

Poll 04.02: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates, $\rho^e > \rho^h$

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$, $a^e \geqslant a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e
  $\frac{dk_t^{e,\tilde{i}}}{k_t^{e,\tilde{i}}} = \left(\Phi(\iota_t^{e,\tilde{i}}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},e}$
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Household Sector

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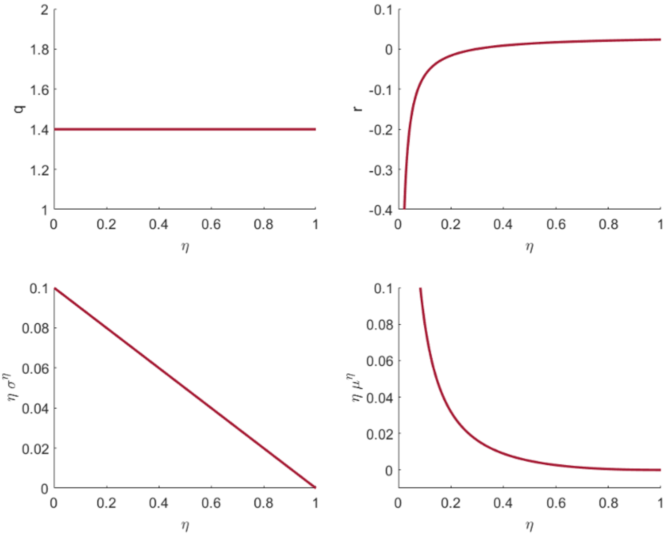
- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log(c_t^h) dt\right]$

Friction: Can only issue

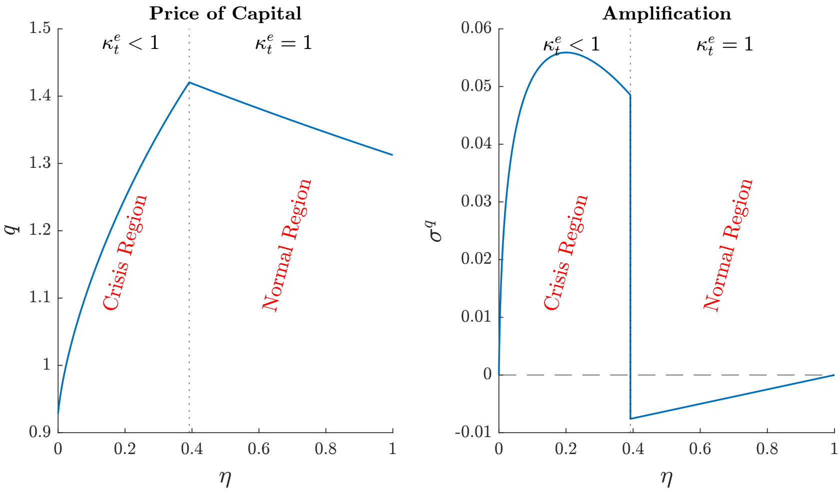
- Risk-free debt only
  Thus, $\chi_t^e = \kappa_t^e$

Recall Previous Lecture: HH can't hold capital or equity



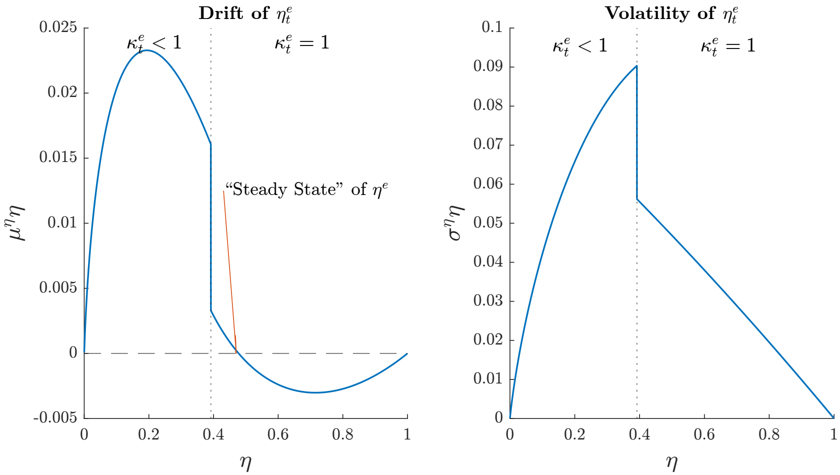
$$\rho = 0.0, a = 0.11, \sigma = 0.1, \Phi(\iota) = \frac{\log(\phi\iota+1)}{\phi},\ \phi = 10$$

Preview of New, Extended Model



$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta =0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$.

Preview of μ_η & σ_η



Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given C/N-ratio and SDF processes for each i **finance block**

   Toolbox 1: Martingale approach, ~~HJB vs. Stochastic Maximum Principle Approach~~

   Fisher separation theorem

   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + asset market clearing or

      Asset allocation κ & risk allocation χ

      Toolbox 2: "Price-taking" social planner approach

   Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω

      Special case: log-utility $c = \rho n, \varsigma = \sigma^n$
4. Numerical model solution
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Individual capital evolution:

$$\frac{\mathrm{d}\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i},i}) - \delta\right)\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},i}$$

  - where $\Delta_t^{k,\tilde{i},i}$ is the individual cumulative capital purchase process
- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i}\mathrm{d}\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$
  - Capital share: $\kappa_t^i = K_t^i/K_t$, $\quad \frac{\mathrm{d}K_t}{K_t} = (\Phi(\iota_t) - \delta)\,\mathrm{d}t + \sigma \mathrm{d}Z_t$
- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{i},i}\mathrm{d}\tilde{i}$
  - Across sectors: $N_t = \Sigma_i N_t^i$
  - Net worth share: $\eta_t^i = N_t^i/N_t$,
- Value of capital stock: $q_t K_t$, $\quad dq_t/q_t = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}Z_t$
- Postulated SDF-process: $\quad \frac{\mathrm{d}\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i}\mathrm{d}t + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i}\mathrm{d}Z_t$

0. Postulate Aggregates and Processes

- ... from price processes to return processes (using Itô)
  - Use Ito product rule to obtain capital gain rate (in absence of purchases/sales)
    - Define $\check{k}_t^{\tilde{i}}$: $\frac{d\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = \left(\Phi(\iota_t^{\tilde{i},i}) - \delta\right) dt + \sigma dZ_t + d\Delta_t^{k,\tilde{i},i}$ (without purchases/sales)

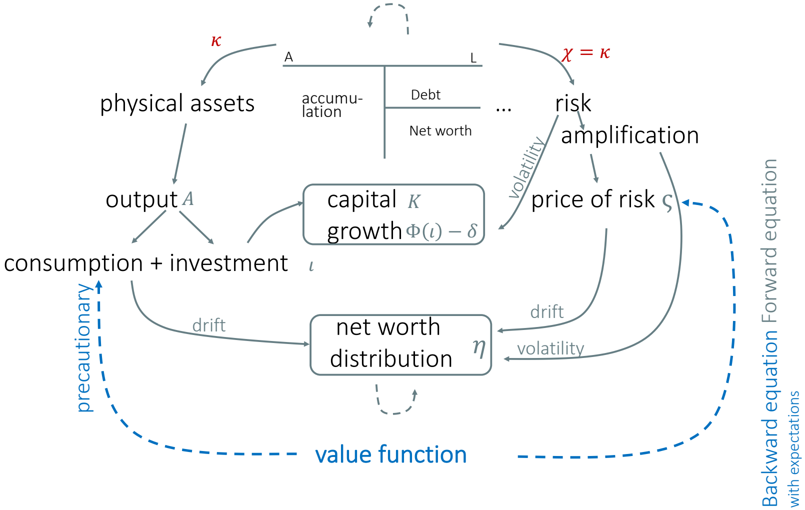
$$dr_t^k(\iota_t^{\tilde{i},i}) = \left(\overbrace{\frac{a^i - \iota_t^i}{q_t}}^{\text{Dividend yield}} + \overbrace{\Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q}^{\text{E[Capital gain rate]} = \frac{d(q_t k_t)}{q_t k_t}} \right) dt + (\sigma + \sigma_t^q) dZ_t$$

For aggregate capital return, Replace a^i with $A(\kappa)$

- Postulate SDF-process: (Example: $\xi_t^i = e^{-\rho t} V'(n_t^i)$)

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t^i dt - \varsigma_t^i dZ_t, \ \varsigma_t^i : \text{price of risk}, \ \& \ e^{-r_f} = \mathbb{E}[SDF]$$

The Big Picture



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1. For given C/N-ratio and SDF processes for each i **finance block**
   Toolbox 1: Martingale approach, ~~HJB vs. Stochastic Maximum Principle Approach~~
   Fisher separation theorem
   a. Real investment ι + Goods market clearing *(static)*
   b. Portfolio choice θ + asset market clearing or
      Asset allocation κ & risk allocation χ
      Toolbox 2: "Price-taking" social planner approach
   Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω
      Special case: log-utility $c = \rho n, \varsigma = \sigma^n$
4. Numerical model solution
5. KFE: Stationary distribution, Fan charts

1a. Individual Agent Choice of ι

- Choice of ι is static problem (and separable) for each t

$$\max_{\iota_t^i} \mathrm{d}r_t^k(\iota_t^i) = \max_{\iota_t^i} \left(\frac{a^i - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu^q + \sigma\sigma^q \right)$$

- FOC: $\frac{1}{q_t} = \Phi'(\iota_t^i)$ Tobin's q
  - All agents: $\iota_t^i = \iota_t \Rightarrow \frac{\mathrm{d}K_t}{K_t} = (\Phi(\iota_t) - \delta)\mathrm{d}t + \sigma \mathrm{d}Z_t$
  - Special functional form:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \Phi\iota = q - 1$$

- Goods market clearing condition: $(A(\kappa) - \iota_t)K_t = \sum_i C_t^i$

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1b. θ-Choice: Martingale Approach

- **Approach 1:** Portfolio Optimization
  - Step 1: Optimization e.g. via Martingale Approach – recall: $\boxed{\mu_t^A = r_t^i + \varsigma_t^i \sigma_t^A}$
    - Of experts' capital choice

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q = r_t + \varsigma_t^e(\sigma + \sigma_t^q),$$

    - Of households' capital choice:

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q \leqslant r_t + \varsigma_t^h(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1$$

    - Step 2: Capital market clearing to obtain asset/risk allocation κ_t^e, χ_t^e from portfolio weights θs

Poll 04.03: Where are the θs? (a) in ς^is?, (b) in μ^A?

- **Approach 2:** Price-taking Social Planner Approach

1b. θ-Choices: Stochastic Maximum Principle

- Experts' problem: $\max_{c_t^e, \iota_t^e, \theta_t^{e,K}} \mathbb{E}\left[\int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t\right]$ s.t.

$$\mathrm{d}n_t^e = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota^e) - r_t)\right)\right]\mathrm{d}t + n_t^e \theta_t^{e,K}(\sigma + \sigma_t^q)\mathrm{d}Z_t$$

- Households' problem: $\max_{c_t^h, \iota_t^h, \theta_t^h} \mathbb{E}\left[\int_s^\infty e^{-\rho^h t} u(c_t^h) \mathrm{d}t\right]$, s.t. $\theta_t^{h,K} \geqslant 0$,

$$\mathrm{d}n_t^h = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K} - r_t)\right)\right]\mathrm{d}t + n_t^h \theta_t^{h,K}(\sigma + \sigma_t^q)\mathrm{d}Z_t,$$

- The Hamiltonians can be constructed as

$$\mathcal{H}_t^e = e^{-\rho^e t} u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e \left(r_t + \theta_t^{e,K}(r_t^{e,K}(\iota_t^e) - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} - \varsigma_t^e \xi_t^e \overbrace{n_t^e \theta_t^{e,K}(\sigma + \sigma_t^q)}^{\sigma_t^{n^e} n_t^e}$$

$$\mathcal{H}_t^h = e^{-\rho^h t} u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h \left(r_t + \theta_t^{h,K}(r_t^{h,K}(\iota_t^h) - r_t)\right)\right] - \varsigma_t^h \xi_t^h n_t^e \theta_t^{h,K}(\sigma + \sigma_t^q) + \xi_t^h n_t^h \lambda_t^h \theta_t^{h,K}$$

- Objective functions are linear in θ (divide through $\xi_t^i n_t^i$) $\Rightarrow$ bang-bang (or indifferent)
- FOC w.r.t. c_t is separated/de-coupled from FOC w.r.t. θ_ts as well as ι_t^e
  $\Rightarrow$ Fisher Separation Theorem btw. $c_t^i, \theta_t^i, \iota_t^i$

1b. θ-Choices

- Experts: $\boldsymbol{\theta}^e = (\theta^{e,K}, \theta^{e,D})$ for capital and debt. $\theta^{e,K} \geqslant 0$. Maximize:

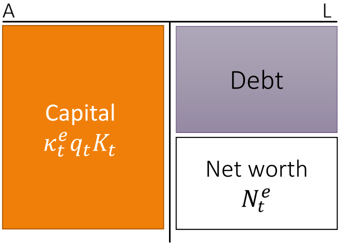
$$\theta_t^{e,K} \mathbb{E}[\mathrm{d}r_t^{e,K}]/\mathrm{d}t + \theta_t^{e,D} r_t - \varsigma_t^e \theta_t^{e,K} \sigma^{r^{e,K}}$$

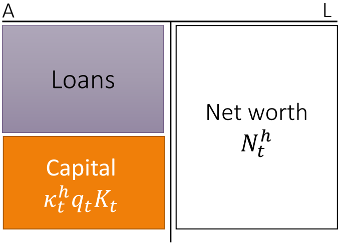
- Households: $\boldsymbol{\theta}^h = (\theta^{h,K}, \theta^{h,D})$, $\theta^{h,K} \geqslant 0$. Maximize:

$$\theta_t^{h,K} \mathbb{E}[\mathrm{d}r_t^{h,K}]/\mathrm{d}t + \theta_t^{h,D} r_t - \varsigma_t^h \theta_t^{h,K} \sigma^{r^{h,K}}$$

- Expert sector

Household sector





1b. *Toolbox 2:* Price Taking Social Planner (2 Types) $\Rightarrow$ Asset and Risk Allocation

- Individual optimization problems are equivalent to optimizing aggregate η-weighted sum of expert + HH maximization problems:

$$\eta^e\{...\} + \eta^h\{...\}$$

- η-weights are s.t. zero-sum assets drop out, positive sum assets' θ become κ, χ

$$\overbrace{\eta_t^e \theta_t^{e,K}}^{\equiv \kappa_t^e} \mathbb{E}[\mathrm{d}r_t^{e,K}]/\mathrm{d}t + \overbrace{\eta_t^h \theta_t^{h,K}}^{\equiv \kappa_t^h} \mathbb{E}[\mathrm{d}r_t^{h,K}]/\mathrm{d}t + \overbrace{(\eta_t^e \theta_t^{e,D} + \eta_t^h \theta_t^{h,D})}^{=0} r_t$$
$$- \varsigma_t^e \underbrace{\eta_t^e \theta_t^{e,K}}_{\equiv \chi_t^e} \sigma_t^{r^K} - \varsigma_t^h \underbrace{\eta_t^h \theta_t^{h,K}}_{\equiv \chi_t^h} \sigma_t^{r^K}$$

Poll 04.04: Why = 0?

a) because marginal benefits = marginal costs at optimum

b) due to martingale behavior

c) debt is in zero net supply

1b. *Toolbox 2:* Price Taking Social Planner (2 Types) $\Rightarrow$ Asset and Risk Allocation

- Planner maximizes η-weighted objectives of experts and households

$$\max_{\{\boldsymbol{\kappa},\boldsymbol{\chi}\}} \mathbb{E}\left[\mathrm{d}r^N\right]/\mathrm{d}t - \varsigma\sigma^{r^N}, \; s.t. \; \chi_t^e = \kappa_t^e, \chi_t^h = \kappa_t^h, \kappa_t^e + \kappa_t^h = 1$$

- Price-taking social planner's problem:

$$\max_{\{\kappa_t^e,\kappa_t^h=1-\kappa_t^e,\chi_t^e=\kappa_t^e,\chi_t^h=\kappa_t^h\}} \left[\frac{\kappa_t^e a^e + \kappa_t^h a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta\right] - (\varsigma_t^e \chi_t^e + \varsigma_t^h \chi_t^h)\sigma_t^{r^K}$$

- Linear objective
- First order condition

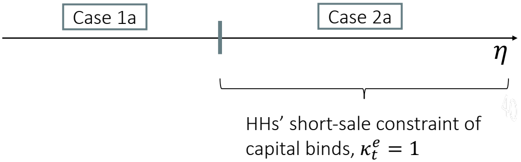
$$\frac{a^e - a^h}{q_t} \geqslant (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q), \text{ with equality if } \kappa_t^e < 1.$$

1b. *Toolbox 2:* Price Taking Social Planner (2 Types) $\Rightarrow$ Asset and Risk Allocation

Cases	1a	2a
allocation risk premia	$\kappa_t^e < 1$ $\frac{a^e - a^h}{q_t} = (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$	$\kappa_t^e = 1$ $\frac{a^e - a^h}{q_t} > (\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$

complementary slackness conditions

Occasionally binding constraint
(HH's shot-sale constraint of capital)



1b. Multiple Assets vs Shocks (θ-space vs χ, κ-space)

- One productive capital with one Brownian shock but different claims (indexed by j, with net-zero supply) on it
  - Individual's problem (complicated by J classes of tradable risky claims):

$$\max_{\boldsymbol{\theta}_t^e} \ \theta_t^{e,K} \mathbb{E}[\mathrm{d}r_t^{e,K}]/\mathrm{d}t + \theta_t^{e,D} r_t + \sum_{j=1}^{J} \theta_t^{e,i} \mathbb{E}[\mathrm{d}r_t^{e,i}]/\mathrm{d}t - \varsigma_t^e \theta_t^{e,K} \sigma^{r^K} - \varsigma_t^e \sum_{j=1}^{J} \theta_t^{e,i} \sigma^{r^J}$$

  - Planner's problem: unchanged because of the net-zero supply property.
- Multiple Brownian shocks and few claims on assets
  - Individual's problem is simple as θ's dimension is low.
  - Planner's problem: more complicated because more risks should be allocated.

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1b. *Toolbox 3:* Change of Numeraire

- x_t^A is a value of a self-financing strategy/asst in \$
- Y_t price of € in \$ (exchange rate):

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t$$

- x_t^A / Y_t value of the self-financing strategy/asst in €: $e^{-\rho t} u'(c_t) Y_t \frac{x_t^A}{Y_t^A}$ follows a martingale.

$$\mu_t^A - \mu_t^B = (-\sigma_t^\xi)(\sigma_t^A - \sigma_t^B) \Rightarrow \mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{\text{price of risk}} \underbrace{(\sigma_t^A - \sigma_t^Y - (\sigma_t^B - \sigma_t^Y))}_{\text{risk}}$$

- Price of risk in €: $\varsigma^{€} = \varsigma^{\$} - \sigma^Y$

1b. *Toolbox 3:* Change of Numeraire

- x_t^A is a value of a self-financing strategy/asst in \$
- Y_t price of € in \$ (exchange rate):

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_t^Y \mathrm{d}t + \sigma_t^Y \mathrm{d}Z_t$$

- x_t^A/Y_t value of the self-financing strategy/asst in €: $e^{-\rho t} u'(c_t) Y_t \frac{x_t^A}{Y_t^A}$ follows a martingale.

$$\mu_t^A - \mu_t^B = (-\sigma_t^\xi)(\sigma_t^A - \sigma_t^B) \Rightarrow \mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{\text{price of risk}} \underbrace{(\sigma_t^A - \sigma_t^Y - (\sigma_t^B - \sigma_t^Y))}_{\text{risk}}$$

- Price of risk in €: $\varsigma^{€} = \varsigma^{\$} - \sigma^Y$

Poll 04.05: Why does the price of risk change, though real risk remains the same?

a) because risk-free rate might not stay risk-free

b) because covariance structure changes

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   Fisher separation theorem
   - a. Real investment ι + Goods market clearing *(static)*
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   Toolbox 3: Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
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2. GE: Markov States and Equilibria

- Equilibrium is a map

  Histories of shocks $\{\boldsymbol{Z}_s, s \in [0,t]\}$ ⇢ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^e$

  ↘ net worth distribution ↗

$$\eta_t^e = \frac{N_t^e}{q_t K_t} \in (0,1)$$

  net worth share

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- **Method 1**: Using Itô's quotient rule $\eta^i_t = N^i_t/(q_t K_t)$
  - Recall:

$$\frac{dN^i_t}{N^i_t} = -\frac{C^i_t}{N^i_t}\mathrm{d}t + r_t \mathrm{d}t + \underbrace{\frac{\chi^i_t}{\eta^i_t}(\sigma + \sigma^q_t)}_{\text{risk}}\varsigma^i_t \mathrm{d}t + \frac{\chi^i_t}{\eta^i_t}(\sigma + \sigma^q_t)\mathrm{d}Z_t$$

  - $\frac{d\eta^i_t}{\eta^i_t} = \ldots$ (lots of algebra)
- **Method 2**: Change of Numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η^i_t: $\mu^{\eta^i}_t$
      Note that change of numeraire does not affect ratio η^i!

2. μ_t^η Drift of Wealth Share: Many Types

- New Numeraire
  - "Total net worth" in the economy N_t (without superscript)
  - Type i's portfolio net worth = net worth share
- Martingale Approach with new numeraire
  - Asset A = i's portfolio return in terms of total wealth

$$\left(\frac{C_t^i}{N_t^i} + \mu_t^{\eta^i}\right) \mathrm{d}t + \sigma_t^{\eta^i} \mathrm{d}Z_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^m \mathrm{d}t + \sigma_t^m \mathrm{d}Z_t$$

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B = \varsigma_t^i(\sigma_t^A - \sigma_t^B)$$

Poll 04.06: Is risk-free asset, risk free in the new numeraire?

a) Yes

b) No

2. μ_t^η Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = (\varsigma_t^i - \sigma_t^N)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

- Add up across types (weighted), (capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'} \eta_t^{i'} \mu_t^{i'}}_{=0} + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta_t^{i'} (\varsigma_t^{i'} - \sigma_t^N)\left(\sigma_t^{\eta^{i'}} - \sigma_t^m\right)$$

- Benchmark asset is an asset everyone can trade $\sigma_t^m = -\sigma_t^N$

Poll 04.07: why = 0?

a) Because we have stationary distribution

b) Because ηs sum up to 1

c) Because ηs follow martingale

2. μ_t^η Drift of Wealth Share: 2 Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = (\varsigma_t^i - \sigma_t^N)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

- Add up across types (weighted), (capital letters without superscripts are aggregates)

$$(\mu_t^e \mu_t^{\eta^e} + \mu_t^h \mu_t^{\eta^h}) + \frac{C_t}{N_t} - r_t^m = \eta_t^e(\varsigma_t^e - \sigma_t^N)(\sigma_t^{\eta^e} - \sigma_t^m) + \eta_t^h(\varsigma_t^h - \sigma_t^N)(\sigma_t^{\eta^h} - \sigma_t^m)$$

- Subtract from each other yield net worth share dynamics

$$\mu_t^{\eta^e} = (1-\eta_t^e)(\varsigma_t^e - \sigma_t^N)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) - (1-\eta_t^e)(\varsigma_t^h - \sigma_t^N)\left(\sigma_t^{\eta^h} - \sigma_t^m\right) - \left(\frac{C_t^e}{N_t^e} - \frac{C_t}{q_t K_t}\right)$$

2. σ^η Volatility of Wealth Share

- Recall Itô quotient rule (only volatility term)
- Since $\eta^e_t = N^e_t / N_t$,

$$\sigma^{\eta^e}_t = \sigma^{N^e}_t - \sigma^N_t = \sigma^{N_i}_t - \sum_{i'} \eta^{i'}_t \sigma^{N^{i'}}_t = (1 - \eta^i_t)\sigma^{N^i}_t - \sum_{i^- \neq i} \eta^{i^-}_t \sigma^{N^{i^-}}_t$$

- Note for (Change in notation in 2 types setting, networth is $n^i = N^i$)

$$\sigma^{\eta^e}_t = (1 - \eta^e_t)(\sigma^{n^e}_t - \sigma^{n^h}_t), \text{ where } \begin{cases} \sigma^{n^e}_t = \frac{\chi^e_t}{\eta^e_t}(\sigma + \sigma^q_t) \\ \sigma^{n^h}_t = \frac{\chi^h_t}{\eta^h_t}(\sigma + \sigma^q_t) = \frac{1-\chi^e_t}{1-\eta^e_t}(\sigma + \sigma^q_t) \end{cases}$$

$$\Rightarrow \qquad \sigma^{\eta^e}_t = \frac{\chi^e_t - \eta^e_t}{\eta^e_t}(\sigma + \sigma^q_t)$$

- Note also: $\eta^e_t \sigma^{\eta^e}_t + \eta^h_t \sigma^{\eta^h}_t = 0 \Rightarrow \sigma^{\eta^h}_t = -\frac{\eta^e_t}{\eta^h_t}\sigma^{\eta^e}_t = -\frac{\eta^e_t}{1-\eta^e_t}\sigma^{\eta^e}_t$

2. Amplification Formula: Loss Spiral

$$\left.\begin{aligned} \text{Recall} \quad \sigma_t^{\eta^e} &= \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q) \\ \text{By Itô's Lemma} \quad \sigma_t^q &= \frac{q'(\eta_t^e)}{q(\eta_t^e)}\eta_t^e \sigma_t^{\eta^e} \end{aligned}\right\} \Rightarrow \sigma_t^q = \underbrace{\frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}}_{\text{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

- Total Volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

- Loss spiral
  - Market illiquidity
    (price impact elasticity)

2. Amplification Formula: Loss Spiral

$$\left.\begin{array}{lr} \text{Recall} & \sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q) \\ \text{By Itô's Lemma} & \sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}\eta_t^e \sigma_t^{\eta^e} \end{array}\right\} \Rightarrow \sigma_t^q = \underbrace{\frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}}_{\text{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

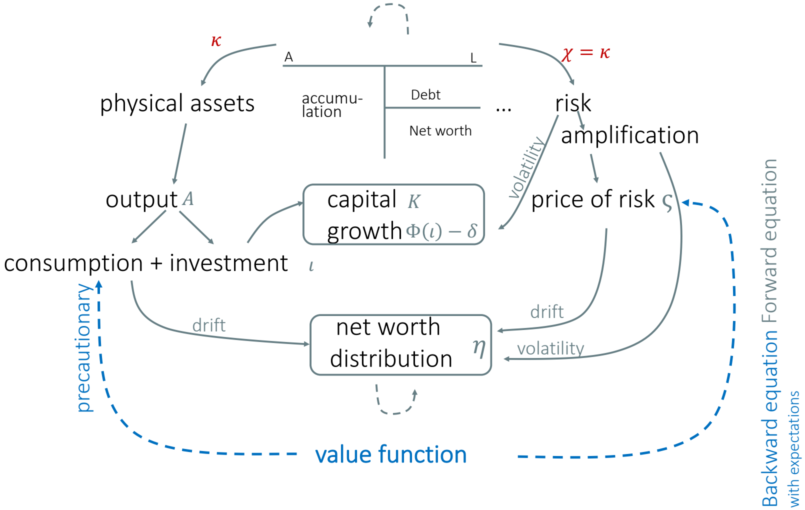
- Total Volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

Poll 04.08: Where is the spiral?
a) Sum of infinite geometric series (denominator)
b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity
    (price impact elasticity)

The Big Pricture



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Toolbox 1: Martingale approach, HJB vs. Stochastic Maximum Principle Approach

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- a Value fcn. as fcn. of individual investment opportunities ω
  Special case: log-utility

4 Numerical model solution

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4a. Obtain κ for Goods Market Clearing

- Determination of κ_t (part of ςs)
  - Based on difference in risk premia: $(\varsigma_t^e - \varsigma_t^h)(\sigma + \sigma_t^q)$
  - For log utility: $(\sigma_t^{n^e} - \sigma_t^{n^h})(\sigma + \sigma_t^q) = \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)$

    Since: $\sigma_t^{n^e} - \sigma_t^{n^h} = \sigma_t^{\eta^e} - \sigma_t^{\eta^h}$ and $\sigma_t^{\eta^e} = \frac{\kappa_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q), \sigma_t^{\eta^h} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$
- Hence,

$$\boxed{(a^e - a^h)/q_t \geqslant \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2, \text{with equality if } \kappa_t^e < 1}$$

4a. Investments and Capital Prices q

- Replacing ι_t.
  - Recall from optimal re-investment $\Phi'(\iota) = 1/q_t$:

$$\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1) \Rightarrow \boxed{\phi\iota = q - 1}$$

- Recall from "amplification slide"

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q(\eta_t^e)/\eta_t^e}\frac{\kappa_t^e - \eta_t^e}{\eta_t^e}} \Rightarrow \boxed{\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q)}$$

4a. Market Clearing

- Output good market:

$$(\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t)K_t = C_t$$

$$\Rightarrow \boxed{\kappa_t^e a^e + (1-\kappa_t^e)a^h - \iota_t = [\eta_t \rho^e + (1-\eta_t)\rho^h]q_t}$$

- Capital market is taken care of by price-taking social planner approach.
- Risk-free debt market also taken care of by price taking social planner approach and by Walras Law

4b. Algorithm – Static Step

- We have four static conditions

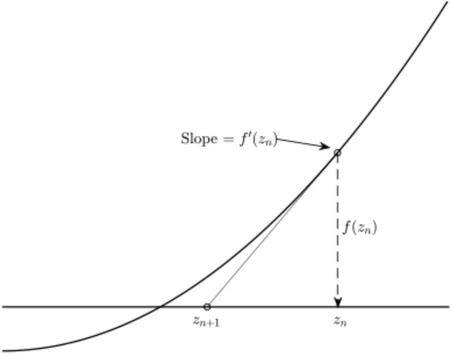
1. Tobin's q: $\phi\iota_t = q_t - 1$
2. Planner condition for κ_t^e: $\frac{a^e - a^h}{q_t} \geqslant \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e}(\sigma + \sigma_t^q)^2$
3. Goods market clearing: $\kappa_t^e a_t^e + (1-\kappa_t^e)a^h - \iota(q_t) = [\eta_t^e \rho^e + (1-\eta_t^e)\rho^h]q_t$
4. Amplification: $\sigma^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)}(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q)$

$\Rightarrow$ Get $q(\eta^e), \kappa^e(\eta^e), \sigma^q(\eta^e)$.

- Start at $q(0)$, solve to the right,
use different procedure for two η^e regions depending on κ^e:

1. While $\kappa^e < 1$, solve ODE for $q(\eta^e)$
   - For given $q(\eta)$, plug optimal investment (1) into (3)
   - Solve ODE using three equilibrium condition (2),(3) and (4) via Newton's method
2. When $\kappa^e = 1$, (2) is no longer informative, solve (1) (3) for $q(\eta^e)$

4b. Aside: Newton's Method

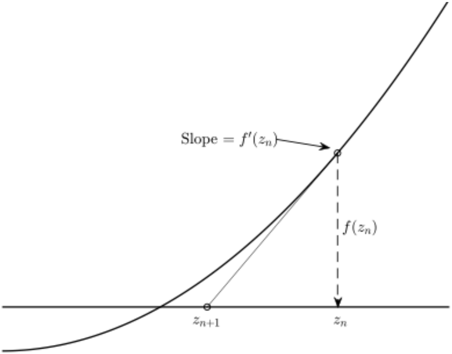


- Find the root of equation system $F(\boldsymbol{z}_n) = 0$ via iterative method:

$$\boldsymbol{z}_{n+1} = \boldsymbol{z}_n - \mathbf{J}_n^{-1}(\boldsymbol{z}_n)$$

  where $\mathbf{J}_n$ is the Jacobian matrix, i.e., $\mathbf{J}_{i,j} = \partial f_i(\boldsymbol{z})/\partial z_j$
- Newton's method does not guarantee global convergence.
- commonly take several-step iteration.

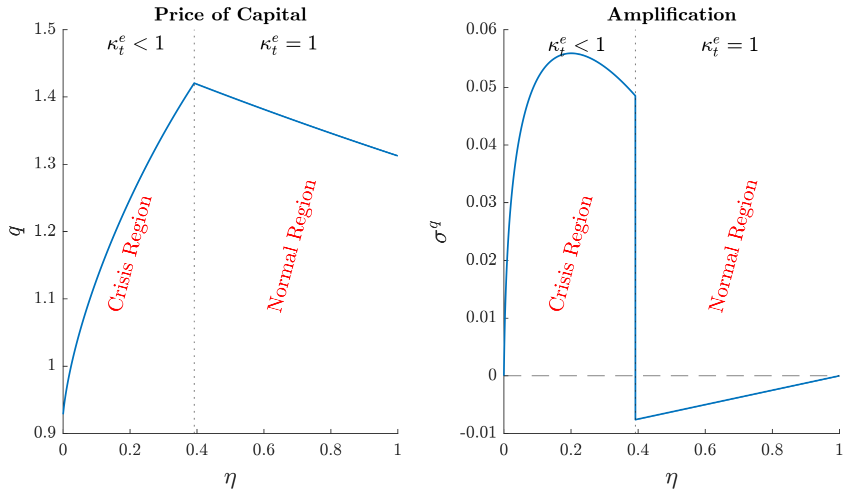
4b. Aside: Newton's Method



$$\boldsymbol{z}_n = \begin{bmatrix} q_t \\ \kappa_t^e \\ \sigma + \sigma_t^q \end{bmatrix}, F(\boldsymbol{z}_n) = \begin{bmatrix} \kappa_t^e a_t^e + (1 - \kappa_t^e) a^h - \iota(q_t) - q_t[\eta_t^e \rho^e + (1 - \eta_t^e)\rho^h] \\ q'(\eta_t^e)(\kappa_t^e - \eta_t^e)(\sigma + \sigma_t^q) - \sigma^q q(\eta_t^e) \\ (a^e - a^h) - q_t \frac{\kappa_t^e - \eta_t^e}{(1-\eta_t^e)\eta_t^e} (\sigma + \sigma_t^q)^2 \end{bmatrix}, \begin{bmatrix} \text{goods mkt} \\ \text{amplif} \\ \text{Planner.} \end{bmatrix}$$

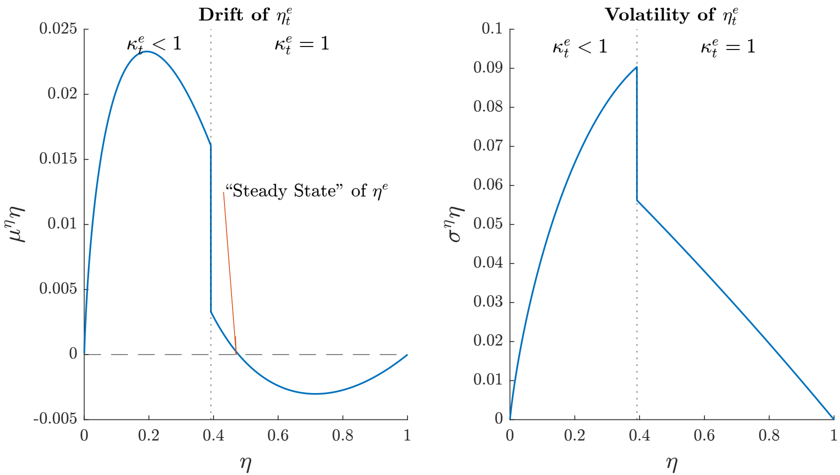
Replace red terms from Tobin's Q ι and planner κ^e condition.

Solution for $q(\eta)$ and Volatility of q



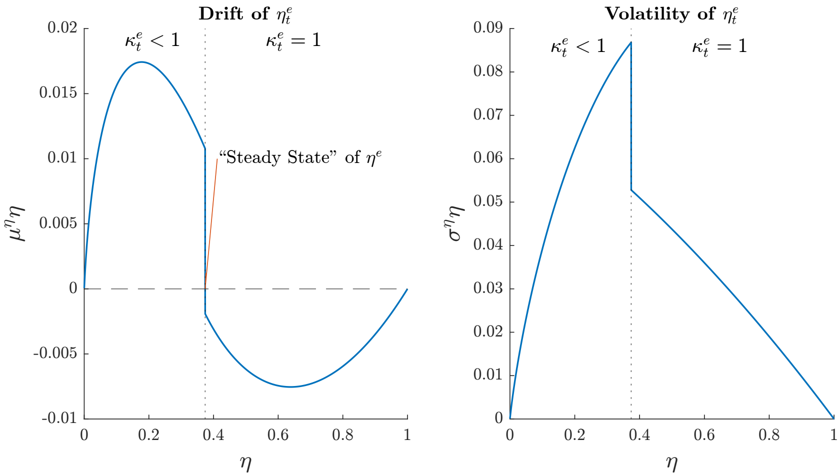
$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta =0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$.

Solutions: Drift and Volatility of η^e



$\rho^e = 0.06$, $\rho^h = 0.04$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$.

Solutions: η^e-Drift/Volatility with SS at Region Boundary



$\rho^e = 0.06$, $\rho^h = 0.02$, $\delta = 0.05$, $a^e = 0.11$, $a^h = 0.03$, $\sigma = 0.10$, $\phi = 10$.

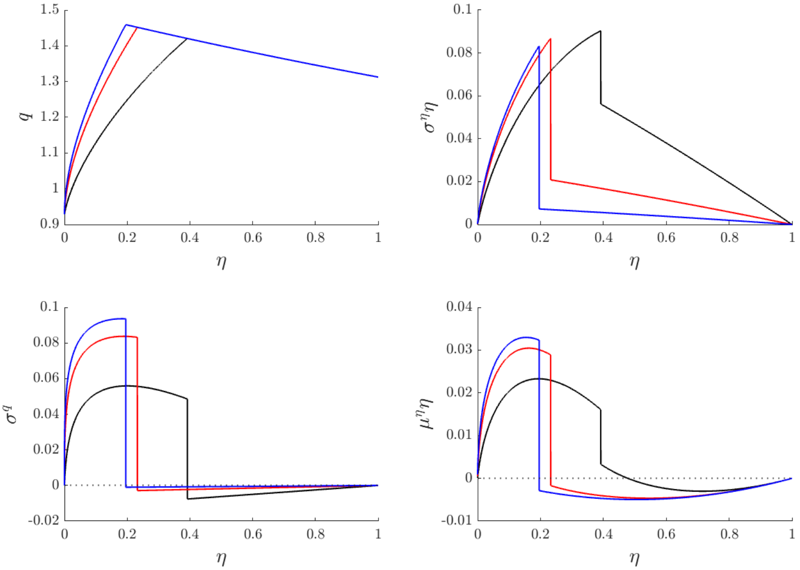
Poll 04.08: Is it possible for "steady state" lie in $\kappa_t^e < 1$?

a) yes

b) no

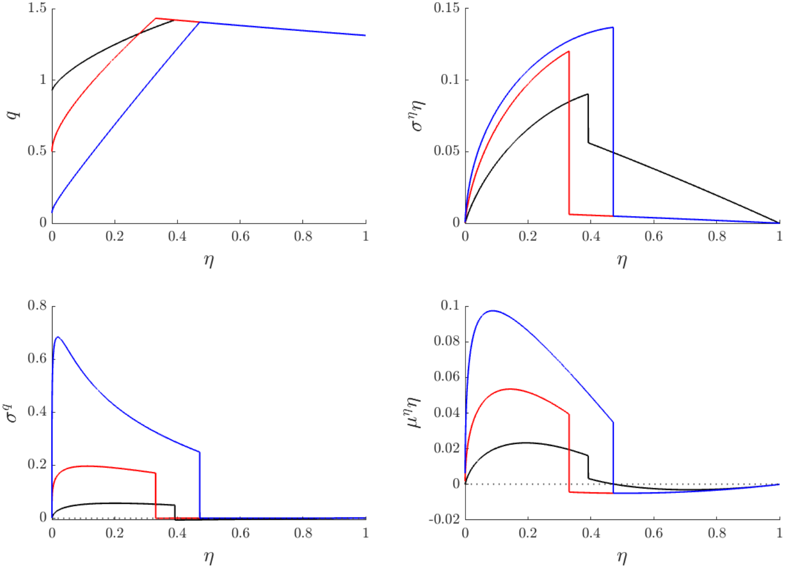
Volatility Paradox

$\sigma = 0.1$, $\sigma = 0.03$, $\sigma = 0.01$



Market Liquidity

$a_h = 0.03$, $a_h = -0.03$, $a_h = -0.09$



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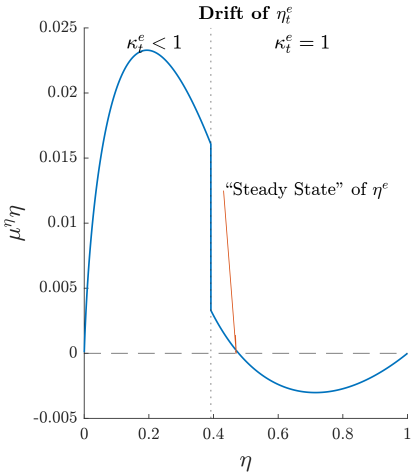
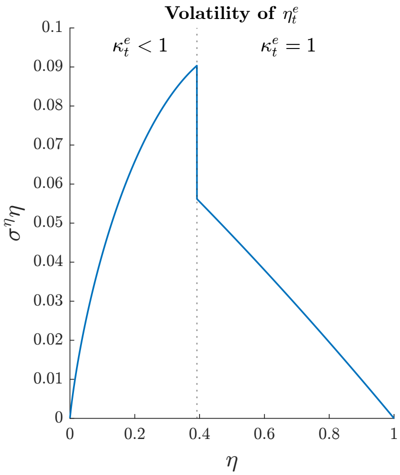
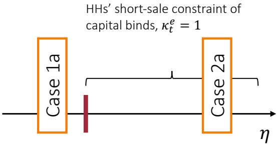
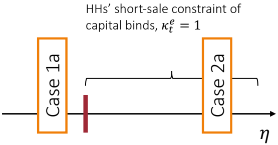
From μ_η, σ_η to Stationary Distribution









5. Kolmogorov Forward Equation

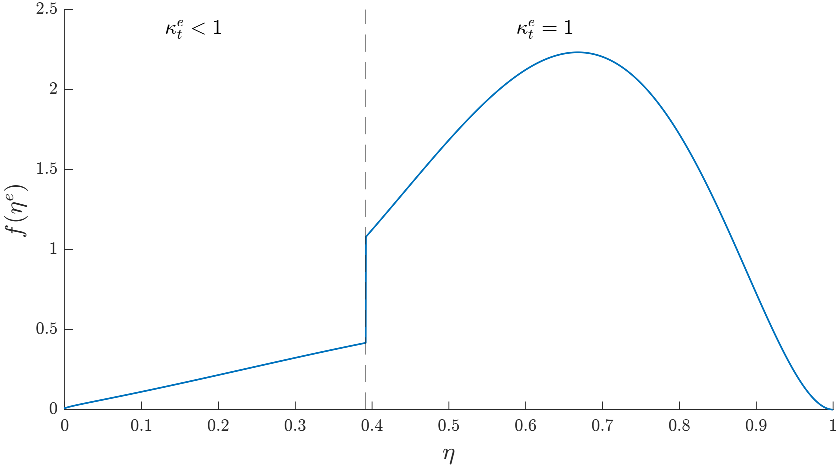
- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density distribution follows:

$$\frac{\partial f(\eta, t)}{\partial t} = -\frac{\partial[f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"
- Corollary: If stationary distribution $f(\eta)$ exists, it satisfies ODE:

$$0 = -\frac{\partial[f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}$$

5. Stationary Distribution

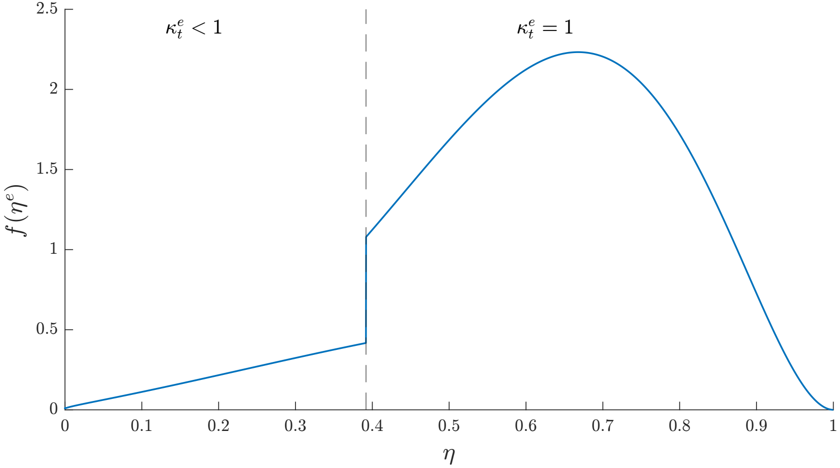


Poll 04.09: Is the constraint always (not just occasionally) binding

a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution



Poll 04.10: What happens for $\rho^e = \rho^h$

a) experts take over the economy $\eta \to 1$

b) there is a steady state