Consider an economy with two productive assets: intangibles that carry idiosyncratic risk, and capital. The total supply of intangibles is denoted by \( L_t \), and the total supply of capital by \( K_t \). Intangibles and capital are combined to produce output according to the Cobb-Douglas production function

\[
a K_t^\alpha L_t^{1-\alpha}.
\]

All units of capital are identical, and total capital in the economy follows

\[
dK_t/K_t = (\Phi(\iota_t) - \delta) \, dt
\]

where \( \iota_t \) is investment per unit of capital. Assume that \( \Phi(\iota) \) is an increasing and concave function with \( \Phi(0) = 0 \), and that \( \delta > 0 \). Individuals can trade capital at price \( q_t \), determined endogenously.

Intangible asset combines all other determinants of business success aside from physical capital, i.e. the number of existing customers, customer loyalty, brand reputation, company culture, worker skills, etc. Intangibles carry idiosyncratic risk, with intangibles of individual \( i \) following

\[
dl_i^t/l_i^t = \tilde{\sigma} \, d\tilde{Z}_i^t,
\]

where \( \tilde{Z}_i^t \) is idiosyncratic. Idiosyncratic risks cancel out in the aggregate, so the total amount of intangibles in the economy grows at the constant rate \( g \),

\[
L_t = \int l_i^t \, di = e^{gt} L_0.
\]

That is, customers move between firms, some skills become more valuable and some become obsolete, same happens with technologies, but the total economic potential grows at rate \( g \).

Following Aiyagari, given the model so far, we could interpret \( l_i^t \) as labor endowment of individual \( i \). However, for this problem, we’ll take an interpretation that makes \( l_i^t \) tradable. For example, firms could sell a part of their business to reduce the number of customers, or buy other businesses to grow. Let the value of all intangible assets in the economy to be \( R_t \).
Overall, the situation we face is as follows. Individuals have intangible assets which carry idiosyncratic risk. To self insure they can hold capital, which is a productive asset that carries no idiosyncratic risk.

Is there enough capital to satisfy the individuals’ demand for insurance through savings? Can they always build more capital to satisfy their demand? Is there room for a third asset (e.g. virtual currency) that can also be used to save? In this problem, we’ll consider the possibility of a third asset, which is unproductive but whose supply is fixed, “coins.” Denote the value of all coins by \( P_t \geq 0 \). Then the total wealth of all individuals is \( R_t + q_t K_t + P_t \).

In this exercise we are interested in the stationary growth equilibrium of this economy, in which \( K_t, P_t \) and \( R_t \) grow at rate \( g \), and \( q_t \) is constant.

Given total output (1), in competitive inputs markets, the portion of the output that goes to intangibles is

\[
(1 - \alpha) a K_t^\alpha L_t^{1 - \alpha}.
\]

The rest goes to capital. Hence, given the total investment of \( \iota_t K_t \), the dividend yield on capital is

\[
\frac{\alpha a K_t^\alpha L_t^{1 - \alpha} - \iota_t K_t}{q_t K_t}.
\]

The optimal investment rate is given by the first-order condition

\[
\Phi'(\iota) q_t = 1.
\]

Individuals have logarithmic utility with discount rate \( \rho \).

1. What is the correct market-clearing condition for all output in this economy?
   
   (a) \( \rho (P_t + R_t + q_t K_t) = aK_t^\alpha L_t^{1 - \alpha} \)
   
   (b) \( \rho q_t K_t = \alpha a K_t^\alpha L_t^{1 - \alpha} - \iota (q_t) K_t \)
   
   (c) \( \rho (P_t + R_t + q_t K_t) = aK_t^\alpha L_t^{1 - \alpha} - \iota (q_t) K_t \)
   
   (d) \( \rho R_t = (1 - \alpha) a K_t^\alpha L_t^{1 - \alpha} \)

2. Using output as numeraire, what are the returns on capital, intangibles and money in the stationary growth equilibrium of the economy? For money return, assume money has value, so its return is well defined.
\[
(a) \quad dr_t^K = \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota t K_t}{q_t K_t} \, dt + gdt, \quad dr_t^L = \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} \, dt + gdt + \sigma d\hat{Z}_t, \quad dr_t^M = gdt.
\]
\[
(b) \quad dr_t^K = \alpha a K_t^\alpha L_t^{1-\alpha} \, dt, \quad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} \, dt + \sigma d\hat{Z}_t, \quad dr_t^M = 0.
\]
\[
(c) \quad dr_t^K = \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota t K_t}{q_t K_t} \, dt + (\Phi(\iota) - \delta) \, dt, \quad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} \, dt + gdt, \quad dr_t^M = 0.
\]
\[
(d) \quad dr_t^K = g \, dt, \quad dr_t^L = g \, dt + \sigma d\hat{Z}_t, \quad dr_t^M = g \, dt.
\]

3. What determines the price of capital \( q_t \) as well as the investment rate in the stationary growth equilibrium?

(a) The productivity parameter \( a \) and the depreciation rate \( \delta \).

(b) Parameters \( a, \alpha \) and \( \delta \).

(c) The conditions \( \Phi(\iota) - \delta = g \) and \( \Phi'(\iota) = q \).

(d) The condition that the expected returns on capital and intangibles are the same.

4. Write down the pricing equation for intangibles relative to capital in the stationary growth equilibrium of this economy

\[
(a) \quad \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha} - (\alpha a K_t^\alpha L_t^{1-\alpha} - \iota t K_t)}{R_t + q_t K_t} = \frac{R_t}{R_t + \hat{P}_t + q_t K_t} \sigma^2.
\]
\[
(b) \quad \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} - \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota t K_t}{q_t K_t} = \frac{R_t}{R_t + \hat{P}_t + q_t K_t} \sigma^2.
\]
\[
(c) \quad \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha} - \iota t K_t}{R_t} = \frac{R_t^2}{(R_t + \hat{P}_t + q_t K_t)^2} \sigma^2.
\]
\[
(d) \quad \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} = g + \frac{R_t}{R_t + \hat{P}_t + q_t K_t} \sigma^2.
\]

5. For the case when \( \sigma = 0 \), write down the quadratic equation that characterizes the level of output per unit of capital \( Y/K = a(L_t/K_t)^{1-\alpha} \) in the steady state. Your equations will have in it \( \iota \) and \( q \) from part 3 (which you can take as given). Hint: You should use the market-clearing condition for output from part 1 and the pricing equation from part 4, taking \( \iota \) and \( q \) as given.
6. Now, consider the case of large enough idiosyncratic risk, so that there exists an equilibrium with money. What level must $Y/K$ attain in all these equilibria?

7. If money has value in equilibrium, characterize $R_t$ and $P_t$ as functions of $\tilde{\sigma}$. For what range of $\tilde{\sigma}$ is there an equilibrium in which money has value?