Economies with Financial Frictions – Review and Exercises

The 2022 Princeton Initiative

Asset pricing: implications

\[ SDF \xi_t = e^{-\mu' t} \]

- **Valuation (e.g. the price of capital q)**
- **Asset allocation:** \( \mu_t^A + \mu_t^F + \sigma_t^A \sigma_t^F \leq 0 \),
- if agent i holds \( > 0 \) of asset A
- **Evolution of wealth and wealth distribution**

Review: Asset pricing

\[ SDF \xi_t = e^{-\mu' t} \]

∀ self-financing trading strategy with value \( A_t, A_t \xi_t \) is a martingale

\[ \frac{d\xi_t}{\xi_t} = \mu_t^A dt + \sigma_t^A dZ_t \]

\[ \frac{dA_t}{A_t} = \mu_t^A dt + \sigma_t^A dZ_t \]

\[ \Rightarrow \mu_t^A + (\mu_t^F + \sigma_t^A \sigma_t^F) = 0 \]

Two assets:

\[ \mu_t^A - \mu_t^B + (\sigma_t^A - \sigma_t^B) \sigma_t^F = 0 \]

If asset B is risk-free:

\[ \mu_t^A - \mu_t^B = \sigma_t^A \sigma_t^F = 0 \]

Example: valuation

- **Representative agent, CRRA utility, so \( \xi_t = e^{\mu_t c_t} \)**
- Single asset with divided \( D_t \), exp. growth \( g \), volatility \( \sigma \)
- **Find asset price and risk-free rate**

\[ \frac{dq_t}{q_t} = \frac{dD_t}{D_t} = g dt + \sigma dZ_t \]

\[ \Rightarrow \frac{dq_t}{q_t} = -\gamma g dt + \frac{\gamma (\gamma + 1) t^{\gamma\sigma} - \gamma \sigma}{2} dt - \gamma \sigma dZ_t \]

\[ dr_t = \frac{D_t}{q_t} dt + g dt + \sigma dZ_t \]

\[ \Rightarrow \frac{Dr_t}{D_t} = \frac{D_t}{q_t} g - \gamma g - \frac{\gamma (\gamma + 1) t^{\gamma\sigma} - \gamma \sigma}{2} dt - \gamma \sigma dZ_t = 0 \]

**Note:** if \( \gamma = 1, D_t/q_t = p \)
Example: evolution of wealth and $\eta$: logarithmic utility

<table>
<thead>
<tr>
<th>Consumption = $\rho$ wealth</th>
</tr>
</thead>
</table>
| **Price of risk:** $\sigma_i^f = \sigma_i^{C'} = \sigma_i^H = -\frac{\psi}{\eta}(\sigma + \sigma_i^f)$  
  $\eta_i = \frac{N_i}{N_i + \Delta}$ |
| **Hence:**  
  $\frac{dN_i}{N_i} = r_i^F dt - \frac{C_i}{N_i} dt + \frac{\psi}{\eta}(\sigma + \sigma_i^f) \left( dZ_t + \frac{\psi}{\eta}(\sigma + \sigma_i^f) dt \right)$  
  $\frac{dN_i}{N_i} = r_i^F dt - \frac{C_i}{N_i} dt + \frac{1 - \psi}{\eta} (\sigma + \sigma_i^f) \left( dZ_t + \frac{1 - \psi}{\eta} (\sigma + \sigma_i^f) dt \right)$ |
| Ito implies that $\frac{d\eta_i}{\eta_i} = \ldots$ |

Example: allocation, logarithmic utility

<table>
<thead>
<tr>
<th>Consumption = $\rho$ wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi + (1 - \psi)q - c(q) = (\rho \eta + \varphi(1 - \eta))q$</td>
</tr>
<tr>
<td><strong>Price of risk (experts):</strong> $\sigma_i^f = \sigma_i^{C'} = \sigma_i^H = -\frac{\psi}{\eta}(\sigma + \sigma_i^f)$</td>
</tr>
<tr>
<td><strong>Asset pricing (experts):</strong> $a - \frac{\nu}{\eta} + \text{cap. gains rate} - r_i^F (\sigma + \sigma_i^f) - \left( \frac{\psi}{\eta} (\sigma + \sigma_i^f) \right) = 0$</td>
</tr>
<tr>
<td><strong>households:</strong></td>
</tr>
<tr>
<td>$\frac{a}{\eta} + \text{cap. gains rate} - r_i^F (\sigma + \sigma_i^f) - \left( \frac{\psi}{\eta} (\sigma + \sigma_i^f) \right) = 0$</td>
</tr>
<tr>
<td>subtracting $\frac{a - \beta}{\eta} = \left( \frac{\psi}{\eta} - \frac{1 - \psi}{1 - \eta} \right)(\sigma + \sigma_i^f)^2$</td>
</tr>
</tbody>
</table>

Asset pricing: implications

- Valuation (e.g. the price of capital $q_i$)
- Asset allocation: $\mu_i^{A,i} + \mu_i^{C,i} + \sigma_i^{A,i} \sigma_i^{C,i} \leq 0$.
  with equality if agent $i$ holds asset $A$ in positive amount
- Evolution of wealth and wealth distribution
  
  Also, to solve the model we need understanding of
  
  - endogenous risk
  
  - value functions (for numerical solutions)
Model from online lectures

\[ \frac{dK_t}{K_t} = (\Phi(t) - \delta) \, dt + \sigma \, dZ_t \]

Output per unit of capital \( a - \lambda \)

CRRA utility, discount rate \( \rho \)

Liquid markets for capital \( k \) with endogenous price per unit \( q \)

\[ dq/q = \mu \, dt + \sigma \, dZ_t \]

Assume log utility and \( \chi = 1 \), i.e. experts can issue only debt. Let’s solve it!

Now, today’s exercise

- Households + experts, \( a = 2 \)
- Capital
  \[ \frac{dK_t}{K_t} = (\Phi(t) - \delta) \, dt + \sigma \, dZ_t + r \Phi \, d\tilde{Z}_t \]
- Log utility (consumption = \( \rho \) • wealth)
- Households
  \[ E \left[ \int_0^\infty e^{-\rho t} \log c_t \, dt \right] \]
- Experts
  \[ E \left[ \int_0^\infty \log c_t \, dt \right] \]
  \[ \frac{dc_t}{c_t} = -\rho \, dt + \sigma \, dZ_t \]
- Risk \( dZ \) is tradable (price of risk \( c_t \))

Logarithmic utility: model solution

Consumption = \( \rho \) wealth

(2) \[ \psi(a + (1 - \psi)g - \iota(q)) = (\rho \eta + \iota'(1 - \eta))q \]

Allocation:

(2) \[ \frac{a - g}{\eta} = \left( \frac{\psi}{\eta} - \frac{1 - \psi}{1 - \eta} \right) (\sigma + \sigma_t)^2 \]

Endogenous risk

(3) \[ \sigma_t^2 = \frac{1 - \eta}{\eta} (\sigma + \sigma_t^2) \]

\[ \rho \eta q = a - \iota(q) \]

Big picture

- Valuation (e.g. the price of capital \( q_t \))
- Allocation
  - capital
    \[ a - \iota/\eta + \text{cap. gains rate} - r_t^f + (\sigma + \sigma_t^2)q_t + \phi \]
    \[ \frac{\psi}{\eta} \sigma - \Phi \hat{\sigma} = 0 \]
    \[ \text{price of idiosyncratic risk, E} \]
    \[ \frac{a - \iota/\eta}{\eta} + \text{cap. gains rate} - r_t^f + (\sigma + \sigma_t^2)q_t + \phi \]
    \[ \frac{\psi}{\eta} \sigma - \Phi \hat{\sigma} = 0 \]
    \[ \text{price of idiosyncratic risk, R} \]
  - aggregate risk
    \[ \sigma_t^2 = \text{volatility of marginal utility} = \alpha \]
- Evolution of the wealth distribution
Question 1

• What is the price of aggregate risk for households and experts, if households’ wealth is \( N_h \) and experts’, \( N_e \)?

(a) \(-\sigma^H \) and \(-\sigma^N \)
(b) \(-\sigma^H \) and \(-\sigma^N + \sigma^e \)
(c) \(\sigma^H \) and \(\sigma^N - \sigma^e \)
(d) \(\sigma^H \) and \(\sigma^N + \sigma^e \)
(e) none of the above

Question 2

(2) Given the fraction \( \psi_t \) of capital held by entrepreneurs, what are the prices of idiosyncratic risk \( \zeta_t \) and \( \zeta_i \) for entrepreneurs and households?

(a) \( \zeta_t = \frac{\psi_t}{\eta_t} \sigma^e \) \( \zeta_i = \frac{1}{\eta_t} \sigma^e \)
(b) \( \zeta_t = \frac{\psi_t}{\eta_t} \varphi \sigma^e \) \( \zeta_i = \frac{1}{\eta_t} \varphi \sigma^e \)
(c) \( \zeta_t = \frac{\psi_t}{\eta_t} \varphi \sigma^e \) \( \zeta_i = \frac{1}{\eta_t} \varphi \sigma^e \)
(d) \( \zeta_t = \zeta_i = \varphi \sigma^e \)

Question 3

(3) As a function of the entrepreneurs’ wealth share \( \eta_t \), determine the fraction of capital \( \psi_t \) that entrepreneurs hold in equilibrium.

- **Hint:**
  \[ \frac{\eta_t - \psi_t}{\eta_t} + \text{cap. gains rate} - r_t^e + (\sigma + \sigma^e)q_t + \varphi \sigma^e \frac{\psi_t}{\eta_t} = 0 \]
  \[ \frac{1 - \psi_t}{\eta_t} + \text{price of idiosyncratic risk, } E \]
  \[ \frac{\eta_t - \psi_t}{\eta_t} + \text{cap. gains rate} - r_t^e + (\sigma + \sigma^e)q_t + \varphi \sigma^e \frac{1}{1 - \psi_t} = 0 \]
  \[ \frac{1 - \psi_t}{\eta_t} + \text{price of idiosyncratic risk, } H \]
Question 3

(3) As a function of the entrepreneurs’ wealth share \( \eta \), determine the fraction of capital \( \psi \) that entrepreneurs hold in equilibrium.

* Hint: 
\[
\begin{align*}
\frac{a - \eta}{\eta} - \text{gains rate} &= r^a + (\sigma + \sigma^*) \eta + \rho \\
\frac{-\eta}{\eta} - \text{gains rate} &= r^a + (\sigma + \sigma^*) \eta + \rho \\
\end{align*}
\]

Which expression gives the law of motion of total wealth \( q^a \)? Recall that \( \sigma^a = \zeta_1 \)

\[
\begin{align*}
\frac{d}{dt} \sigma^a &= \zeta_1 \\
\end{align*}
\]

(4) If the risk-free rate is \( r^F \), which of the following expressions gives the law of motion of total wealth \( q^a \)? Recall that \( \sigma^a = \zeta_1 \)

\[
\begin{align*}
\frac{d}{dt} \sigma^a &= \zeta_1 \\
\end{align*}
\]

Question 5

(5) Which expression gives the law of motion of total wealth \( q^a \)? Recall that \( \sigma^a = \zeta_1 \)

\[
\begin{align*}
\frac{d}{dt} \sigma^a &= \zeta_1 \\
\end{align*}
\]

(a) \( \frac{d}{dt} \sigma^a = \rho^a + \Phi(\zeta_1) - \delta + \sigma \eta dZ_t - \rho d \eta \)

(b) \( \frac{d}{dt} \sigma^a = \rho^a + \Phi(\zeta_1) - \delta + \sigma \eta dZ_t - \rho d \eta \)

(c) \( \frac{d}{dt} \sigma^a = \rho^a + \Phi(\zeta_1) - \delta + \sigma \eta dZ_t - \rho d \eta \)

(d) all of the above

Question 6

Which equation should we use to derive the law of motion of \( \eta \)?

* Derive the law of motion of \( \eta \). Recall that

\[
\begin{align*}
\frac{d}{dt} \eta &= \left( \rho + \eta \sigma^a \right) \eta + \eta \sigma \eta dZ_t \\
\frac{d}{dt} \eta &= \left( \rho + \eta \sigma^a \right) \eta + \eta \sigma \eta dZ_t \\
\end{align*}
\]

(6) If the risk-free rate is \( r^F \), which of the following expressions gives the law of motion of total wealth \( q^a \)? Recall that \( \sigma^a = \zeta_1 \)

\[
\begin{align*}
\frac{d}{dt} \sigma^a &= \zeta_1 \\
\end{align*}
\]
Question 6
\[
\frac{d(X_t / Y_t)}{X_t / Y_t} = (\mu_t - \mu^* - \sigma_t^2(\sigma^2_t - \sigma^2^*) dt + (\sigma_t^2 - \sigma^2^*) dZ_t
\]

- Derive the law of motion of \( \eta \). You can use
\[
\frac{d\eta_t}{\eta_t} = \left( \tau^2_t - \rho + (\omega + \sigma^2)\eta_t + \frac{\sigma^2}{\eta_t} \right) dt + (\omega + \sigma^2) dZ_t
\]
\[
\frac{d(/KI)}{\varphi K_t} = (\tau^2_t - \rho + (\omega + \sigma^2)\eta_t + \frac{\sigma^2}{\eta_t} \right) dt + (\omega + \sigma^2) dZ_t
\]
\[
\frac{d\eta}{\eta} = \left( (1 - \eta) \frac{\omega^2(1 - \omega^2)\sigma^2}{\eta(1 - \eta)\omega^2 \sigma^2} + \eta(\eta - 1)(\sigma^2)^2 \right) + (1 - \eta)\sigma^2 dZ_t
\]

Summary
- Valuation (e.g. the price of capital \( q_t \))
  \[
  \rho q_t = a - \lambda(q_t)
  \]
- Capital allocation
  \[
  \psi_t = \frac{\eta_t}{\eta_t + (1 - \eta_t)\omega^2}
  \]
- Wealth distribution
  \[
  \frac{d\eta}{\eta} = \left( (1 - \eta) \frac{\omega^2(1 - \omega^2)\sigma^2}{\eta(1 - \eta)\omega^2 \sigma^2} + \eta(\eta - 1)(\sigma^2)^2 \right) + (1 - \eta)\sigma^2 dZ_t
  \]

Bonus question
- Suppose we have a Markov diffusion process
  \[
  dX_t = \mu(X_t) dt + \sigma(X_t) dZ_t
  \]
then density over \( X \) follows
\[
g(x, t) = \frac{1}{\sigma(x)^2} \exp \left( \int \frac{\partial \mu(y)}{\sigma(y)^2} dy \right)
\]
Stationary density satisfies
\[
g(x) = \text{const} \frac{1}{\sigma(x)^2} \exp \left( \int \frac{\partial \mu(y)}{\sigma(y)^2} dy \right)
\]
For today's exercise, under what conditions does the stationary density exist?

Economies with Financial Frictions — Exercise Continued

The 2022 Princeton Initiative