The 2020 Princeton Initiative.
Problem Set 2.
Saturday, September 5, 10:30am Eastern Time.

Problem. Consider an economy with two types of agents: entrepreneurs and households. When managed by households, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \tilde{\sigma}\, d\tilde{Z}_t,$$

where shocks dZ_t are aggregate and $d\tilde{Z}_t$ are idiosyncratic (agent-specific, and canceling out in the aggregate). When managed by entrepreneurs, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \phi\tilde{\sigma}\, d\tilde{Z}_t,$$

where $\phi \in (0, 1)$. That is, entrepreneurs are able to diversify and reduce their idiosyncratic risk exposure.

Capital k_t produces output ak_t if managed by any agent, so after investment of ι_t, the remaining output that can be consumed (or traded) is given by $(a - \iota_t)k_t$. Assume that the investment function $\Phi(\iota)$ satisfies $\Phi(0) = 0$, $\Phi > 0$ and $\Phi'' \leq 0$. Denote the aggregate supply of capital by K_t and the price of capital per unit by q_t.

Assume that aggregate risk dZ_t can be traded between the households and entrepreneurs and carries the equilibrium price of risk of ς_t. Denote the price of idiosyncratic risk of entrepreneurs by $\tilde{\varsigma}_t$ and that of households by $\underline{\tilde{\varsigma}}_t$. Thus, the budget constraint/law of motion of wealth of any entrepreneur is given by

$$\frac{dn_t}{n_t} = (r_t^F + \varsigma_t \sigma_t^n + \tilde{\varsigma}_t \tilde{\sigma}_t^n)\, dt + \sigma_t^n\, dZ_t + \tilde{\sigma}_t^n\, d\tilde{Z}_t - \frac{c_t}{n_t}\, dt,$$

where r_t^F is the equilibrium risk-free rate and c_t is the entrepreneur's consumption. The budget constraint of a household is written similarly. Agents control their idiosyncratic risk exposure $\tilde{\sigma}_t^n \geq 0$ by choosing how much capital to hold, and aggregate risk exposure σ_t^n by also trading derivatives. Agents cannot use derivatives to insure idiosyncratic risk. Aggregate-risk derivatives and the risk-free asset are in zero net supply.

Assume that households have utility

$$E\left[\int_0^\infty e^{-\rho t} \log \underline{c}_t\, dt\right],$$

where $\rho > 0$ is a discount rate and $\underline{c}_t$ is the consumption of a household. Entrepreneurs have utility

$$E\left[\int_0^\infty \theta_t \log c_t \, dt\right],$$

where θ_t is the process of discounting and *taste shocks* that follows

$$\theta_0 = 1, \quad \frac{d\theta_t}{\theta_t} = -\rho \, dt + (\sigma^\theta) \, dZ_t,$$

where ρ is the same discount rate as that of households and $(\sigma^\theta) \geq 0$ is an exogenous parameter. In order to solve this problem, you can rely (without proof) on the fact that the optimal consumption rate (of entrepreneurs as well) is given by the discount rate ρ times net worth.

The goal of this problem is to characterize the equilibrium in this environment.

(1) Given ς_t, what is the households' choice aggregate risk exposure $\underline{\sigma}^n_t$? How about that of entrepreneurs, σ^n_t?

(2) Given the fraction κ_t of capital held by entrepreneurs, what are the prices of idiosyncratic risk $\tilde{\varsigma}_t$ and $\tilde{\underline{\varsigma}}_t$ for entrepreneurs and households?

(3) As a function of the entrepreneurs' wealth share η_t, determine the fraction of capital κ_t that entrepreneurs hold in equilibrium.

(4) If the risk-free rate is r^F_t, which of the following expressions gives the law of motion of experts' wealth N_t?

$$(a) \quad \frac{dN_t}{N_t} = \left(r^F_t + (\varsigma_t + (\sigma^\theta))\varsigma_t + \frac{\kappa_t}{\eta_t}\phi\tilde{\sigma}^2\right) dt + (\varsigma_t + (\sigma^\theta)) \, dZ_t.$$

$$(b) \quad \frac{dN_t}{N_t} = \left(r^F_t - \rho + (\varsigma_t + (\sigma^\theta))\varsigma_t + \frac{\kappa_t^2}{\eta_t^2}\phi^2\tilde{\sigma}^2\right) dt + (\varsigma_t + (\sigma^\theta)) \, dZ_t.$$

$$(c) \quad \frac{dN_t}{N_t} = \left(r^F_t - \rho + (\varsigma_t + (\sigma^\theta))^2 + \frac{\kappa_t^2}{\eta_t^2}\phi^2\tilde{\sigma}^2\right) dt + (\varsigma_t + (\sigma^\theta)) \, dZ_t.$$

$$(d) \quad \frac{dN_t}{N_t} = \left(r^F_t - \rho + \frac{\kappa_t}{\eta_t}\sigma\varsigma_t + \frac{\kappa_t}{\eta_t}\phi\tilde{\sigma}^2\right) dt + \frac{\kappa_t}{\eta_t}\sigma \, dZ_t.$$

(5) If the risk-free rate is r_t^F, which of the following expressions gives the law of motion of total wealth $q_t K_t$?

(a) $\frac{d(q_t K_t)}{q_t K_t} = \frac{a-\iota_t}{q_t}\,dt + (\mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^q)\,dt + (\sigma + \sigma_t^q)\,dZ_t - \rho\,dt$

(b) $\frac{d(q_t K_t)}{q_t K_t} = (\mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^q)\,dt + (\sigma + \sigma_t^q)\,dZ_t$

(c) $\frac{d(q_t K_t)}{q_t K_t} = \left(r_t^F - \rho + (\varsigma_t + \eta(\sigma^\theta))\varsigma_t + \frac{\kappa_t^2}{\eta_t}\phi^2\tilde{\sigma}^2 + \frac{(1-\kappa_t)^2}{1-\eta_t}\tilde{\sigma}^2\right)\,dt + (\varsigma_t + \eta(\sigma^\theta))\,dZ_t$

(d) all of the above

(e) (a) and (b) but not (c)

(6) Derive the law of motion of η_t. Please express your answer in terms of model primitives (no endogenous variables). Recall that for processes

$$dX_t/X_t = \mu_t^X\,dt + \sigma_t^X\,dZ_t \quad \text{and} \quad dY_t/Y_t = \mu_t^Y\,dt + \sigma_t^Y\,dZ_t,$$

we have

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + \sigma_t^Y(\sigma_t^Y - \sigma_t^X))\,dt + (\sigma_t^X - \sigma_t^Y)\,dZ_t.$$

(7) Under what conditions does the stationary distribution of η exist? You may use directly the fact (which you can derive from part (6)) that

$$\frac{\mu^\eta \eta}{(\sigma^\eta \eta)^2} = \left(\frac{1}{1-\eta} + \frac{1}{\eta}\right)\frac{\phi^2(1-\phi^2)}{(\eta + (1-\eta)\phi^2)^2}\frac{\tilde{\sigma}^2}{(\sigma^\theta)^2} - \frac{1}{1-\eta}.$$

Hint: you have to make sure that the stationary density integrates to a finite number both near $\eta = 0$ and 1.

You can also determine the stationary density (up to an integration constant) in closed form, but we will not do it in class.