The 2021 Princeton Initiative
Saturday Morning Exercise

The Setting

- Output a $k_t^{\alpha} \, l_t^{1-\alpha}$
- Capital

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt$$

- l_t: 1st interpretation - intangibles (customers, know-how, etc)
  - 2nd interpretation: labor

$$dl_t^i = \tilde{\sigma}\, d\tilde{Z}_t^i, \quad L_t = \int l_t^i \, di = 1$$

- With interpretation 1: individuals can trade coins (in fixed supply), k and intangibles (e.g. acquire customers of a struggling business). Values

$$P_t \quad q_t K_t \quad \text{and} \quad R_t$$

- Utility: logarithmic, discount rate ρ

The Setting

- Values (coins, capital, intangibles)

$$P_t \quad q_t K_t \quad \text{and} \quad R_t$$

- Dividend yield

$$0, \quad \frac{\alpha a K_t^{\alpha} L_t^{1-\alpha} - \iota_t K_t}{q_t K_t}, \quad \frac{(1-\alpha) a K_t^{\alpha} L_t^{1-\alpha}}{R_t}$$

- Optimal investment FOC

$$\Phi'(\iota_t) q_t = 1$$

Question 1

1. What is the correct market-clearing condition for all output in this economy?

(a) $\rho(P_t + R_t + q_t K_t) = a K_t^{\alpha} L_t^{1-\alpha}$

(b) $\rho q_t K_t = \alpha a K_t^{\alpha} L_t^{1-\alpha} - \iota(q_t) K_t$

(c) $\rho(P_t + R_t + q_t K_t) = a K_t^{\alpha} L_t^{1-\alpha} - \iota(q_t) K_t$

(d) $\rho R_t = (1-\alpha) a K_t^{\alpha} L_t^{1-\alpha}$

Question 2

2. Using output as numeraire, what are the returns on capital, intabgibles and money in the stationary growth equilibrium of the economy? For money return, assume money has value, so its return is well defined.

$$(a) \quad dr_t^K = \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} dt + g dt, \;\; dr_t^L = \frac{(1-\alpha) a K_t^\alpha L_t^{1-\alpha}}{R_t} dt + g dt + \tilde{\sigma} d\tilde{Z}_t, \;\; dr_t^M = g dt.$$

$$(b) \quad dr_t^K = \alpha a K_t^\alpha L_t^{1-\alpha}\, dt, \qquad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} dt + \tilde{\sigma}\, d\tilde{Z}_t, \qquad dr_t^M = 0.$$

$$(c) \quad dr_t^K = \frac{\alpha a K_t^\alpha L_t^{1-\alpha} - \iota_t K_t}{q_t K_t} dt + (\Phi(\iota_t) - \delta) dt, \qquad dr_t^L = (1-\alpha) a K_t^\alpha L_t^{1-\alpha} dt + g dt, \qquad dr_t^M = 0.$$

$$(d) \quad dr_t^K = g\, dt, \qquad dr_t^L = g\, dt + \tilde{\sigma}\, d\tilde{Z}_t, \qquad dr_t^M = g\, dt.$$

Question 3

3. What determines the price of capital q_t as well as the investment rate in the stationary growth equilibrium?

(a) The productivity parameter a and the depreciation rate δ.

(b) Parameters a, α and δ.

(c) The conditions $\Phi(\iota) - \delta = g$ and $\Phi'(\iota) = q$.

(d) The condition that the expected returns on capital and intabgibles are the same.

Question 4

4. Write down the pricing equation for intabgibles relative to capital in the stationary growth equilirium of this economy

$$(a) \quad \frac{(1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}-(\alpha a K_t^{\alpha}L_t^{1-\alpha}-\iota_t K_t)}{R_t+q_tK_t} = \frac{R_t}{R_t+P_t+q_tK_t}\tilde{\sigma}^2.$$

$$(b) \quad \frac{(1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}}{R_t} - \frac{\alpha a K_t^{\alpha}L_t^{1-\alpha}-\iota_t K_t}{q_tK_t} = \frac{R_t}{R_t+P_t+q_tK_t}\tilde{\sigma}^2.$$

$$(c) \quad \frac{(1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}}{R_t} - \frac{\alpha a K_t^{\alpha}L_t^{1-\alpha}-\iota_t K_t}{q_tK_t} = \frac{R_t^2}{(R_t+P_t+q_tK_t)^2}\tilde{\sigma}^2.$$

$$(d) \quad \frac{(1-\alpha)aK_t^{\alpha}L_t^{1-\alpha}}{R_t} = g + \frac{R_t}{R_t+P_t+q_tK_t}\tilde{\sigma}^2.$$

Equilibrium with or without money

- Next, we'll characterize the key ratio of output to capital $Y/K = a(L/K)^{1-\alpha}$, from which we can back out everything else. We'll do it for the cases of
  - no idiosyncratic risk
  - large enough idiosyncratic risk so money has value
- Then we'll answer the ultimate question: how large idiosyncratic risk needs to be for coins to have value in equilibrium

Question 5

5. For the case when $\tilde{\sigma} = 0$, write down the quadratic equation that characterizes the level of output per unit of capital $Y/K = a(L_t/K_t)^{1-\alpha}$ in the steady state. Your equations will have in it ι and q from part 3 (which you can take as given). Hint: You should use the market-clearing condition for output from part 1 and the pricing equation from part 4, taking ι and q as given.

- If we know $L_t = e^{-gt} L_0$, we can use Y/K to determine K_t

Question 5

- Quadratic equation for Y/K, no idiosyncratic risk

Solution. The market clearing and pricing conditions become

$$\frac{(1-\alpha)Y}{R_t} = \frac{\alpha Y - \iota_t K_t}{q_t K_t} \quad \text{and} \quad \rho(R + qK) = Y - \iota K.$$

Combining these two equations, we obtain

$$\frac{(1-\alpha)Y}{\frac{Y-\iota K}{\rho} - qK} = \frac{\alpha Y - \iota_t K_t}{q_t K_t}$$

$$\rho(1-\alpha)Y/K = (\alpha Y/K - \iota)\underbrace{\left(\frac{Y/K - \iota}{q} - \rho\right)}_{R/(qK)}.$$

We are interested in the solution such that $R/(qK) > 0$, *and so* $\alpha Y/K > \iota$. *Hence, money cannot have any value in this case.*

Question 6

6. Now, consider the case of large enough idiosyncratic risk, so that there exists an equilibrium with money. What level must Y/K attain in all these equilibria?

Question 6

- Y/K when money has value

Solution. Capital and money are both risk-free assets, and must have equal return if money is to have value. Since the dividend yield on capital must be 0, we must have $\alpha Y/K = \iota$. Hence,

$$Y/K = \frac{\iota}{\alpha}.$$

In contrast, with $\tilde{\sigma} = 0$, $Y/K > \iota/\alpha$ and satisfies

$$\rho(1-\alpha)Y/K = (\alpha Y/K - \iota)\underbrace{\left(\frac{Y/K - \iota}{q} - \rho\right)}_{R/(qK)}.$$

As $\tilde{\sigma}$ increases, the amount of capital K grows for each L, until dividend yield on capital erodes to 0.

Question 7

If money has value in equilibrium, characterize R_t and P_t as functions of $\tilde{\sigma}$. For what range of $\tilde{\sigma}$ is is there an equilibrium in which money has value?

Question 7

- Condition for money to have value

Solution. *Plugging* $\alpha Y/K = \iota$ *into the market clearing condition and the pricing equation, we obtain*

$$\rho(P + R + qK) = \frac{1-\alpha}{\alpha}\iota K = \frac{R^2}{R + P + qK}\tilde{\sigma}^2.$$

Hence, we have

$$\sqrt{\rho} = \frac{R}{R + P + qK}\tilde{\sigma}, \quad P + R + qK = \frac{1-\alpha}{\alpha}\frac{\iota}{\rho}K$$

$$R = \frac{\sqrt{\rho}}{\tilde{\sigma}}\frac{1-\alpha}{\alpha}\frac{\iota}{\rho}K, \quad P = \left(\frac{\tilde{\sigma} - \sqrt{\rho}}{\tilde{\sigma}}\frac{1-\alpha}{\alpha}\frac{\iota}{\rho} - q\right)K.$$

Money has value when

$$\boxed{\frac{\tilde{\sigma} - \sqrt{\rho}}{\tilde{\sigma}}\frac{1-\alpha}{\alpha}\frac{\iota}{\rho} > q}.$$

Bonus: numerical graph