Economies with Financial Frictions – Review and Exercises

The 2022 Princeton Initiative

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Review: Asset pricing

$$\text{SDF } \xi_t = e^{-\rho t} u'(c_t)$$

$\forall$self-financing trading strategy with value A_t, $A_t\xi_t$ is a martingale

$$\frac{d\xi_t}{\xi_t} = \mu_t^\xi \, dt + \sigma_t^\xi \, dZ_t \qquad \frac{dA_t}{A_t} = \mu_t^A \, dt + \sigma_t^A \, dZ_t \quad \Rightarrow \quad \mu_t^A + \mu_t^\xi + \sigma_t^A \sigma_t^\xi = 0$$

Two assets: $\mu_t^A - \mu_t^B + \underbrace{(\sigma_t^A - \sigma_t^B)}_{\text{independent of numeraire}} \sigma_t^\xi = 0$

If asset B is risk-free: $\mu_t^A - r_t^F + \sigma_t^A \sigma_t^\xi = 0$

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Asset pricing: implications

$$\text{SDF } \xi_t = e^{-\rho t} u'(c_t) \qquad \mu_t^A + \mu_t^\xi + \sigma_t^A \sigma_t^\xi = 0$$

- Valuation (e.g. the price of capital q_t)
- Asset allocation: $\mu_t^{A,i} + \mu_t^{\xi,i} + \sigma_t^{A,i} \sigma_t^{\xi,i} \leq 0,$

= if agent i holds > 0 of asset A

- Evolution of wealth and wealth distribution

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Example: valuation

- Representative agent, CRRA utility, so $\xi_t = e^{-\rho t} c_t^{-\gamma}$
- Single asset with divided D_0, exp. growth g, volatility σ
- Find asset price and risk-free rate

$$\frac{dc_t}{c_t} = \frac{dD_t}{D_t} = g \, dt + \sigma \, dZ_t \quad \Rightarrow \quad \frac{dc_t^{-\gamma}}{c_t^{-\gamma}} = -\gamma g \, dt + \frac{\gamma(\gamma+1)}{2}\sigma^2 \, dt - \gamma\sigma \, dZ_t$$

$$dr_t^A = \frac{D_0}{q_0} \, dt + g \, dt + \sigma \, dZ_t \quad \Rightarrow \quad \underbrace{\frac{D_0}{q_0} + g}_{\mu^A} \underbrace{-\rho - \gamma g + \frac{\gamma(\gamma+1)}{2}\sigma^2}_{+\mu^\xi = -r^F} - \gamma\sigma\sigma = 0$$

Note: if $\gamma = 1$, $D_0/q_0 = \rho$

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Model from online lectures

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\,dt + \sigma\,dZ_t$$

experts	households
Output per unit of capital $a - \iota_t$	$\underline{a} - \iota_t$, $\underline{a} \le a$
CRRA utility, discount rate ρ	$\underline{\rho} < \rho$
issue debt & equity, but keep risk $\chi_t \ge \underline{\chi}$	

Liquid markets for capital k_t with endogenous price per unit q_t

$dq_t/q_t = \mu_t^q\,dt + \sigma_t^q\,dZ_t$

Question about allocation: Assume log utility and $\underline{\chi} = 1$, i.e. experts can issue only debt.
What equation describes ψ, the fraction of capital held by experts?

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Example: allocation, logarithmic utility

Consumption = ρ wealth

$$\psi a + (1-\psi)\underline{a} - \iota(q) = (\rho\eta + \underline{\rho}(1-\eta))q$$

Price of risk (experts): $\sigma_t^\xi = -\sigma_t^C = -\sigma_t^N = -\frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)$

Asset pricing (experts): $\frac{a - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma^q)\left(-\frac{\psi_t}{\eta_t}(\sigma + \sigma^q)\right) = 0$

households: $\frac{\underline{a} - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma^q)\left(-\frac{1-\psi_t}{1-\eta_t}(\sigma + \sigma^q)\right) = 0$

subtracting

$$\frac{a - \underline{a}}{q_t} = \left(\frac{\psi_t}{\eta_t} - \frac{1-\psi_t}{1-\eta_t}\right)(\sigma + \sigma_t^q)^2$$

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Example: evolution of wealth and η: logarithmic utility

Consumption = ρ wealth

Price of risk: $\sigma_t^\xi = -\sigma_t^C = -\sigma_t^N = -\frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)$ $\qquad \eta_t = \frac{N_t}{N_t + \underline{N}_t}$

Hence, $$\frac{dN_t}{N_t} = r_t^F\,dt - \underbrace{\frac{C_t}{N_t}}_{\rho}\,dt + \underbrace{\frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)}_{\text{risk}}\left(dZ_t + \frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)\,dt\right)$$

$$\frac{d\underline{N}_t}{\underline{N}_t} = r_t^F\,dt - \underbrace{\frac{\underline{C}_t}{\underline{N}_t}}_{\underline{\rho}}\,dt + \underbrace{\frac{1-\psi_t}{1-\eta_t}(\sigma + \sigma_t^q)}_{\text{risk}}\left(dZ_t + \frac{1-\psi_t}{1-\eta_t}(\sigma + \sigma_t^q)\,dt\right)$$

Ito implies that $\frac{d\eta_t}{\eta_t} = \ldots$

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Asset pricing: implications

- Valuation (e.g. the price of capital q_t)
- Asset allocation: $\mu_t^{A,i} + \mu_t^{\xi,i} + \sigma_t^{A,i}\sigma_t^{\xi,i} \le 0,$

with equality if agent i holds asset A in positive amount

- Evolution of wealth and wealth distribution

Also, to solve the model we need understanding of

- endogenous risk
- value functions (for numerical solutions)

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Model from online lectures

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)\,dt + \sigma\,dZ_t$$

experts	households
Output per unit of capital $a - \iota_t$	$\underline{a} - \iota_t$, $\underline{a} \leq a$
CRRA utility, discount rate ρ	$\underline{\rho} < \rho$
issue debt & equity, but keep risk $\chi_t \geq \underline{\chi}$	

Liquid markets for capital k_t with endogenous price per unit q_t

$dq_t/q_t = \mu_t^q\,dt + \sigma_t^q\,dZ_t$

Assume log utility and $\underline{\chi} = 1$, i.e. experts can issue only debt. **Let's solve it!**

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Logarithmic utility: model solution

Consumption = ρ wealth

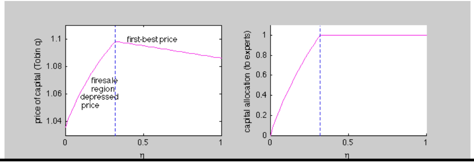
(1) $$\psi a + (1-\psi)\underline{a} - \iota(q) = (\rho\eta + \underline{\rho}(1-\eta))q$$

Allocation: (2) $$\frac{a - \underline{a}}{q_t} = \left(\frac{\psi_t}{\eta_t} - \frac{1-\psi_t}{1-\eta_t}\right)(\sigma + \sigma_t^q)^2$$

Endogenous risk (3) $$\sigma_t^\eta = \frac{\psi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q), \quad q_t\sigma_t^q = q'(\eta)\underbrace{(\psi_t - \eta_t)(\sigma + \sigma_t^q)}_{\sigma_t^\eta \eta_t}$$

$\eta = 0$, $\psi = 0$, $q(0) = \ldots$

η, $q(\eta)$ → (1): ψ → (2): σ^q → (3) $q'(\eta)$



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Now, today's exercise

- Households + experts, $a = \underline{a}$
- Capital

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\,dt + \sigma\,dZ_t + \varphi\,\tilde{\sigma}\,d\tilde{Z}_t$$

- Log utility (consumption = ρ · wealth)
- Households $$E\left[\int_0^\infty e^{-\rho t}\log \underline{c}_t\,dt\right]$$
- Experts

$$E\left[\int_0^\infty \zeta_t \log c_t\,dt\right] \qquad \frac{d\zeta_t}{\zeta_t} = -\rho\,dt + \sigma^\zeta\,dZ_t$$

- Risk dZ is tradable (price of risk ς_t)

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Big picture

- Valuation (e.g. the price of capital q_t) $$\rho q_t = a - \iota(q_t)$$
- Allocation
  - capital

$$\frac{a - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \varphi\tilde{\sigma} \qquad \underbrace{\frac{\psi_t}{\eta_t}\varphi\tilde{\sigma}}_{\text{price of idiosyncratic risk, E}} \qquad = 0$$

$$\frac{a - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \tilde{\sigma} \qquad \underbrace{\frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}}_{\text{price of idiosyncratic risk, H}} \qquad = 0$$

  - aggregate risk

$$\sigma_t^\xi = -\text{volatility of marginal utility} = \varsigma_t$$

- Evolution of the wealth distribution

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Question 1

- What is the price of aggregate risk for households and experts, if households' wealth is $\underline{N}_t$ and experts', N_t?

(a) $-\sigma_t^{\underline{N}}$ and $-\sigma_t^N$
(b) $-\sigma_t^{\underline{N}}$ and $-\sigma_t^N + \sigma^\zeta$
(c) $\sigma_t^{\underline{N}}$ and $\sigma_t^N - \sigma^\zeta$
(d) $\sigma_t^{\underline{N}}$ and $\sigma_t^N + \sigma^\zeta$
(e) none of the above

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Question 1

- For experts, MU = ζ_t/c_t, volatility $\sigma^\zeta - \sigma_t^c = \sigma^\zeta - \sigma_t^N$.
- Price of risk = - volatility of MU, $\sigma_t^N - \sigma^\zeta$ for E and $\sigma_t^{\underline{N}}$ for H.
- Both must equal ς_t since aggregate risk is tradable

$$\sigma_t^N - \sigma^\zeta = \sigma_t^{\underline{N}} = \varsigma_t$$
$$\varsigma_t = \eta(\sigma_t^N - \sigma^\zeta) + (1 - \eta)\sigma_t^{\underline{N}} = \sigma - \eta\, \sigma^\zeta$$

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Question 2

(2) Given the fraction ψ_t of capital held by entrepreneurs, what are the prices of idiosyncratic risk $\tilde{\varsigma}_t$ and $\underline{\tilde{\varsigma}}_t$ for entrepreneurs and households?

(a) $\tilde{\varsigma}_t = \frac{\psi_t}{\eta_t}\tilde{\sigma} \quad \underline{\tilde{\varsigma}}_t = \frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}$
(b) $\tilde{\varsigma}_t = \frac{\psi_t}{\eta_t}\varphi\tilde{\sigma} \quad \underline{\tilde{\varsigma}}_t = \frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}$
(c) $\tilde{\varsigma}_t = \frac{\psi_t}{\eta_t}\varphi\tilde{\sigma} \quad \underline{\tilde{\varsigma}}_t = \frac{1-\psi_t}{1-\eta_t}\varphi\tilde{\sigma}$
(d) $\tilde{\varsigma}_t = \underline{\tilde{\varsigma}}_t = \varphi\tilde{\sigma}$

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Question 3

(3) As a function of the entrepreneurs' wealth share η_t, determine the fraction of capital ψ_t that entrepreneurs hold in equilibrium.

- Hint: $$\frac{a - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \varphi\tilde{\sigma} \qquad \underbrace{\frac{\psi_t}{\eta_t}\varphi\tilde{\sigma}}_{\text{price of idiosyncratic risk, E}} \qquad = 0$$
$$\frac{a - \iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \tilde{\sigma} \qquad \underbrace{\frac{1 - \psi_t}{1 - \eta_t}\tilde{\sigma}}_{\text{price of idiosyncratic risk, H}} \qquad = 0$$

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Question 3

(3) As a function of the entrepreneurs' wealth share η_t, determine the fraction of capital ψ_t that entrepreneurs hold in equilibrium.

- Hint:

$$\frac{a-\iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \varphi\tilde{\sigma} \underbrace{\frac{\psi_t}{\eta_t}\varphi\tilde{\sigma}}_{\text{price of idiosyncratic risk, E}} = 0$$

$$\frac{a-\iota_t}{q_t} + \text{cap. gains rate} - r_t^F + (\sigma + \sigma_t^q)\varsigma_t + \tilde{\sigma} \underbrace{\frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}}_{\text{price of idiosyncratic risk, H}} = 0$$

$$\frac{\psi_t}{\eta_t}\varphi^2\tilde{\sigma}^2 = \frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}^2 \quad \Rightarrow \quad \psi_t = \frac{\eta_t}{\eta_t + (1-\eta_t)\varphi^2}$$

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Question 4

(4) If the risk-free rate is r_t^F, which of the following expressions gives the law of motion of experts' wealth N_t? Recall that

$$\sigma_t^{\underline{N}} = \varsigma_t, \quad \sigma_t^{\overline{N}} = \varsigma_t + \sigma^\zeta.$$

$$(a) \quad \frac{dN_t}{N_t} = \left(r_t^F + (\varsigma_t + \sigma^\zeta)\varsigma_t + \frac{\psi_t}{\eta_t}\varphi\tilde{\sigma}^2\right) dt + (\varsigma_t + \sigma^\zeta)\, dZ_t$$

$$(b) \quad \frac{dN_t}{N_t} = \left(r_t^F - \rho + (\varsigma_t + \sigma^\zeta)\varsigma_t + \frac{\psi_t^2}{\eta_t^2}\varphi^2\tilde{\sigma}^2\right) dt + (\varsigma_t + \sigma^\zeta)\, dZ_t$$

$$(c) \quad \frac{dN_t}{N_t} = \left(r_t^F - \rho + (\varsigma_t + \sigma^\zeta)^2 + \frac{\psi_t^2}{\eta_t^2}\varphi^2\tilde{\sigma}^2\right) dt + (\varsigma_t + \sigma^\zeta)\, dZ_t$$

$$(d) \quad \frac{dN_t}{N_t} = \left(r_t^F - \rho + \frac{\psi_t}{\eta_t}\sigma\varsigma_t + \frac{\psi_t}{\eta_t}\varphi\tilde{\sigma}^2\right) dt + \frac{\psi_t}{\eta_t}\sigma\, dZ_t$$

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Question 5

(5) Which expression gives the law of motion of total wealth q_tK_t?
Recall that

$$\sigma_t^{\underline{N}} = \varsigma_t, \quad \sigma_t^{\overline{N}} = \varsigma_t + \sigma^\zeta.$$

$$(a) \quad \frac{d(q_tK_t)}{q_tK_t} = \frac{a-\iota_t}{q_t}\, dt + (\mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t - \rho\, dt$$

$$(b) \quad \frac{d(q_tK_t)}{q_tK_t} = (\mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t$$

$$(c) \quad \frac{d(q_tK_t)}{q_tK_t} = \left(r_t^F - \rho + (\varsigma_t + \eta\sigma^\zeta)\varsigma_t + \frac{\psi_t^2}{\eta_t}\varphi^2\tilde{\sigma}^2 + \frac{(1-\psi_t)^2}{1-\eta_t}\tilde{\sigma}^2\right) dt + (\varsigma_t + \eta\sigma^\zeta) dZ_t$$

(d) all of the above

(e) (a) and (b) but not (c)

- Which equation should we use to derive the law of motion of η?

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Question 6

- Derive the law of motion of η. Recall that

$$\frac{dN_t}{N_t} = \left(r_t^F - \rho + (\varsigma_t + \sigma^\zeta)\varsigma_t + \frac{\psi_t^2}{\eta_t^2}\varphi^2\tilde{\sigma}^2\right) dt + (\varsigma_t + \sigma^\zeta)\, dZ_t$$

$$\frac{d(q_tK_t)}{q_tK_t} = (r_t^F - \rho + (\varsigma_t + \eta\sigma^\zeta)\varsigma_t + \underbrace{\frac{\psi_t^2}{\eta_t}\varphi^2\tilde{\sigma}^2 + \frac{(1-\psi_t)^2}{1-\eta_t}\tilde{\sigma}^2}_{\frac{\psi_t}{\eta_t}\varphi^2\tilde{\sigma}^2 \quad \text{since } \frac{\psi_t}{\eta_t}\varphi^2\tilde{\sigma}^2 = \frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}^2})\, dt + (\varsigma_t + \eta\sigma^\zeta)\, dZ_t$$

$$\psi_t = \frac{\eta_t}{\eta_t + (1-\eta_t)\varphi^2}$$

and

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y - \sigma_t^Y(\sigma_t^X - \sigma_t^Y))\, dt + (\sigma_t^X - \sigma_t^Y)\, dZ_t$$

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Question 6

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y - \sigma_t^Y(\sigma_t^X - \sigma_t^Y))\,dt + (\sigma_t^X - \sigma_t^Y)\,dZ_t$$

- Derive the law of motion of η. You can use

$$\tfrac{dN_t}{N_t} = \left(r_t^F - \rho + (\varsigma_t + \sigma^\zeta)\varsigma_t + \tfrac{\psi_t^2}{\eta_t^2}\varphi^2\tilde{\sigma}^2\right)\,dt + (\varsigma_t + \sigma^\zeta)\,dZ_t \qquad \psi_t = \frac{\eta_t}{\eta_t + (1-\eta_t)\varphi^2}$$

$$\frac{d(q_t K_t)}{q_t K_t} = (r_t^F - \rho + (\varsigma_t + \eta\sigma^\zeta)\varsigma_t + \underbrace{\frac{\psi_t^2}{\eta_t}\varphi^2\tilde{\sigma}^2 + \frac{(1-\psi_t)^2}{1-\eta_t}\tilde{\sigma}^2}_{\frac{\psi_t}{\eta_t}\varphi^2\tilde{\sigma}^2 \quad \text{since } \frac{\psi_t}{\eta_t}\varphi^2\tilde{\sigma}^2 = \frac{1-\psi_t}{1-\eta_t}\tilde{\sigma}^2})\,dt + (\varsigma_t + \eta\sigma^\zeta)\,dZ_t$$

$$\boxed{\frac{d\eta_t}{\eta_t} = \left((1-\eta)\frac{\varphi^2(1-\varphi^2)\tilde{\sigma}^2}{(\eta + (1-\eta)\varphi^2)^2} + \eta(\eta-1)(\sigma^\zeta)^2\right) + (1-\eta)\sigma^\zeta\,dZ_t}$$

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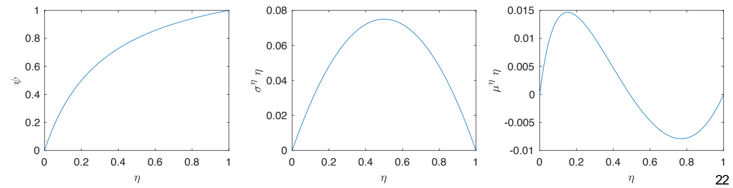
Summary

- Valuation (e.g. the price of capital q_t) $\qquad \rho q_t = a - \iota(q_t)$
- Capital allocation $\qquad \psi_t = \frac{\eta_t}{\eta_t + (1-\eta_t)\varphi^2}$
- Wealth distribution $\qquad \frac{d\eta_t}{\eta_t} = \left((1-\eta)\frac{\varphi^2(1-\varphi^2)\tilde{\sigma}^2}{(\eta + (1-\eta)\varphi^2)^2} + \eta(\eta-1)(\sigma^\zeta)^2\right) + (1-\eta)\sigma^\zeta\,dZ_t$



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Bonus question

- Suppose we have a Markov diffusion process

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$$

then density over X follows

$$g_t(x,t) = -\frac{\partial}{\partial x}(\mu(x)g(x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma(x)^2 g(x,t))$$

Stationary density satisfies

$$g(x) = \text{const}\frac{1}{\sigma(x)^2}\exp\left(\int \frac{2\mu(y)}{\sigma(y)^2}\,dy\right)$$

For today's exercise, under what conditions does the stationary density exist?

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Economies with Financial Frictions – Exercise Continued

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