The 2021 Princeton Initiative.
Problem Set 2: Idiosyncratic risk, capital and money.

Saturday, September 11, 11am Eastern Time.

Consider an economy with two productive assets: intangibles that carry idiosyncratic risk, and capital. The total supply of intangibles is denoted by L_t, and the total supply of capital by K_t. Intangibles and capital are combined to produce output according to the Cobb-Douglas production function

$$aK_t^{\alpha} L_t^{1-\alpha}. \tag{1}$$

All units of capital are identical, and total capital in the economy follows

$$dK_t/K_t = (\Phi(\iota_t) - \delta)\, dt$$

where ι_t is investment per unit of capital. Assume that $\Phi(\iota)$ is an increasing and concave function with $\Phi(0) = 0$, and that $\delta > 0$. Individuals can trade capital at price q_t, determined endogenously.

Intangible asset combines all other determinants of business success aside from physical capital, i.e. the number of existing customers, customer loyalty, brand reputation, company culture, worker skills, etc. Intangibles carry idiosyncratic risk, with intangibles of individual i following

$$dl_t^i/l_t^i = \tilde{\sigma}\, d\tilde{Z}_t^i,$$

where $\tilde{Z}_t^i$ is idiosyncratic. Idiosyncratic risks cancel out in the aggregate, so the total amount of intangibles in the economy grows at the constant rate g,

$$L_t = \int l_t^i\, di = e^{gt} L_0.$$

That is, customers move between firms, some skills become more valuable and some become obsolete, same happens with technologies, but the total economic potential grows at rate g.

Following Aiyagari, given the model so far, we could interpret l_t^i as labor endowment of individual i. However, for this problem, we'll take an interpretation that makes l_t^i tradable. For example, firms could sell a part of their business to reduce the number of customers, or buy other businesses to grow. Let the value of all intangible assets in the economy to be R_t.

Overall, the situation we face is as follows. Individuals have intangible assets which carry idiosyncratic risk. To self insure they can hold capital, which is a productive asset that carries no idiosyncratic risk.

Is there enough capital to satisfy the individuals' demand for insurance through savings? Can they always build more capital to satisfy their demand? Is there room for a third asset (e.g. virtual currency) that can also be used to save? In this problem, we'll consider the possibility of a third asset, which is unproductive but whose supply is fixed, "coins." Denote the value of all coins by $P_t \geq 0$. Then the total wealth of all individuals is $R_t + q_t K_t + P_t$. In this exercise we are interested in the stationary growth equilibrium of this economy, in which K_t, P_t and R_t grow at rate g, and q_t is constant.

Given total output (1), in competitive inputs markets, the portion of the output that goes to intangibles is

$$(1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}.$$

The rest goes to capital. Hence, given the total investment of $\iota_t K_t$, the dividend yield on capital is

$$\frac{\alpha a K_t^{\alpha}L_t^{1-\alpha} - \iota_t K_t}{q_t K_t}.$$

The optimal investment rate is given by the first-order condition

$$\Phi'(\iota_t)q_t = 1.$$

Individuals have logarithmic utility with discount rate ρ.

1. What is the correct market-clearing condition for all output in this economy?

(a) $\rho(P_t + R_t + q_t K_t) = aK_t^{\alpha}L_t^{1-\alpha}$
(b) $\rho q_t K_t = \alpha a K_t^{\alpha}L_t^{1-\alpha} - \iota(q_t)K_t$
(c) $\rho(P_t + R_t + q_t K_t) = aK_t^{\alpha}L_t^{1-\alpha} - \iota(q_t)K_t$
(d) $\rho R_t = (1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}$

2. Using output as numeraire, what are the returns on capital, intabgibles and money in the stationary growth equilibrium of the economy? For money return, assume money has value, so its return is well defined.

(a) $dr_t^K = \dfrac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} dt + g dt, \;\; dr_t^L = \dfrac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} dt + g dt + \tilde{\sigma} d\tilde{Z}_t, \;\; dr_t^M = g dt.$

(b) $dr_t^K = \alpha a K_t^\alpha L_t^{1-\alpha}\, dt, \quad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} dt + \tilde{\sigma}\, d\tilde{Z}_t, \quad dr_t^M = 0.$

(c) $dr_t^K = \dfrac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} dt + (\Phi(\iota_t) - \delta) dt, \quad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} dt + g dt, \quad dr_t^M = 0.$

(d) $dr_t^K = g\, dt, \quad dr_t^L = g\, dt + \tilde{\sigma}\, d\tilde{Z}_t, \quad dr_t^M = g\, dt.$

3. What determines the price of capital q_t as well as the investment rate in the stationary growth equilibrium?

(a) The productivity parameter a and the depreciation rate δ.

(b) Parameters a, α and δ.

(c) The conditions $\Phi(\iota) - \delta = g$ and $\Phi'(\iota) = q$.

(d) The condition that the expected returns on capital and intabgibles are the same.

4. Write down the pricing equation for intabgibles relative to capital in the stationary growth equilirium of this economy

(a) $\frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha} - (\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t)}{R_t + q_t K_t} = \frac{R_t}{R_t + P_t + q_t K_t} \tilde{\sigma}^2.$

(b) $\frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} - \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} = \frac{R_t}{R_t + P_t + q_t K_t} \tilde{\sigma}^2.$

(c) $\frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} - \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} = \frac{R_t^2}{(R_t + P_t + q_t K_t)^2} \tilde{\sigma}^2.$

(d) $\frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} = g + \frac{R_t}{R_t + P_t + q_t K_t} \tilde{\sigma}^2.$

5. For the case when $\tilde{\sigma} = 0$, write down the quadratic equation that characterizes the level of output per unit of capital $Y/K = a(L_t/K_t)^{1-\alpha}$ in the steady state. Your equations will have in it ι and q from part 3 (which you can take as given). Hint: You should use the market-clearing condition for output from part 1 and the pricing equation from part 4, taking ι and q as given.

6. Now, consider the case of large enough idiosyncratic risk, so that there exists an equilibrium with money. What level must Y/K attain in all these equilibria?

7. If money has value in equilibrium, characterize R_t and P_t as functions of $\tilde{\sigma}$. For what range of $\tilde{\sigma}$ is is there an equilibrium in which money has value?