

## The 2021 Princeton Initiative.

### Problem Set 1.

Friday, September 9, 10:30am Eastern Time.

**Problem.** Consider an economy with two types of agents: entrepreneurs and households. When managed by households, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t,$$

where shocks  $dZ_t$  are aggregate and  $d\tilde{Z}_t$  are idiosyncratic (agent-specific, and canceling out in the aggregate). When managed by entrepreneurs, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t + \phi\tilde{\sigma} d\tilde{Z}_t,$$

where  $\phi \in (0, 1)$ . That is, entrepreneurs are able to diversify and reduce their idiosyncratic risk exposure.

Capital  $k_t$  produces output  $ak_t$  if managed by any agent, so after investment of  $\iota_t$ , the remaining output that can be consumed (or traded) is given by  $(a - \iota_t)k_t$ . Assume that the investment function  $\Phi(\iota)$  satisfies  $\Phi(0) = 0$ ,  $\Phi > 0$  and  $\Phi'' \leq 0$ . Denote the aggregate supply of capital by  $K_t$  and the price of capital per unit by  $q_t$ .

Assume that aggregate risk  $dZ_t$  can be traded between the households and entrepreneurs and carries the equilibrium price of risk of  $\varsigma_t$ . Denote the price of idiosyncratic risk of entrepreneurs by  $\tilde{\zeta}_t$  and that of households by  $\underline{\tilde{\zeta}}_t$ . Thus, the budget constraint/law of motion of wealth of any entrepreneur is given by

$$\frac{dn_t}{n_t} = (r_t^F + \varsigma_t \sigma_t^n + \tilde{\zeta}_t \tilde{\sigma}_t^n) dt + \sigma_t^n dZ_t + \tilde{\sigma}_t^n d\tilde{Z}_t - \frac{c_t}{n_t} dt,$$

where  $r_t^F$  is the equilibrium risk-free rate and  $c_t$  is the entrepreneur's consumption. The budget constraint of a household is written similarly. Agents control their idiosyncratic risk exposure  $\tilde{\sigma}_t^n \geq 0$  by choosing how much capital to hold, and aggregate risk exposure  $\sigma_t^n$  by also trading derivatives. Agents cannot use derivatives to insure idiosyncratic risk. Aggregate-risk derivatives and the risk-free asset are in zero net supply.

Assume that households have utility

$$E \left[ \int_0^\infty e^{-\rho t} \log c_t dt \right],$$

where  $\rho > 0$  is a discount rate and  $c_t$  is the consumption of a household. Entrepreneurs have utility

$$E \left[ \int_0^\infty \theta_t \log c_t dt \right],$$

where  $\theta_t$  is the process of discounting and *taste shocks* that follows

$$\theta_0 = 1, \quad \frac{d\theta_t}{\theta_t} = -\rho dt + (\sigma^\theta) dZ_t,$$

where  $\rho$  is the same discount rate as that of households and  $(\sigma^\theta) \geq 0$  is an exogenous parameter. In order to solve this problem, you can rely (without proof) on the fact that the optimal consumption rate (of entrepreneurs as well) is given by the discount rate  $\rho$  times net worth.

The goal of this problem is to characterize the equilibrium in this environment.

(1) Given  $\varsigma_t$ , what is the households' choice aggregate risk exposure  $\sigma_t^n$ ? How about that of entrepreneurs,  $\sigma_t^n$ ?

(2) Given the fraction  $\kappa_t$  of capital held by entrepreneurs, what are the prices of idiosyncratic risk  $\tilde{\zeta}_t$  and  $\underline{\tilde{\zeta}}_t$  for entrepreneurs and households?

(3) As a function of the entrepreneurs' wealth share  $\eta_t$ , determine the fraction of capital  $\kappa_t$  that entrepreneurs hold in equilibrium.

(4) If the risk-free rate is  $r_t^F$ , which of the following expressions gives the law of motion of experts' wealth  $N_t$ ?

- (a)  $\frac{dN_t}{N_t} = \left( r_t^F + (\varsigma_t + \sigma^\theta)\varsigma_t + \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}^2 \right) dt + (\varsigma_t + (\sigma^\theta)) dZ_t.$
- (b)  $\frac{dN_t}{N_t} = \left( r_t^F - \rho + (\varsigma_t + \sigma^\theta)\varsigma_t + \frac{\kappa_t^2}{\eta_t^2} \phi^2 \tilde{\sigma}^2 \right) dt + (\varsigma_t + (\sigma^\theta)) dZ_t.$
- (c)  $\frac{dN_t}{N_t} = \left( r_t^F - \rho + (\varsigma_t + \sigma^\theta)^2 + \frac{\kappa_t^2}{\eta_t^2} \phi^2 \tilde{\sigma}^2 \right) dt + (\varsigma_t + (\sigma^\theta)) dZ_t.$
- (d)  $\frac{dN_t}{N_t} = \left( r_t^F - \rho + \frac{\kappa_t}{\eta_t} \sigma \varsigma_t + \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}^2 \right) dt + \frac{\kappa_t}{\eta_t} \sigma dZ_t.$

(5) If the risk-free rate is  $r_t^F$ , which of the following expressions gives the law of motion of total wealth  $q_t K_t$ ?

- (a)  $\frac{d(q_t K_t)}{q_t K_t} = \frac{a - \iota_t}{q_t} dt + (\mu_t^q + \Phi(\iota_t) - \delta + \sigma \sigma_t^q) dt + (\sigma + \sigma_t^q) dZ_t - \rho dt$
- (b)  $\frac{d(q_t K_t)}{q_t K_t} = (\mu_t^q + \Phi(\iota_t) - \delta + \sigma \sigma_t^q) dt + (\sigma + \sigma_t^q) dZ_t$
- (c)  $\frac{d(q_t K_t)}{q_t K_t} = \left( r_t^F - \rho + (\varsigma_t + \eta \sigma^\theta) \varsigma_t + \frac{\kappa_t^2}{\eta_t} \phi^2 \tilde{\sigma}^2 + \frac{(1 - \kappa_t)^2}{1 - \eta_t} \tilde{\sigma}^2 \right) dt + (\varsigma_t + \eta \sigma^\theta) dZ_t$
- (d) all of the above
- (e) (a) and (b) but not (c)

(6) Derive the law of motion of  $\eta_t$ . Please express your answer in terms of model primitives (no endogenous variables). Recall that for processes

$$dX_t/X_t = \mu_t^X dt + \sigma_t^X dZ_t \quad \text{and} \quad dY_t/Y_t = \mu_t^Y dt + \sigma_t^Y dZ_t,$$

we have

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X)) dt + (\sigma_t^X - \sigma_t^Y) dZ_t.$$

(7) Under what conditions does the stationary distribution of  $\eta$  exist? You may use directly the fact (which you can derive from part (6)) that

$$\frac{\mu^\eta \eta}{(\sigma^\eta \eta)^2} = \left( \frac{1}{1 - \eta} + \frac{1}{\eta} \right) \frac{\phi^2 (1 - \phi^2)}{(\eta + (1 - \eta) \phi^2)^2 (\sigma^\theta)^2} - \frac{1}{1 - \eta}.$$

Hint: you have to make sure that the stationary density integrates to a finite number both near  $\eta = 0$  and 1.