Modern Macro, Money, and International Finance
Eco529
Lecture 05: Endogenous Risk Dynamics in Real Macro Model with Heterogenous Agents

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents
1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models
1. A Simple Money Model
2. Cashless vs. Cash Economy and “The I Theory of Money”
3. Welfare Analysis & Optimal Policy
   1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models
1. International Financial Architecture

Digital Money
Risk premia, price of risk

- Risk premia = price of risk * (endogenous + **exogenous risk**)
  - Exogenous risk – shock from outside
  - Endogenous risk
    - Amplification: adverse feedback loops
    - Multiple equilibria: Run, Sudden Stops

- Non-linearities are key for financial stability
  - Around vs. away from steady state
Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- (“Net worth trap” double-humped stationary distribution)
Persistence Leads to Dynamic Amplification

- *Static* amplification occurs because fire-sales of capital from productive sector to less productive sector depress asset prices
  - Importance of *market liquidity* of physical capital
- *Dynamic* amplification occurs because a temporary shock translates into a persistent decline in output and asset prices
  - Forward *grow net worth*
  - Backward *asset pricing*
“Single Shock Critique”

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed

- In reality an adverse shock may be followed by additional adverse shocks
  - Build-up extra safety cushion for an additional shock in a crisis

- Impulse response vs. volatility dynamics
Endogenous Volatility & Volatility Paradox

- **Endogenous Risk/Volatility Dynamics in BruSan**
  - Beyond Impulse responses
  - Input: constant volatility
  - Output: endogenous risk time-varying volatility

⇒ Precautionary savings
  - Role for money/safe asset
    - *Later: in Money lecture*

⇒ Nonlinearities in crisis ⇒ endogenous fat tails, skewness

- **Volatility Paradox**
  - Low exogenous (measured) volatility leads to high build-up of (hidden) endogenous volatility (Minsky)
Two Type/Sector Model with Outside Equity

- **Expert sector**

- **Household sector**

- **Skin in the Game Constraint:**
  Experts must hold fraction \( \chi^e_t \geq \alpha \kappa^e_t \) of aggregate capital risk with \( \alpha \in (0,1) \) (\( \kappa^e_t > \kappa^e_t \) never happens in equilibrium)

- Return on inside equity \( N_t \) can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

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BruSan 2017: Handbook of Macroeconomics, Lecture Notes, Chapter 3
Financial Frictions and Distortions  UPDATE!

- Skin in the game constraint
  - Retain certain fraction of risk

- Incomplete markets
  - “natural” leverage constraint \((\text{BruSan})\)
  - Costly state verification \((\text{BGG})\)

- + Leverage constraints
  (no “liquidity creation”)
  - Exogenous limit \((\text{Bewley/Ayagari})\)

- Collateral constraints
  - Next period’s price \((\text{KM})\)
    \[
    Rb_t \leq q_{t+1} k_t
    \]
  - Next periods volatility \((\text{VaR, JG})\)
  - Current price
Two Type Model Setup

Expert sector

- Output: \( y_t^e = a^e k_t^e \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \nu_t \)
- Capital share of experts:

\[
(\kappa) = \kappa^e a^e + (1 - \kappa^e) a^h
\]

Household sector

- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \nu_t \)
- Capital share of experts:

Poll 10: Why is it important that households can hold capital?

a) to capture fire-sales
b) for households to speculate
c) to obtain stationary distribution
Two Type Model Setup

**Expert sector**
- Output: \( y_t^e = a^e k_t^e \) \( a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \ell_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = (\Phi(\ell_t^{i,e}) - \delta)dt + \sigma dZ_t + d\Delta k_t^{i,e}
\]

**Household sector**
- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \ell_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = (\Phi(\ell_t^{i,h}) - \delta)dt + \sigma dZ_t + d\Delta k_t^{i,h}
\]

Physical capital evolution absent market transactions/fire-sales
Two Type Model Setup

**Expert sector**
- Output: \( y_t^e = a^e k_t^e \) \( a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( \ell_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = (\Phi(\ell_t^{i,e}) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,e}
\]

**Household sector**
- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( \ell_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = (\Phi(\ell_t^{i,h}) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,h}
\]

Poll 12: What are the modeling tricks to obtain stationary distribution?

a) switching types
b) agents die, OLG/perpetual youth models (without bequest motive)
c) different preference discount rates
Two Type Model Setup

**Expert sector**

- **Output:** \( y_t^e = a^e k_t^e \) \( a^e \geq a^h \)
- **Consumption rate:** \( c_t^e \)
- **Investment rate:** \( \dot{l}_t^e \)

\[
\frac{d k_t^{i,e}}{k_t^{i,e}} = (\Phi(l_t^{i,e}) - \delta) dt + \sigma dZ_t + d\Delta_{t}^{k,e}
\]

\[
E_0\left[ \int_0^\infty e^{-\rho^e t} \left( \frac{c_t^e}{1-\gamma} \right)^{1-\gamma} dt \right] \quad \rho^e \geq \rho^h
\]

**Household sector**

- **Output:** \( y_t^h = a^h k_t^h \)
- **Consumption rate:** \( c_t^h \)
- **Investment rate:** \( \dot{l}_t^h \)

\[
\frac{d k_t^{i,h}}{k_t^{i,h}} = (\Phi(l_t^{i,h}) - \delta) dt + \sigma dZ_t + d\Delta_{t}^{k,h}
\]

\[
E_0\left[ \int_0^\infty e^{-\rho^h t} \left( \frac{c_t^h}{1-\gamma} \right)^{1-\gamma} dt \right]
\]
Two Type Model Setup

**Expert sector**
- Output: \( y_t^e = a^e k_t^e \quad a^e \geq a^h \)
- Consumption rate: \( c_t^e \)
- Investment rate: \( i_t^e \)

\[
\frac{dk_t^{i,e}}{k_t^{i,e}} = (\Phi(i_t^{i,e}) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,e}
\]

\[
E_0\left[ \int_0^\infty e^{-\rho^e t}\frac{c_t^{e(1-\gamma)}}{1-\gamma}dt \right] \quad \rho^e \geq \rho^h
\]

**Friction:** Can only issue
- Risk-free debt
- Equity, but must hold \( \chi_t^e \geq \alpha \kappa_t \)

**Household sector**
- Output: \( y_t^h = a^h k_t^h \)
- Consumption rate: \( c_t^h \)
- Investment rate: \( i_t^h \)

\[
\frac{dk_t^{i,h}}{k_t^{i,h}} = (\Phi(i_t^{i,h}) - \delta)dt + \sigma dZ_t + d\Delta_t^{k,h}
\]

\[
E_0\left[ \int_0^\infty e^{-\rho^h t}\frac{c_t^{h(1-\gamma)}}{1-\gamma}dt \right]
\]
Recall Previous Lecture: HH can’t hold capital or equity

\[ a = .11, \rho = 5\%, \sigma = .1, \Phi(i) = \frac{\log \phi + 1}{\phi}, \phi = 10 \]
Preview of new, extended model

- Price of capital

\[ \kappa^e \leq 1 \quad \kappa^e = 1 \]

Parameters:
\[ \rho_e = 0.06, \rho^h = 0.05, a^e = 0.11, a^h = 0.03, \]
\[ \delta = 0.05, \sigma = 0.01, \alpha = 0.50, \gamma = 2, \phi = 10 \]

- Amplification

\[ \kappa^e \leq 1 \quad \kappa^e = 1 \]
Drift and Volatility of $\eta^e$

"Steady state" $\eta^*$,

$\eta^e = \alpha \kappa^e$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$, **finance block**
   a. Real investment $\iota +$ Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta +$ Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) **forward equation**

3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta) u(K) (n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
The Big Picture

allocation of physical assets

output $A(\kappa)$

consumption + investment

$\kappa$

$A$ $L$

accumulation Debt Outside equity

risk amplification

price of risk $\zeta$

value function

precautionary

drift

net worth distribution $\eta$

volatility

drift

Backward equation Forward equation with expectations

$\chi \geq \chi$

$A(\kappa)$

capital growth $\Phi(\iota) - \delta$

$\nu$

Federal Reserve

$\mu$

$\nu$

$\zeta$

$\eta$

Drift

Volatility
1a. Individual Agent Choice of \( \iota, \theta, c \)

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

\[
\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma \sigma^q_t - r_t = \left[ \zeta^e_t \chi^e_t / \kappa^e_t + \zeta^h_t (1 - \chi^e_t / \kappa^e_t) \right] (\sigma + \sigma^q) 
\]

new compared to Basak-Cuoco

- Of households' capital choice

\[
\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma \sigma^q_t - r_t \leq \zeta^h_t (\sigma + \sigma^q) 
\]

with equality if \( \kappa^e_t < 1 \)

- Note: alternative approach replaces this step with Fisher Separation Social Planners' choice (see Lecture Notes)
1b. Asset/Risk Allocation across \( I \) Types

- Sketch of Proof of Theorem

1. **Fisher Separation Theorem**: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent \((i, \tilde{\iota})\) portfolio maximization is equivalent to the maximization problem of a firm

\[
\max_{\{\theta^j, i\}} E_t \left[ \frac{d r^{n(i, \tilde{i})}}{d t} \right] - \zeta \sigma^{r_n}
\]

\[
d r^{n(i, \tilde{i})} = \sum_j \theta^{j,i} d r^j = \sum_j \theta^{j,i} E[d r^j] + \sum_j \theta^{j,i} \sigma^j d Z_t
\]

- is linear in \( \theta \)s
  - Either bang-bang solution for \( \theta \)s s.t. portfolio constraints bind
  - Or prices/returns/risk premia are s.t. that firm is indifferent

2. **Aggregate**
   - Taking \( \eta \)-weighted sum to obtain return on aggregate wealth

3. Use market clearing to relate \( \theta \)s to \( \kappa \)s & \( \chi \)s (incl. \( \theta \)-constraint)
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      - Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      - Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      - Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)
   - forward equation

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
**Toolbox 3: Change of Numeraire**

- $x_t^A$ is a value of a self-financing strategy/asset in $\$
- $Y_t$ price of € in $\$$(exchange rate)$
  
  \[
  \frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t
  \]

- $x_t^A/Y_t$ value of the self-financing strategy/asset in €
  
  \[
  e^{-\rho t} u'(c_t) Y_t \frac{x_t^A}{Y_t} \text{ follows a martingale}
  \]

Recall $\mu_t^A - \mu_t^B = (-\sigma_t^\xi)(\sigma^A - \sigma_t^B)$

\[
\mu_t^{A/Y} - \mu_t^{B/Y} = (-\sigma_t^\xi - \sigma_t^{Y}) (\sigma^A - \sigma_t^{'Y} - \sigma_t^B + \sigma_t^{Y})
\]

- Price of risk $\xi^€ = \xi^\$ - \sigma^Y$

Poll 23: Why does the price of risk change, though real risk remains the same
- a) because risk-free rate might not stay risk-free
- b) because covariance structure changes
Toolbox 3: Change of Numeraire

- $x_t^A$ is a value of a self-financing strategy/asset in $\$
- $Y_t$ price of € in $\$\$ (exchange rate)

\[
\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t
\]

- $x_t^A / Y_t$ value of the self-financing strategy/asset in €

\[
e^{-\rho t} u'(c_t) \frac{Y_t^x_A}{Y_t} \text{ follows a martingale}
\]

Recall $\mu_t^A - \mu_t^B = \left( -\sigma_t^x \right) \left( \sigma_t^A - \sigma_t^B \right)$

\[
\mu_t^{A/Y} - \mu_t^{B/Y} = \left( -\sigma_t^x - \sigma_t^Y \right) \left( \sigma_t^A - \sigma_t^Y - \sigma_t^B + \sigma_t^Y \right)
\]

- Price of risk $\xi^€ = \xi^$ - $\sigma^Y$
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$  
   a. Real investment $\iota$ + Goods market clearing (static)  
      - *Toolbox 1*: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or 
      Asset allocation $\kappa$ & risk allocation $\chi$
      - *Toolbox 2*: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\theta$
      - *Toolbox 3*: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$)  
   - *forward equation*
   - *backward equation*

3. Value functions
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      - *Special cases*: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $z$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
2. GE: Markov States and Equilibria

- Equilibrium is a map of histories of shocks on prices:
  \[ \{Z_s, s \in [0, t]\} \rightarrow q_t, \zeta^i_t, \iota^i_t, \theta^i_t \]

- Net worth distribution:
  \[ \eta^e_t = \frac{N^e_t}{q_t K_t} \in (0, 1) \]

- All agents maximize utility:
  - Choose: portfolio, consumption, technology

- All markets clear:
  - Consumption, capital, money, outside equity
2. Law of Motion of Wealth Share $\eta_t$

**Method 1:** Using Ito’s quotation rule $\eta_t^i = N_t^i / (q_t K_t)$

- Recall
  \[
  dN_t^i = r_t N_t^i dt + \frac{x_t\nu_t^i}{\eta_t^i} (\sigma + \sigma_t^q) \quad \text{price of risk}
  \]

- \[
  \frac{d\eta_t^i}{\eta_t^i} = \ldots \quad \text{(lots of algebra)}
  \]

**Method 2:** Change of numeraire + Martingale Approach

- New numeraire: Total wealth in the economy, $N_t$
- Apply Martingale Approach for value of $i$’s portfolio
  - Simple algebra to obtain drift of $\eta_t^i$: $\mu_t^\eta_t^i$
  - Note that change of numeraire does not affect ratio $\eta_t^i$!
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- **New Numeraire**
  - “Total net worth” in the economy, $N_t$ (without superscript)
  - Type $i$’s portfolio net worth = net worth share

- **Martingale Approach with new numeraire**
  - Asset $A = i$’s portfolio return in terms of total wealth,
    \[
    \left( \frac{C_t^i}{N_t^i} + \mu_t^\eta^i \right) dt + \sigma_t^\eta^i dZ_t + \tilde{\sigma}_t^\eta^i d\tilde{Z}_t
    \]
  - Asset $B$ (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))
    \[
    r_t^m dt + \sigma_t^m dZ_t
    \]
  - Apply our martingale asset pricing formula
    \[
    \mu_t^A - \mu_t^B = (\sigma_t^A - \sigma_t^B)
    \]

Poll 28: Is risk-free asset, risk free in the new numeraire?
- a) Yes
- b) No
2. $\mu^\eta$ Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)
\[
\mu^\eta_t i + \frac{C^i_t}{N^i_t} - r^m_t = (\zeta^i_t - \sigma^N_t) \left(\sigma^\eta_t i - \sigma^m_t \right)
\]

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)
\[
\sum_{i'} \eta^i_{t'} \mu^i_{t'} + \frac{C_t}{N_t} - r^m_t = \sum_{i'} \eta^i_{t'} (\zeta^i_{t'} - \sigma^N_t) \left(\sigma^{\eta i'}_t - \sigma^m_t \right)
\]

- Poll 29: Why $= 0$?
  a) Because we have stationary distribution
  b) Because $\eta$s sum up to 1
  c) Because $\eta$s follow martingale

Benchmark asset everyone can trade
\[
\sigma^m_t = -\sigma^N_t
\]
2. $\mu^\eta$ Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)
  \[
  \mu_t^\eta + \frac{C_t^i}{N_t^i} - r_t^m = (\zeta_t^i - \sigma_t^N) (\sigma_t^\eta^i - \sigma_t^m)
  \]

- Add up across types (weighted),
  (capital letters without superscripts are aggregates for total economy)
  \[
  (\eta_t^e \mu_t^{\eta^e} + \eta_t^h \mu_t^{\eta^h}) + \frac{C_t}{N_t} - r_t^m = \\
  \eta_t^e (\zeta_t^e - \sigma_t^N) (\sigma_t^{\eta^e} - \sigma_t^m) + \eta_t^h (\zeta_t^h - \sigma_t^N) (\sigma_t^{\eta^h} - \sigma_t^m)
  \]

- Subtract from each other yield
  \[
  \mu_t^{\eta^e} = (1 - \eta_t^e) (\zeta_t^e - \sigma_t^N) (\sigma_t^{\eta^e} - \sigma_t^m) - (1 - \eta_t^e) (\zeta_t^h - \sigma_t^{N^h}) (\sigma_t^{\eta^h} - \sigma_t^m) \\
  - \left( \frac{C_t^e}{N_t^e} - \frac{C_t}{q_tK_t} \right)
  \]

For benchmark asset: risk-free debt
\[
\sigma_t^m = -\sigma_t^N
\]
2. $\sigma^\eta$ Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term)

- Since $\eta_t = N^e_t / N_t$,

$$
\sigma^\eta_t = \sigma^N_t - \sigma^N_t = \sigma^N_t - \sum_{i'} \eta_t^{i'} \sigma^{N_{i'}} = (1 - \eta_t^i) \sigma^{N_i} - \sum_{i' \neq i} \eta_t^{i'} \sigma^{N_{i'}}
$$

- Note for 2 types example

$$
\sigma^e_t = \frac{\chi^e_t / \eta^e_t = \theta_e, \kappa + \theta_e, \theta_e, O_E}{(\sigma + \sigma^q_t)}
\quad \sigma^h_t = \frac{\chi^h_t}{\eta^h_t} (\sigma + \sigma^q_t) = \frac{1 - \chi^e_t}{1 - \eta^e_t} (\sigma + \sigma^q_t)
$$

Hence,

$$
\sigma^e_t = \frac{\chi^e_t - \eta^e_t}{\eta^e_t} (\sigma + \sigma^q_t)
$$

- Note also, $\eta_t^e \sigma^e_t + \eta_t^h \sigma^h_t = 0 \Rightarrow \sigma^h_t = -\frac{\eta_t^e}{\eta_t^h} \sigma^e_t = \frac{\eta_t^e}{1 - \eta_t^e} \sigma^e_t$
2. $\sigma^\eta$ Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term): $\sigma_t^{X/Y} = \sigma_t^X - \sigma_t^Y$

- Since $\eta_t = N_t^e/N_t$, $N_t = q_t K_t$, by Ito ratio rule (only volatility term)

  $$\sigma_t^\eta = \sigma_t^{N^e} - \sigma_t^N = \sigma_t^{N^e} - \sigma_t^q - \sigma$$

- We also have

  $$\sigma_t^{N^e} = \frac{\chi_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Substituting this in previous formula yields

  $$\sigma_t^{\eta^e} = \frac{\chi_t^{e-\eta_t^e}}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h} \sigma_t^{\eta^e} = -\frac{\eta_t^e}{1-\eta_t^e} \sigma_t^{\eta^e}$
2. Amplification Formula: Loss Spiral

- Recall

\[ \sigma_t \eta^e = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} \left( \sigma + \sigma_t^q \right) \]

leverage

- By Ito’s Lemma on \( q(\eta^e) \)

\[ \sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t \]

- Total volatility

\[ \sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e) \chi_t^e - \eta_t^e}{q/\eta_t^e \eta_t^e}} \]

- Loss spiral
  - Market illiquidity (price impact elasticity)
2. Amplification Formula: Loss Spiral

- Recall

\[ \sigma_t^e = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} \left( \sigma + \sigma_t^q \right) \]

- Leverage

\[ \sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^e \]

- Total volatility

\[ \sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)\chi_t^e - \eta_t^e}{q/\eta_t^e}} \]

- Loss spiral

- Market illiquidity (price impact elasticity)

Poll 34: Where is the spiral?

a) Sum of infinite geometric series (denominator)
b) in \( q' \), since with constant price, no spiral
Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes

1. For given $C/N$-ratio and SDF processes for each $i$ finance block
   a. Real investment $\iota$ + Goods market clearing (static)
      ▪ Toolbox 1: Martingale Approach, HJB vs. Stochastic Maximum Principle Approach
   b. Portfolio choice $\theta$ + Asset market clearing or Asset allocation $\kappa$ & risk allocation $\chi$
      ▪ Toolbox 2: “price-taking social planner approach” – Fisher separation theorem
   c. “Money evaluation equation” $\vartheta$
      ▪ Toolbox 3: Change in numeraire to total wealth (including SDF)

2. Evolution of state variable $\eta$ (and $K$) forward equation

3. Value functions backward equation
   a. Value fcn. as fcn. of individual investment opportunities $\omega$
      ▪ Special cases: log-utility, constant investment opportunities
   b. Separating value fcn. $V^i(n^i; \eta, K)$ into $v^i(\eta)u(K)(n^i/n^i)^{1-\gamma}$
   c. Derive $C/N$-ratio and $\zeta$ price of risk

4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration

5. KFE: Stationary distribution, Fan charts
Solution

- Price of capital

Parameters:

\[ \rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03, \]
\[ \delta = .05, \sigma = .01, \alpha = .50, \gamma = 2, \phi = 10 \]
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .01, .05, .1$
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = .1, .2, .5$
  (skin the game constraint)
Market Liquidity

- Comparative static w.r.t. $a^h = 0.03, -0.03, -0.09$
From $\mu^e (\eta^e)$ & $\sigma^e (\eta^e)$ to Stationary Distribution

- Drift and Volatility of $\eta^e$

\[ \eta^e = \alpha \kappa^e \]
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5.
5. Kolmogorov Forward Equation

- Given an initial distribution \( f(\eta, 0) = f_0(\eta) \), the density diffusion follows PDE

\[
\frac{\partial f(\eta, t)}{\partial t} = \frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2} \frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}
\]

- “Kolmogorov Forward Equation” is in physics referred to as “Fokker-Planck Equation”

- Corollary: if stationary distribution \( f(\eta) \) exists, it satisfies the ODE

\[
0 = \frac{\partial [f(\eta, t)\mu(\eta)]}{\partial \eta} + \frac{1}{2} \frac{\partial^2 [f(\eta, t)\sigma^2(\eta)]}{\partial \eta^2}
\]
5. Stationary Distribution

- Stationary distribution of $\eta^e$

$$\eta^e = \alpha \kappa^e$$

Experts' skin in the game constraint binds $\chi^e_t = \alpha \kappa^e_t$

Perfect risk-sharing region (infeasible)

Poll 43: Is the constraint always (not just occasionally) binding

a) yes

b) no, only for some parameters $\rho^e > \rho^h$
5. Stationary Distribution

- Stationary distribution of $\eta^e$

\[ \eta^e = \alpha \kappa^e \]

Poll 44: What happens for $\rho^e = \rho^h$

- a) experts take over the economy, $\eta \to 1$
- b) there is a steady state at $\eta = \alpha$
5. Fan chart and distributional impulse response

- ... the theory to Bank of England’s empirical fan charts
- Starts at $\eta_0$, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32 \ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution
5. Fan chart and distributional impulse response

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock \( dZ_t = -2.32\, dt \) for a period of \( \Delta t = 1 \).
- Converges back to stationary distribution
5. Density Diffusion

- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32 \, dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution
5. Density Diffusion Movies
5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run

\[ \sigma = 0.15 \]