The 2021 Princeton Initiative
Friday Morning Exercise
The Setting

• Households + experts, \( a = a \)

• Capital

\[
\frac{dk_t}{k_t} = (\Phi(\nu_t) - \delta) \, dt + \sigma \, dZ_t + \varphi \, \sigma \, d\tilde{Z}_t
\]

• Log utility (consumption = \( \rho \cdot \text{wealth} \))

• Households

\[
E \left[ \int_0^\infty e^{-\rho t} \log c_t \, dt \right]
\]

• Experts

\[
E \left[ \int_0^\infty \theta_t \log c_t \, dt \right], \quad \theta_0 = 1, \quad \frac{d\theta_t}{\theta_t} = -\rho \, dt + \sigma^\theta \, dZ_t
\]

• Risk \( dZ \) is tradable (price of risk \( c_t \)), but \( d\tilde{Z}_t \) is not
Quick Review: Asset pricing

\[ SDF \; \xi_t = e^{-\rho t} u'(c_t) \]

∀ self-financing trading strategy w/ value \( A_t, A_t \xi_t \) is a martingale

\[ \frac{d\xi_t}{\xi_t} = \mu_t^\xi \; dt + \sigma_t^\xi \; dZ_t \quad \Rightarrow \quad \mu_t^A + \mu_t^\xi + \sigma_t^A \sigma_t^\xi = 0 \]

Change numeraire \( A_t' = A_t/Y_t, \xi_t' = \xi_t Y_t, A_t' \xi_t' = A_t \xi_t \) is a martingale

Log utility

\[ E \left[ \int_0^\infty e^{-\rho t} \log c_t \; dt \right] \Rightarrow c_t = \rho n_t, \quad \zeta_t = -(-\sigma_t^\xi) = \sigma_t^n \]

\[ E \left[ \int_0^\infty \theta_t \log c_t \; dt \right] \quad \text{with} \quad \frac{d\theta_t}{\theta_t} = -\rho \; dt + (\sigma^\theta) \; dZ_t, \quad \text{also} \; c_t = \rho n_t \]
Evolution of wealth distribution

• Given risk, price of risk, and consumption rates, we can write the law of motion of agents’ wealth and wealth distribution

\[
\frac{dn_t}{n_t} = r^F_t dt + \sigma^n_t (\xi_t dt + dZ_t) - \frac{c_t}{n_t}
\]

• We can
  • write this in any numeraire
  • aggregate this (take weighted average)
  • get law of motion of wealth shares \( \eta \) - using total wealth as numeraire, or from Ito’s formula

\[
\frac{dX_t}{X_t} = \mu^X_t dt + \sigma^X_t dZ_t, \quad \frac{dY_t}{Y_t} = \mu^Y_t dt + \sigma^Y_t dZ_t \quad \Rightarrow \\
\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu^X_t - \mu^Y_t + \sigma^Y_t (\sigma^Y_t - \sigma^X_t)) dt + (\sigma^X_t - \sigma^Y_t) dZ_t
\]
Allocation

- “Price-taking planner:” assets are allocated to maximize their risk-adjusted return
- Price of risk can depend on who holds the asset…
Back to the problem...

• Households + experts, \( a = a \)

• Capital

\[
\frac{dk_t}{k_t} = (\Phi(t_t) - \delta) \, dt + \sigma \, dZ_t + \varphi \, \tilde{\sigma} \, d\tilde{Z}_t
\]

• Households

\[
E \left[ \int_0^\infty e^{-\rho t} \log c_t \, dt \right]
\]

• Experts

\[
E \left[ \int_0^\infty \theta_t \log c_t \, dt \right], \quad \theta_0 = 1, \quad \frac{d\theta_t}{\theta_t} = -\rho \, dt + \sigma^\theta \, dZ_t
\]

• Risk \( dZ \) is tradable (price of risk \( c_t \))

• Let’s characterize **equilibrium**: law of motion of \( \eta \), allocation of capital
Question 1

• Given $\zeta_t$, what is the households choice of $\sigma_t^n$? What about $\sigma_t^n$ for experts? ($\kappa$ is the fraction of capital held by experts)

(a) $\sigma_t^n = (1-\kappa)/(1-\eta) (\sigma + \sigma^q)$
(b) $\sigma_t^n = (1-\kappa)/(1-\eta) \sigma$
(c) $\sigma_t^n = \zeta_t$
(d) $\sigma_t^n = \zeta_t$
(e) More than one answer above is correct
• Given $c_t$, what is the households choice of $\sigma_t^n$? What about $\sigma_t^n$ for experts?

• For HH, $\sigma_t^n = c_t$ (standard log utility). For experts, $\text{MU} = \theta_t/c_t$, volatility $\sigma^\theta - \sigma_t^c = \sigma^\theta - \sigma_t^n$. This has to price risky asset, so

$$c_t = - (\sigma^\theta - \sigma_t^n), \quad \sigma_t^n = c_t + \sigma^\theta$$
Question 2

(2) Given the fraction $\kappa_t$ of capital held by entrepreneurs, what are the prices of idiosyncratic risk $\tilde{\zeta}_t$ and $\tilde{\zeta}_t$ for entrepreneurs and households?

(a) $\tilde{\zeta}_t = \frac{\kappa_t}{\eta_t} \tilde{\sigma}$ \quad $\tilde{\zeta}_t = \frac{1-\kappa_t}{1-\eta_t} \tilde{\sigma}$.

(b) $\tilde{\zeta}_t = \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}$ \quad $\tilde{\zeta}_t = \frac{1-\kappa_t}{1-\eta_t} \tilde{\sigma}$.

(c) $\tilde{\zeta}_t = \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}$ \quad $\tilde{\zeta}_t = \frac{1-\kappa_t}{1-\eta_t} \phi \tilde{\sigma}$.

(d) $\tilde{\zeta}_t = \tilde{\zeta}_t = \phi \tilde{\sigma}$.
Question 3

(3) As a function of the entrepreneurs’ wealth share $\eta_t$, determine the fraction of capital $\kappa_t$ that entrepreneurs hold in equilibrium.

• Hint: think about the price-taking planner:

$$\max E[\text{return}] - \text{cost of risk}$$
(3) As a function of the entrepreneurs’ wealth share \( \eta_t \), determine the fraction of capital \( \kappa_t \) that entrepreneurs hold in equilibrium.

\[
\phi \tilde{\sigma} \frac{\kappa_t}{\eta_t} \bigg( \frac{\phi \tilde{\sigma}}{\zeta_t} \bigg) = \tilde{\sigma} \frac{1 - \kappa_t}{1 - \eta_t} \bigg( \frac{\phi \tilde{\sigma}}{\zeta_t} \bigg) \quad \Rightarrow \quad \kappa = \frac{\eta}{\eta + (1 - \eta)\phi^2}
\]
(4) If the risk-free rate is $r_t^F$, which of the following expressions gives the law of motion of experts’ wealth $N_t$?

(a) $\frac{dN_t}{N_t} = \left( r_t^F + (\zeta_t + \sigma^\theta)\zeta_t + \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}^2 \right) dt + (\zeta_t + \sigma^\theta) dZ_t.$

(b) $\frac{dN_t}{N_t} = \left( r_t^F - \rho + (\zeta_t + \sigma^\theta)\zeta_t + \frac{\kappa_t^2}{\eta_t^2} \phi^2 \tilde{\sigma}^2 \right) dt + (\zeta_t + \sigma^\theta) dZ_t.$

(c) $\frac{dN_t}{N_t} = \left( r_t^F - \rho + (\zeta_t + \sigma^\theta)^2 + \frac{\kappa_t^2}{\eta_t^2} \phi^2 \tilde{\sigma}^2 \right) dt + (\zeta_t + \sigma^\theta) dZ_t.$

(d) $\frac{dN_t}{N_t} = \left( r_t^F - \rho + \frac{\kappa_t}{\eta_t} \sigma \zeta_t + \frac{\kappa_t}{\eta_t} \phi \tilde{\sigma}^2 \right) dt + \frac{\kappa_t}{\eta_t} \sigma dZ_t.$
Question 5

(5) If the risk-free rate is $r_t^F$, which of the following expressions gives the law of motion of total wealth $q_tK_t$?

(a) $\frac{d(q_tK_t)}{q_tK_t} = \frac{a-\nu_t}{q_t} \ dt + (\mu_t^q + \Phi(\nu_t) - \delta + \sigma \sigma_t^q) \ dt + (\sigma + \sigma_t^q) \ dZ_t - \rho \ dt$

(b) $\frac{d(q_tK_t)}{q_tK_t} = (\mu_t^q + \Phi(\nu_t) - \delta + \sigma \sigma_t^q) \ dt + (\sigma + \sigma_t^q) \ dZ_t$

(c) $\frac{d(q_tK_t)}{q_tK_t} = \left( r_t^F - \rho + (\zeta_t + \eta \sigma^\theta) \zeta_t + \frac{\kappa_t^2}{\eta_t} \phi^2 \tilde{\sigma}^2 + \frac{(1-\kappa_t)^2}{1-\eta_t} \tilde{\sigma}^2 \right) \ dt + (\zeta_t + \eta \sigma^\theta) dZ_t$

(d) all of the above

(e) (a) and (b) but not (c)

- Which equation should we use to derive the law of motion of $\eta$?
• Derive the law of motion of $\eta$. You can use

$$\frac{dN_t}{N_t} = \left( r_t^F - \rho + (\zeta_t + \sigma^\theta) \zeta_t + \frac{\kappa_t^2}{\eta_t^2} \phi_t^2 \bar{\sigma}^2 \right) dt + (\zeta_t + \sigma^\theta) dZ_t.$$ 

$$\frac{d(q_t K_t)}{q_t K_t} = \left( r_t^F - \rho + (\zeta_t + \eta \sigma^\theta) \zeta_t + \frac{\kappa_t^2}{\eta_t} \phi_t^2 \bar{\sigma}^2 + \frac{(1 - \kappa_t)^2}{1 - \eta_t} \left( \frac{\kappa_t}{\eta_t} \phi_t^2 \bar{\sigma}^2 \right)^2 \right) dt + (\zeta_t + \eta \sigma^\theta) dZ_t$$

and

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X)) dt + (\sigma_t^X - \sigma_t^Y) dZ_t$$

and

$$\kappa = \frac{\eta}{\eta + (1 - \eta) \phi^2}$$
Question 6

\[ \frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + \sigma_t^Y (\sigma_t^Y - \sigma_t^X)) \, dt + (\sigma_t^X - \sigma_t^Y) \, dZ_t \]

- Derive the law of motion of \( \eta \). You can use

\[ \frac{dN_t}{N_t} = \left( r_t^F - \rho + (\varsigma_t + \sigma^\theta) \varsigma_t + \frac{\kappa_t^2}{\eta_t^2} \phi^2 \tilde{\sigma}^2 \right) dt + (\varsigma_t + \sigma^\theta) \, dZ_t \]

\[ \frac{d(q_t K_t)}{q_t K_t} = \left( r_t^F - \rho + (\varsigma_t + \eta \sigma^\theta) \varsigma_t + \frac{\kappa_t^2}{\eta_t} \phi^2 \tilde{\sigma}^2 + \frac{(1 - \kappa_t)^2}{1 - \eta_t} \tilde{\sigma}^2 \right) dt + (\varsigma_t + \eta \sigma^\theta) \, dZ_t \]

\[ \frac{d\eta_t}{\eta_t} = \left( 1 - \eta \right) \frac{\phi^2 (1 - \phi^2) \tilde{\sigma}^2}{(\eta + (1 - \eta) \phi^2)^2} + \eta (\eta - 1) (\sigma^\theta)^2 \right) + (1 - \eta) \sigma^\theta \, dZ_t \]
Question 7

• Under what conditions does the stationary distribution of $\eta$ exist?

• From 6:

$$\frac{\mu^n \eta}{(\sigma^n \eta)^2} = \left( \frac{1}{1 - \eta} + \frac{1}{\eta} \right) \frac{\phi^2(1 - \phi^2)}{(\eta + (1 - \eta)\phi^2)^2} \frac{\tilde{\sigma}^2}{(\sigma^\theta)^2} - \frac{1}{1 - \eta}$$

• You can use the equation for density from KFE

$$g(\eta) = \text{const} \frac{1}{(\sigma^n \eta)^2} \exp \left( \int \frac{2\mu^n \eta}{(\sigma^n \eta)^2} d\eta \right)$$

• $g(\eta)$ must integrate to a finite number near 0 and 1