Modern Macro, Money, and International Finance

Eco529

Lecture 05: Endogenous Risk Dynamics in Real Macro Model with Heterogenous Agents

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Course Overview

Real Macro-Finance Models with Heterogeneous Agents

1. A Simple Real Macro-finance Model
2. Endogenous (Price of) Risk Dynamics
3. A Model with Jumps due to Sudden Stops/Runs

Money Models

1. A Simple Money Model
2. Cashless vs. Cash Economy and "The I Theory of Money"
3. Welfare Analysis & Optimal Policy
   1. Fiscal, Monetary, and Macroprudential Policy

International Macro-Finance Models

1. International Financial Architecture

Digital Money

Risk premia, price of risk

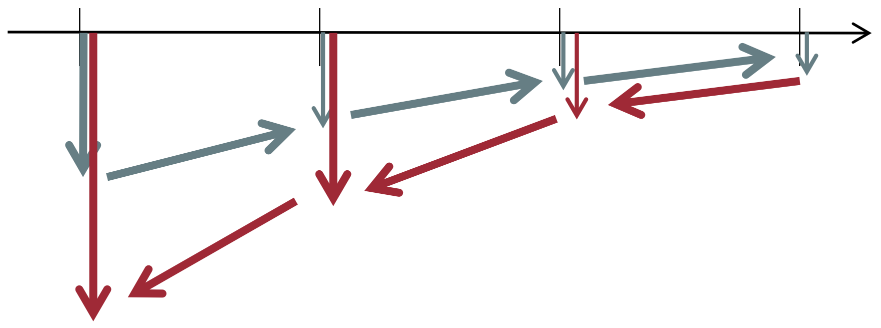
- Risk premia = price of risk * (endogenous + exogenous risk)
  - Exogenous risk – shock from outside
  - Endogenous risk
    - Amplification: adverse feedback loops
    - Multiple equilibria: Run, Sudden Stops
- Non-linearities are key for financial stability
  - Around vs. away from steady state

Desired Model Properties

- Normal regime: stable around steady state
  - Experts are adequately capitalized
  - Experts can absorb macro shock
- Endogenous risk and price of risk
  - Fire-sales, liquidity spirals, fat tails
  - Spillovers across assets and agents
  - Market and funding liquidity connection
  - SDF vs. cash-flow news
- Volatility paradox
- Financial innovation less stable economy
- (“Net worth trap” double-humped stationary distribution)

Persistence Leads to Dynamic Amplification

- *Static* amplification occurs because fire-sales of capital from productive sector to less productive sector depress asset prices
  - Importance of *market liquidity* of physical capital
- *Dynamic* amplification occurs because a temporary shock translates into a persistent decline in output and asset prices
  - Forward grow net worth
  - Backward asset pricing



“Single Shock Critique”

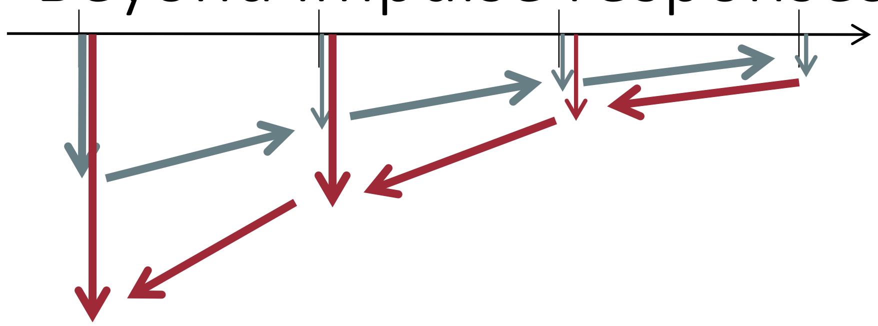
- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed

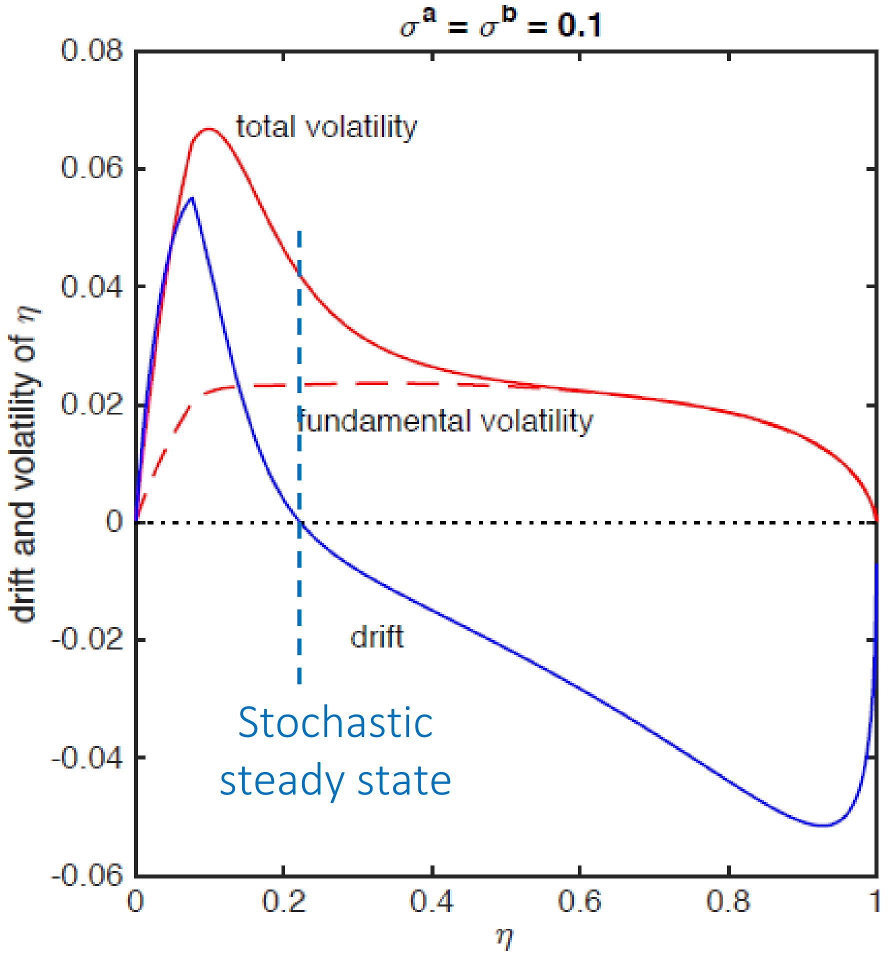
  - In reality an adverse shock may be followed by additional adverse shocks
    - Build-up extra safety cushion for an additional shock in a crisis

- Impulse response vs. volatility dynamics

Endogenous Volatility & Volatility Paradox

- Endogenous Risk/Volatility Dynamics in BruSan
  - Beyond Impulse responses





  - Input: constant volatility
  - Output: endogenous risk
    time-varying volatility

⇒Precautionary savings
  - Role for money/safe asset
    - *Later: in Money lecture*

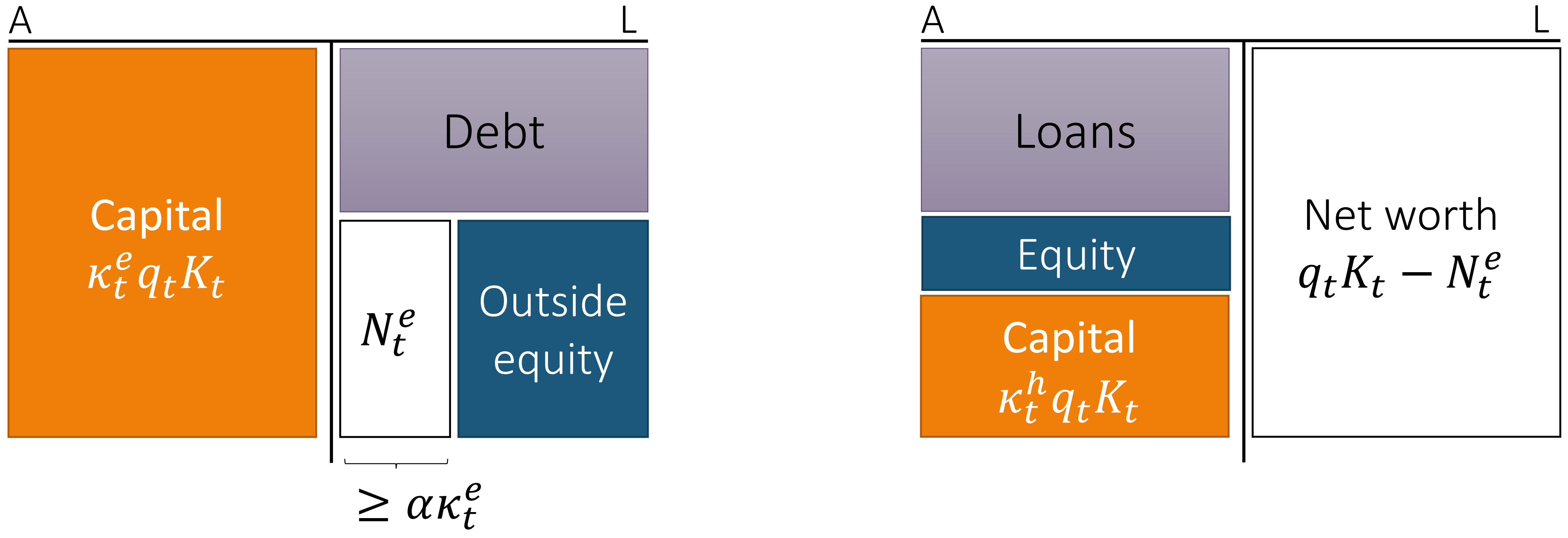
⇒ Nonlinearities in crisis ⇒ endogenous fait tails, skewness

- Volatility Paradox
  - Low exogenous (measured) volatility leads to
    high build-up of (hidden) endogenous volatility (Minksy)

Two Type/Sector Model with Outside Equity

BruSan 2017: Handbook of Macroeconomics, Lecture Notes, Chatper 3

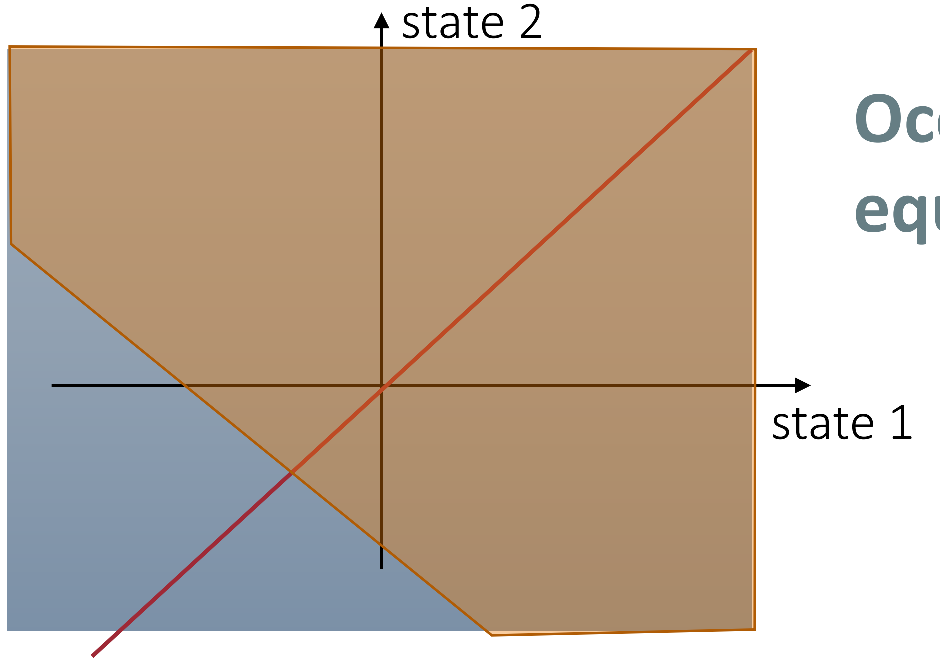
- Expert sector — Household sector



- Skin in the Game Constraint:
  Experts must hold fraction $\chi_t^e \geq \alpha \kappa_t^e$ of aggregate capital risk with $\alpha \in (0,1)$ ($\chi_t^e > \kappa_t^e$ never happens in equilibrium)
- Return on inside equity N_t can differ from outside equity
  - Issue outside equity at required return from HH
  - In related model, He and Krishnamurthy 2013 impose that inside and outside equity have same return

Financial Frictions and Distortions UPDATE!

- Skin in the game constraint
  - Retain certain fraction of risk



Occasionally binding equity constraint

- Incomplete markets
  - "natural" leverage constraint *(BruSan)*
  - Costly state verification *(BGG)*

- \+ Leverage constraints
  (no "liquidity creation")
  - Exogenous limit *(Bewley/Ayagari)*

  - Collateral constraints
    - Next period's price *(KM)*

      $$Rb_t \leq q_{t+1} k_t$$

    - Next periods volatility *(VaR, JG)*
    - Current price

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$

$a^e \geq a^h$

Household sector

- Output: $y_t^h = a^h k_t^h$

$$(\boldsymbol{\kappa}) = \kappa^e a^e + (1 - \kappa^e) a^h$$

Poll 10: Why is it important that households can hold capital?

a) to capture fire-sales

b) for households to speculate

c) to obtain stationary distribution

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,e}$$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,h}$$

Physical capital evolution absent market transactions/fire-sales

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,e}$$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,h}$$

Poll 12: What are the modeling tricks to obtain stationary distribution?

a) switching types

b) agents die, OLG/perpetual youth models (without bequest motive)

c) different preference discount rates

Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
- Consumption rate: c_t^e
- Investment rate: ι_t^e

$$\frac{dk_t^{\tilde{\imath},e}}{k_t^{\tilde{\imath},e}} = \left(\Phi\left(\iota_t^{\tilde{\imath},e}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,e}$$

- $E_0[\int_0^\infty e^{-\rho^e t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}dt]$ $\quad \rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,h}$$

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Two Type Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$ $\quad a^e \geq a^h$
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- $E_0[\int_0^\infty e^{-\rho^e t}\frac{(c_t^e)^{1-\gamma}}{1-\gamma}dt]$ $\quad \rho^e \geq \rho^h$

Household sector

- Output: $y_t^h = a^h k_t^h$
- Consumption rate: c_t^h
- Investment rate: ι_t^h

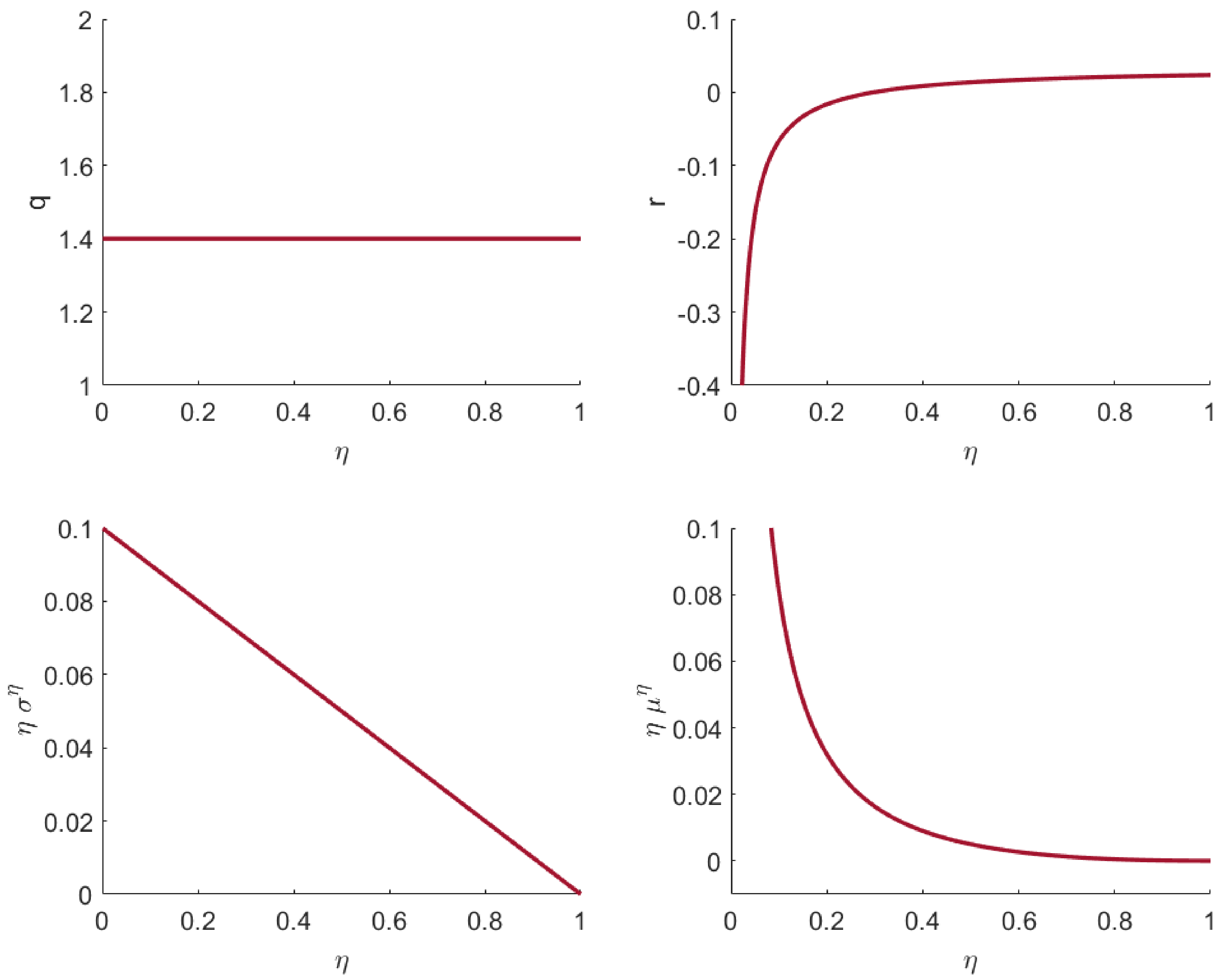
$$\frac{dk_t^{\tilde{\imath},h}}{k_t^{\tilde{\imath},h}} = \left(\Phi\left(\iota_t^{\tilde{\imath},h}\right) - \delta\right)dt + \sigma dZ_t + d\Delta_t^{k,h}$$

- $E_0[\int_0^\infty e^{-\rho^h t}\frac{(c_t^h)^{1-\gamma}}{1-\gamma}dt]$

Friction: Can only issue

- Risk-free debt
- Equity, but must hold $\chi_t^e \geq \alpha\kappa_t$

Recall Previous Lecture: HH can't hold capital or equity

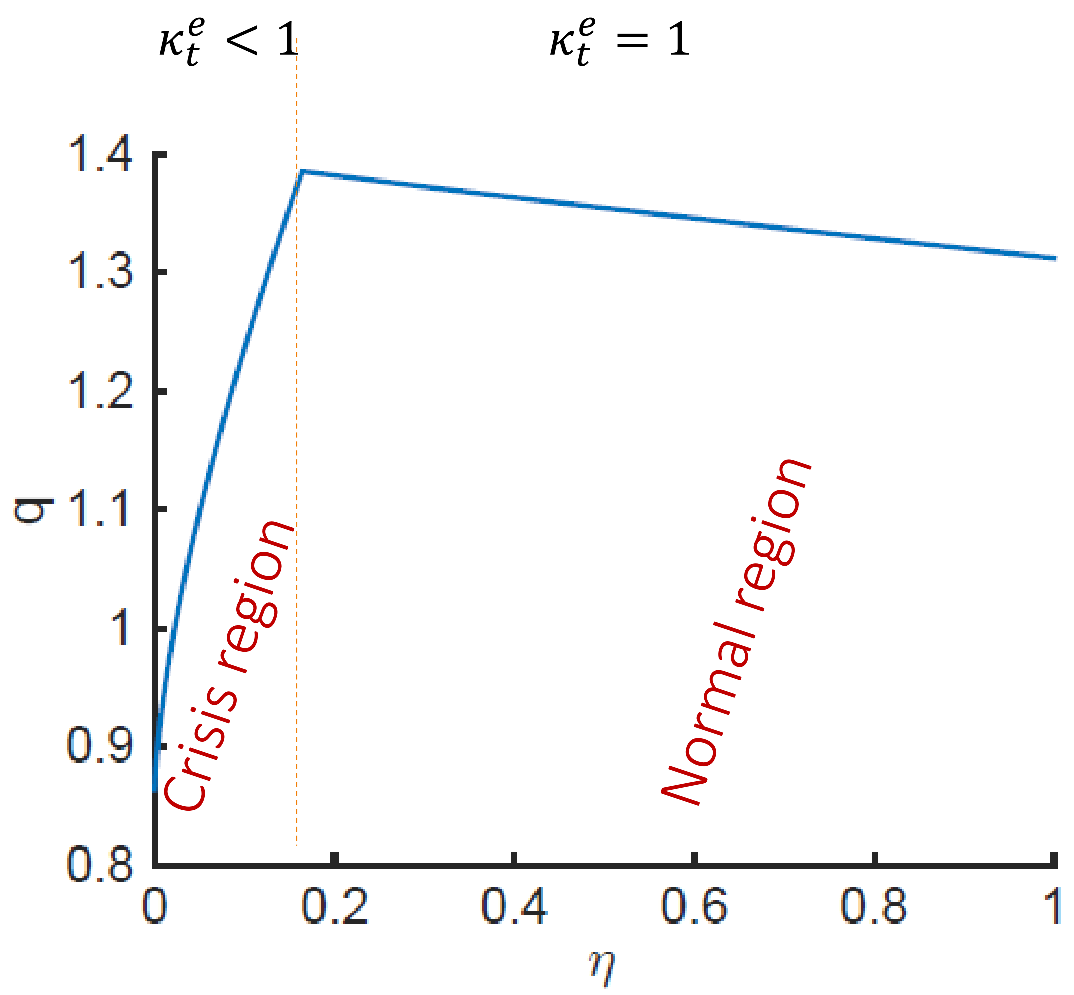


Basak-Cuco

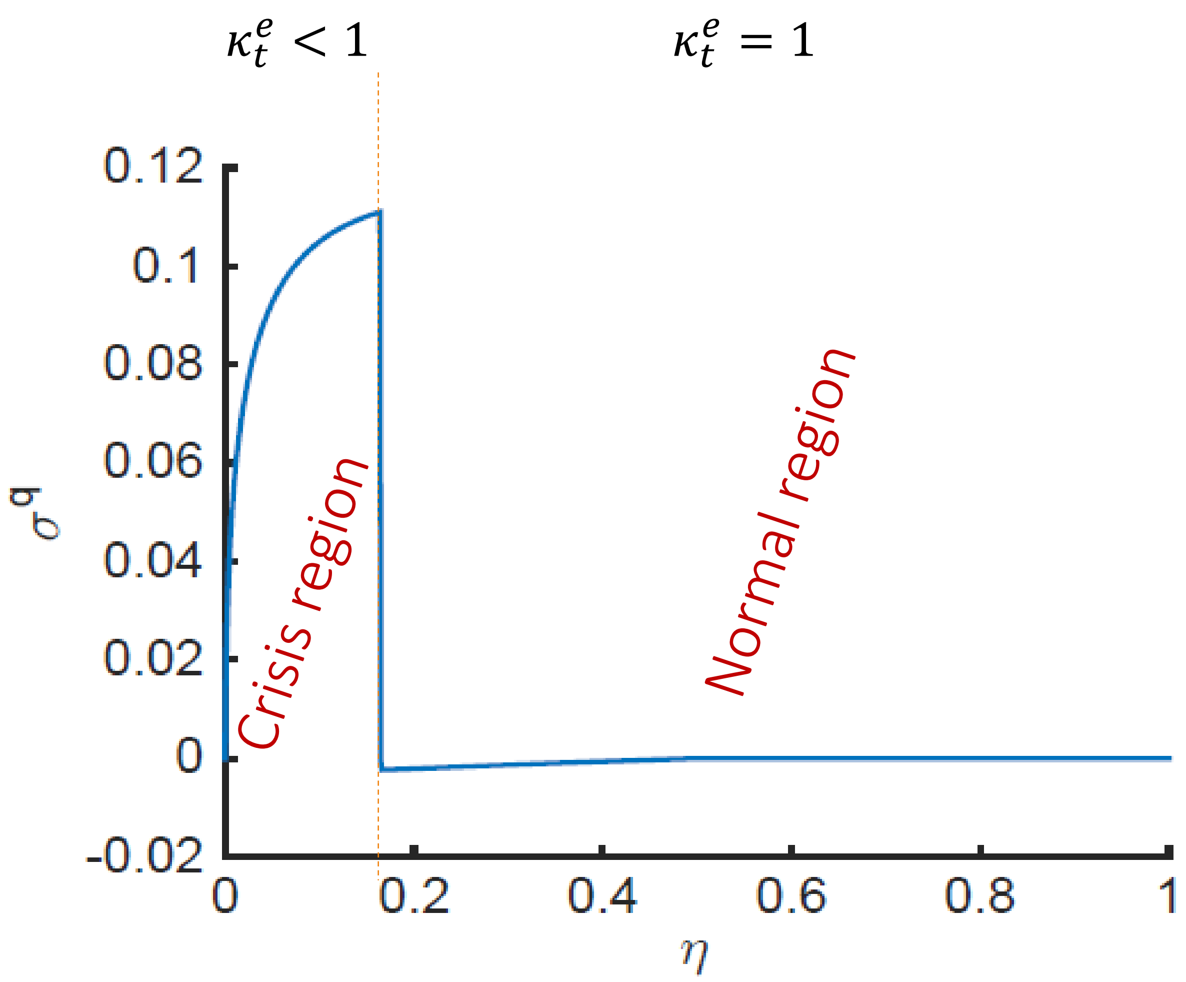
$$a = .11, \rho = 5\%, \sigma = .1, \Phi(\iota) = \frac{\log(\phi\iota + 1)}{\phi}, \phi = 10$$

Preview of new, extended model

- Price of capital



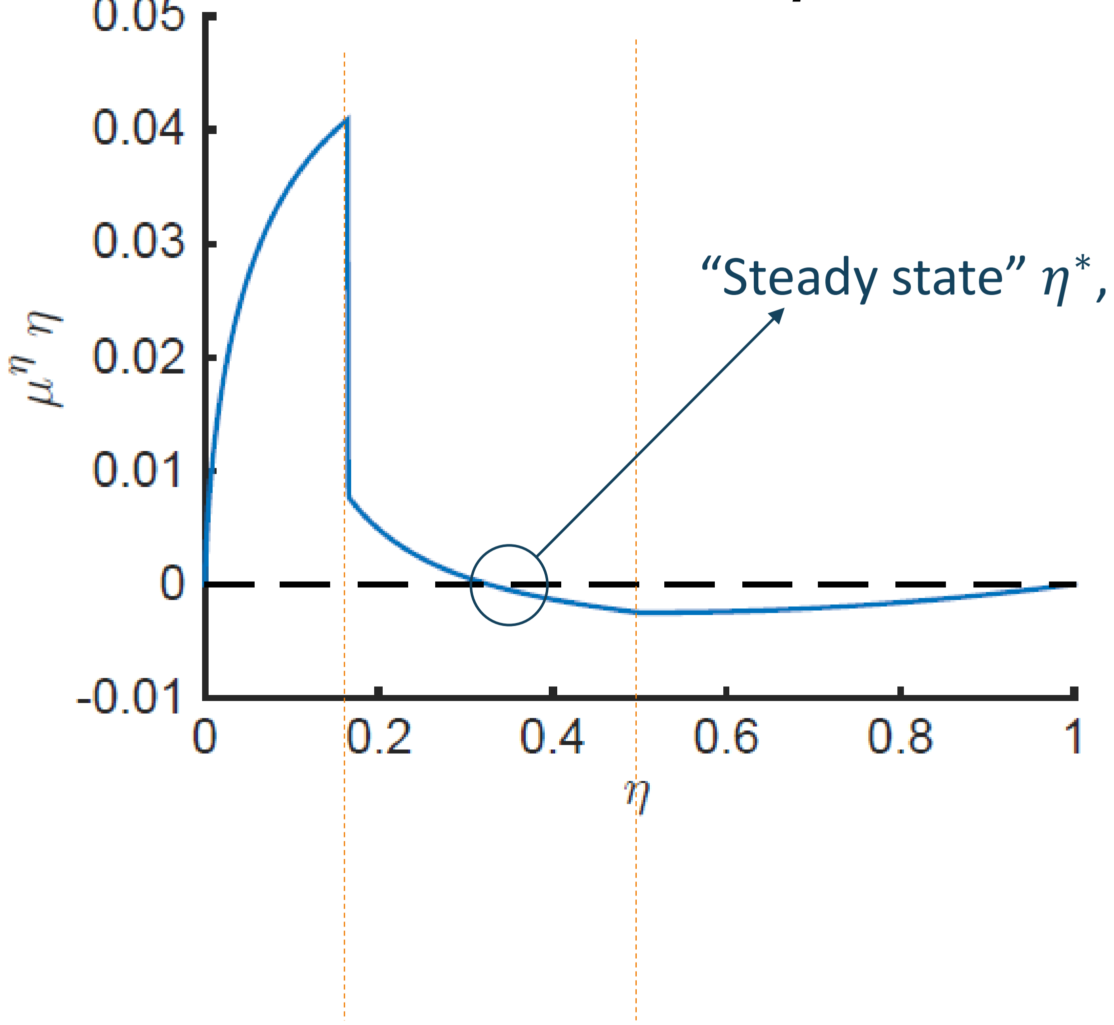
Amplification

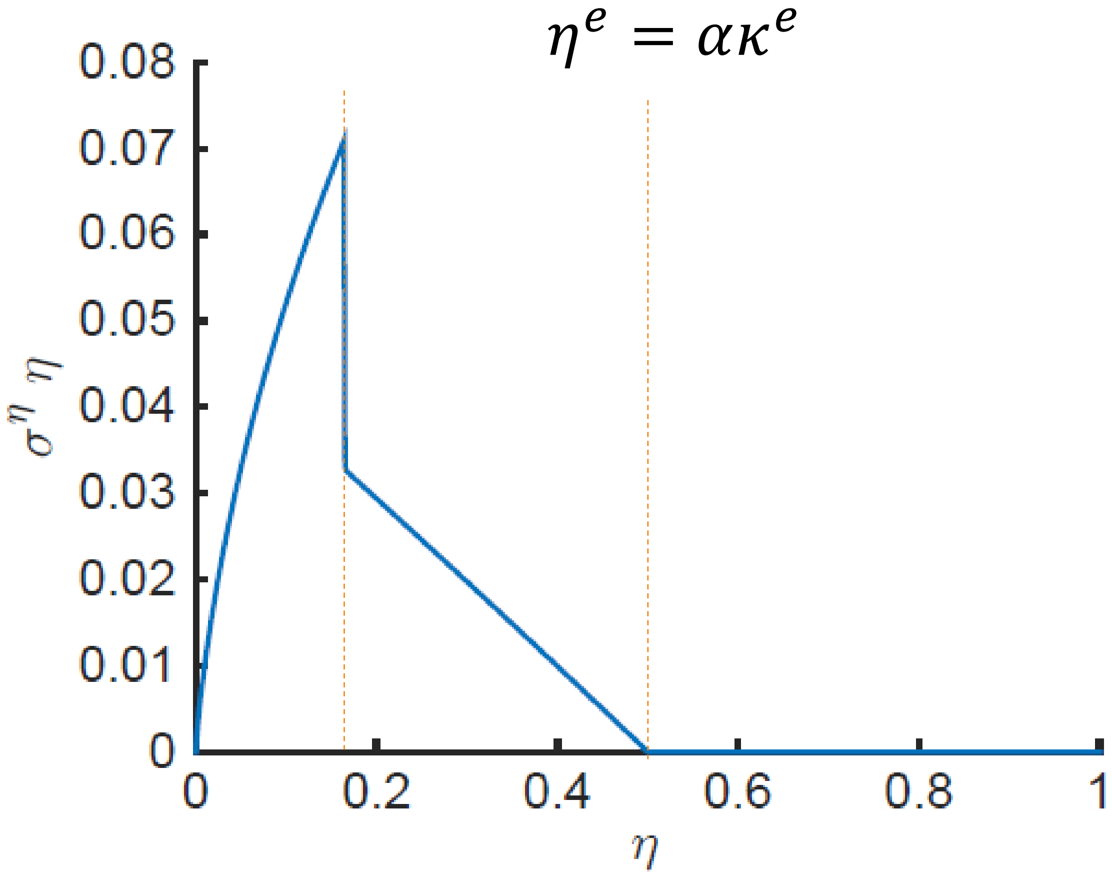


Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .01, \alpha = .50, \gamma = 2, \phi = 10$

Preview $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$

- Drift and Volatility of η^e





Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

   b. Portfolio choice θ + Asset market clearing or
   Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
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4. Numerical model solution
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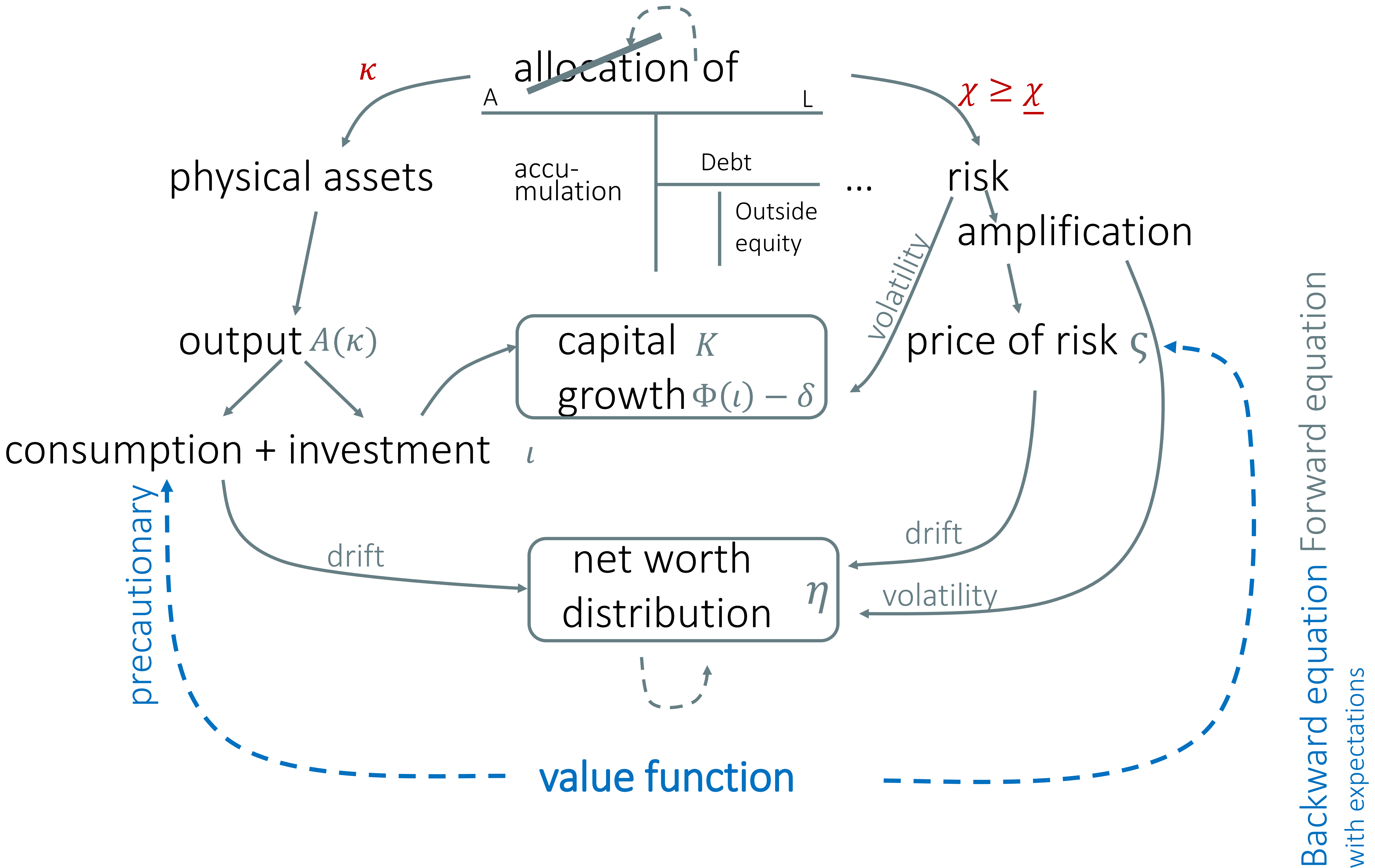
5. KFE: Stationary distribution, Fan charts

The Big Picture



1a. Individual Agent Choice of ι, θ, c

- Of experts with outside equity issuance (after plugging in households' outside equity choice)

$$\frac{a^e - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = [\varsigma_t^e \chi_t^e / \kappa_t^e + \varsigma_t^h (1 - \chi_t^e / \kappa_t^e)](\sigma + \sigma^q)$$

new compared to Basak-Cuoco

- Of households' capital choice

$$\frac{a^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t \leq \varsigma_t^h (\sigma + \sigma^q)$$

with equality if $\kappa_t^e < 1$

- Note: alternative approach replaces this step with Fisher Separation Social Planners' choice (see Lecture Notes)

1b. Asset/Risk Allocation across I Types

- Sketch of Proof of Theorem

1. Fisher Separation Theorem: (delegated portfolio choice by firm)
   - FOC yield the martingale approach solution
   - Each individual agent $(i,\tilde{\imath})$ portfolio maximization is equivalent to the maximization problem of a firm

$$\max_{\{\boldsymbol{\theta}^{j,i}\}} E_t\left[dr^{n^{(i,\tilde{\imath})}}\right]/dt - \varsigma\sigma^{r^n}$$

   - $dr^{n^{(i,\tilde{\imath})}} = \sum_j \theta^{j,i} dr^j = \sum_j \theta^{j,i} E[dr^j] + \sum_j \theta^{j,i}\sigma^j dZ_t$ is linear in θs
     - Either bang-bang solution for θs s.t. portfolio constraints bind
     - Or prices/returns/risk premia are s.t. that firm is indifferent

2. Aggregate
   - Taking η-weighted sum to obtain return on aggregate wealth

3. Use market clearing to relate θs to κs & χs (incl. θ-constraint)

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Toolbox 3: Change of Numeraire

- x_t^A is a value of a self-financing strategy/asset in \$
- Y_t price of € in \$ (exchange rate)

$$\frac{dY_t}{Y_t} = \mu_t^Y dt + \sigma_t^Y dZ_t$$

- x_t^A / Y_t value of the self-financing strategy/asset in €

$$\underbrace{e^{-\rho t} u'(c_t)}_{=\xi_t} Y_t \frac{x_t^A}{Y_t} \text{ follows a martingale}$$

Recall $\mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk}$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \cancel{\sigma_t^Y} - \sigma_t^B + \cancel{\sigma_t^Y})}_{risk}$$

- Price of risk $\varsigma^{€} = \varsigma^{\$} - \sigma^Y$

Poll 23: Why does the price of risk change, though real risk remains the same

a) because risk-free rate might not stay risk-free

b) because covariance structure changes

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Recall $\mu_t^A - \mu_t^B = \underbrace{(-\sigma_t^\xi)}_{=\varsigma_t} \underbrace{(\sigma^A - \sigma_t^B)}_{risk}$

$$\mu_t^{A/Y} - \mu_t^{B/Y} = \underbrace{(-\sigma_t^\xi - \sigma_t^Y)}_{price\ of\ risk} \underbrace{(\sigma^A - \sigma_t^Y - \sigma_t^B + \sigma_t^Y)}_{risk}$$

- Price of risk $\varsigma^€ = \varsigma^\$ - \sigma^Y$

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2. GE: Markov States and Equilibria

- Equilibrium is a

  Histories of shocks $\dashrightarrow$ prices $q_t, \varsigma_t^i, \iota_t^i, \theta_t^i$

  $\{Z_s, s \in [0, t]\}$

  net worth distribution

  $$= \frac{N_t^e}{q_t K_t} \in (0,1)$$

- All agents maximize utility
  - Choose: portfolio, consumption, technology
- All markets clear
  - Consumption, capital, money, outside equity

2. Law of Motion of Wealth Share η_t

- Method 1: Using Ito's quotation rule $\eta_t^i = N_t^i/(q_t K_t)$
  - Recall

$$\frac{dN_t^i}{N_t^i} = r_t dt + \underbrace{\frac{\chi_t^i \kappa_t^i}{\eta_t^i}(\sigma+\sigma_t^q)}_{risk} \underbrace{\varsigma_t^i}_{\substack{price\ of \\ risk}} dt + \frac{\chi_t^i \kappa_t^i}{\eta_t^i}(\sigma + \sigma_t^q) dZ_t - \frac{C_t^i}{N_t^i} dt$$

  - $\frac{d\eta_t^i}{\eta_t^i} = \ldots$ (lots of algebra)

- Method 2: Change of numeraire + Martingale Approach
  - New numeraire: Total wealth in the economy, N_t
  - Apply Martingale Approach for value of i's portfolio
    - Simple algebra to obtain drift of η_t^i: $\mu_t^{\eta^i}$
      Note that <u>change of numeraire does not affect ratio η^i</u>!

2. μ^{η} Drift of Wealth Share: Many Types

- New Numeraire
  - "Total net worth" in the economy, N_t (without superscript)
  - Type i's portfolio net worth = net worth share

- Martingale Approach with new numeraire
  - Asset A $= i$'s portfolio return in terms of total wealth,

$$\Bigg(\underbrace{\frac{C_t^i}{N_t^i}}_{\text{Dividend yield}} + \underbrace{\mu_t^{\eta^i}}_{\text{E[capital gains] rate}}\Bigg)dt + \sigma_t^{\eta^i} dZ_t + \tilde{\sigma}_t^{\eta^i} d\tilde{Z}_t$$

  - Asset B (benchmark asset that everyone can hold, e.g. risk-free asset or money (in terms of total economy wide wealth as numeraire))

$$r_t^m dt + \sigma_t^m dZ_t$$

  - Apply our martingale asset pricing formula

$$\mu_t^A - \mu_t^B = \quad (\sigma_t^A - \sigma_t^B)$$

Poll 28: Is risk-free asset, risk free in the new numeraire?
a) Yes
b) No

2. μ^{η} Drift of Wealth Share: Many Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = \left(\varsigma_t^i - \sigma_t^N\right)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{\sum_{i'}^{I} \eta_t^{i'} \mu_t^{\eta^{i'}}}_{=0} + \frac{C_t}{N_t} - r_t^m = \sum_{i'} \eta_t^{i'} \left(\varsigma_t^{i'} - \sigma_t^N\right)\left(\sigma_t^{\eta^{i'}} - \sigma_t^m\right)$$

- *Poll 29: Why $= 0$?*
  - *a) Because we have stationary distribution*
  - *b) Because ηs sum up to 1*
  - *c) Because ηs follow martingale*

Benchmark asset everyone can trade

$$\sigma_t^m = -\sigma_t^N$$

2. μ^{η} Drift of Wealth Share: Two Types

- Asset pricing formula (relative to benchmark asset)

$$\mu_t^{\eta^i} + \frac{C_t^i}{N_t^i} - r_t^m = \left(\varsigma_t^i - \sigma_t^N\right)\left(\sigma_t^{\eta^i} - \sigma_t^m\right)$$

For benchmark asset: risk-free debt

$$\sigma_t^m = -\sigma_t^N$$

- Add up across types (weighted),

(capital letters without superscripts are aggregates for total economy)

$$\underbrace{\left(\eta_t^e \mu_t^{\eta^e} + \eta_t^h \mu_t^{\eta^h}\right)} + \frac{C_t}{N_t} - r_t^m$$

$$= \eta_t^e\left(\varsigma_t^e - \sigma_t^N\right)\left(\sigma_t^{\eta^e} - \sigma_t^m\right) + \eta_t^h\left(\varsigma_t^h - \sigma_t^N\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

- Subtract from each other yield

$$\mu_t^{\eta^e} = (1 - \eta_t^e)(\varsigma_t^e - \sigma_t^N)\left(\qquad - \sigma_t^m\right) - (1 - \eta_t^e)\left(\varsigma_t^h - \sigma_t^{N^h}\right)\left(\sigma_t^{\eta^h} - \sigma_t^m\right)$$

$$-\left(\frac{C_t^e}{N_t^e} - \frac{C_t}{q_t K_t}\right)$$

2. σ^{η} Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term)
- Since $\eta_t = N_t^e / N_t$,

$$\sigma_t^{\eta} = \sigma_t^{N^e} - \sigma_t^{N} = \sigma_t^{N^i} - \sum_{i'} \eta_t^{i'} \sigma_t^{N^{i'}} = (1 - \eta_t^i)\sigma_t^{N^i} - \sum_{i^- \neq i} \eta_t^{i^-} \sigma_t^{N^{i^-}}$$

Change in notation in 2 type setting
Type-net worth is $n^i = N^i$

- Note for

$$\sigma_t^{\eta^e} = (1 - \eta_t^e)(\sigma_t^{n^e} - \sigma_t^{n^h})$$

$$\sigma_t^{n^e} = \underbrace{\chi_t^e / \eta_t^e}_{=\theta^{e,K} + \theta^{e,OE}} (\sigma + \sigma_t^q) \qquad \sigma_t^{n^h} = \frac{\chi_t^h}{\eta_t^h}(\sigma + \sigma_t^q) = \frac{1 - \chi_t^e}{1 - \eta_t^e}(\sigma + \sigma_t^q)$$

Hence,

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h} \sigma_t^{\eta^e} = -\frac{\eta_t^e}{1 - \eta_t^e} \sigma_t^{\eta^e}$

2. σ^{η} Volatility of Wealth Share

- Recall Ito ratio rule (only volatility term): $\sigma_t^{X/Y} = \sigma_t^X - \sigma_t^Y$
- Since $\eta_t = N_t^e / N_t,\ N_t = q_t K_t$, by Ito ratio rule (only volatility term)

$$\sigma_t^{\eta} = \sigma_t^{N^e} - \sigma_t^{N} = \sigma_t^{N^e} - \sigma_t^{q} - \sigma$$

- We also have

$$\sigma_t^{N^e} = \frac{\chi_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

- Substituting this in previous formula yields

$$\sigma_t^{\eta^e} = \frac{\chi_t^e - \eta_t^e}{\eta_t^e}(\sigma + \sigma_t^q)$$

- Note also, $\eta_t^e \sigma_t^{\eta^e} + \eta_t^h \sigma_t^{\eta^h} = 0 \Rightarrow \sigma_t^{\eta^h} = -\frac{\eta_t^e}{\eta_t^h}\sigma_t^{\eta^e} = -\frac{\eta_t^e}{1-\eta_t^e}\sigma_t^{\eta^e}$

2. Amplification Formula: Loss Spiral

- Recall $$\sigma_t^{\eta^e} = \underbrace{\frac{\chi_t^e - \eta_t^e}{\eta_t^e}}_{\text{leverage}} (\sigma + \sigma_t^q)$$

- By Ito's Lemma on $q(\eta^e)$ $$\sigma_t^q = \frac{q'(\eta_t^e)}{q(\eta_t^e)} \eta_t^e \sigma_t^{\eta^e}$$

$$\sigma_t^q = \frac{q'(\eta_t^e)}{\underbrace{q/\eta_t^e}_{elasticity}} \frac{\chi_t^e - \eta_t^e}{\eta_t^e} (\sigma + \sigma_t^q)$$

- Total volatility

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta_t^e)}{q/\eta_t^e} \frac{\chi_t^e - \eta_t^e}{\eta_t^e}}$$

- Loss spiral
  - Market illiquidity (price impact elasticity)

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- Total volatility

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Poll 34: Where is the spiral?
a) Sum of infinite geometric series (denominator)
b) in q', since with constant price, no spiral

- Loss spiral
  - Market illiquidity (price impact elasticity)

Solving MacroModels Step-by-Step

0. Postulate aggregates, price processes & obtain return processes
1. For given C/N-ratio and SDF processes for each i *finance block*
   a. Real investment ι + Goods market clearing *(static)*
   - *Toolbox 1:* Martingale Approach, HJB vs. Stochastic Maximum Principle Approach

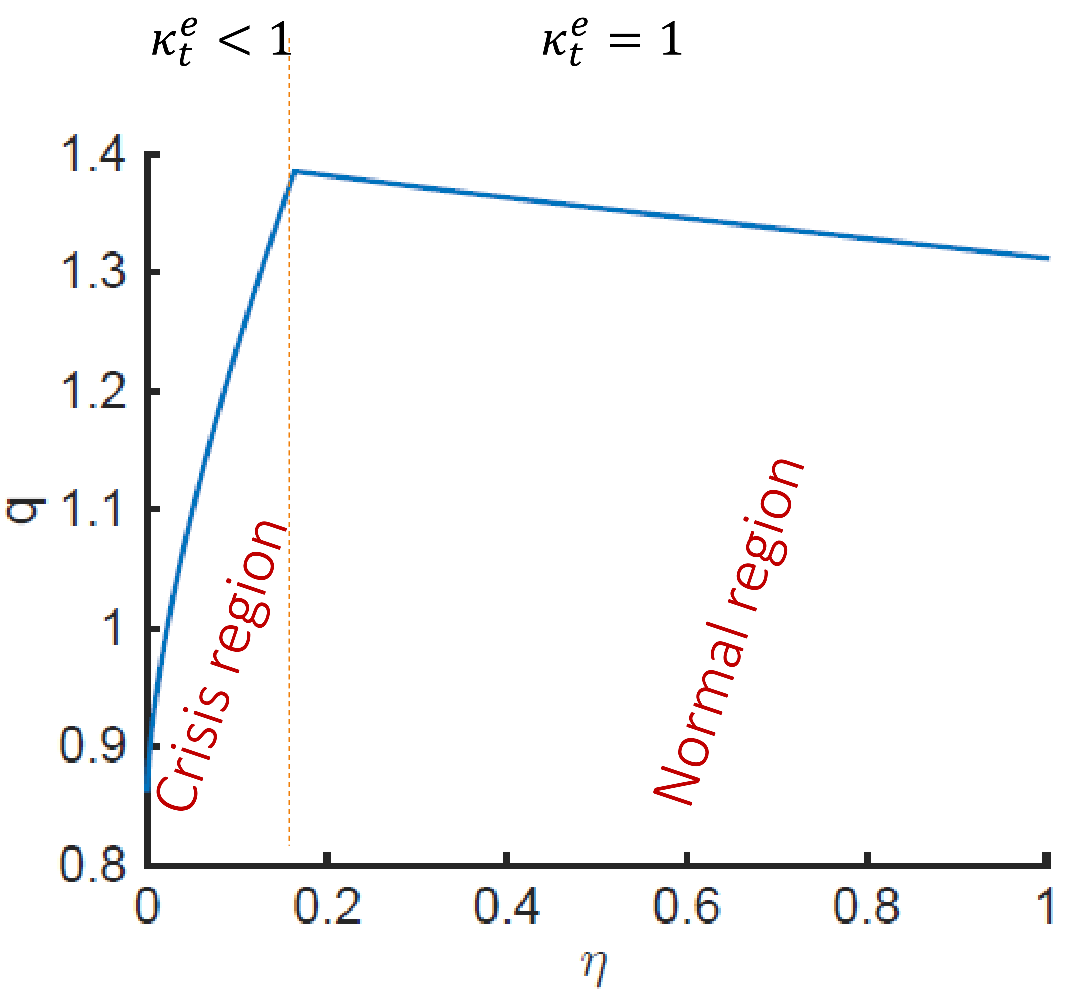
   b. Portfolio choice θ + Asset market clearing or
      Asset allocation κ & risk allocation χ
   - *Toolbox 2:* "price-taking social planner approach" – Fisher separation theorem

   c. "Money evaluation equation" ϑ
   - *Toolbox 3:* Change in numeraire to total wealth (including SDF)
2. Evolution of state variable η (and K) *forward equation*
3. Value functions *backward equation*
   a. Value fcn. as fcn. of individual investment opportunities ω
   - *Special cases: log-utility, constant investment opportunities*

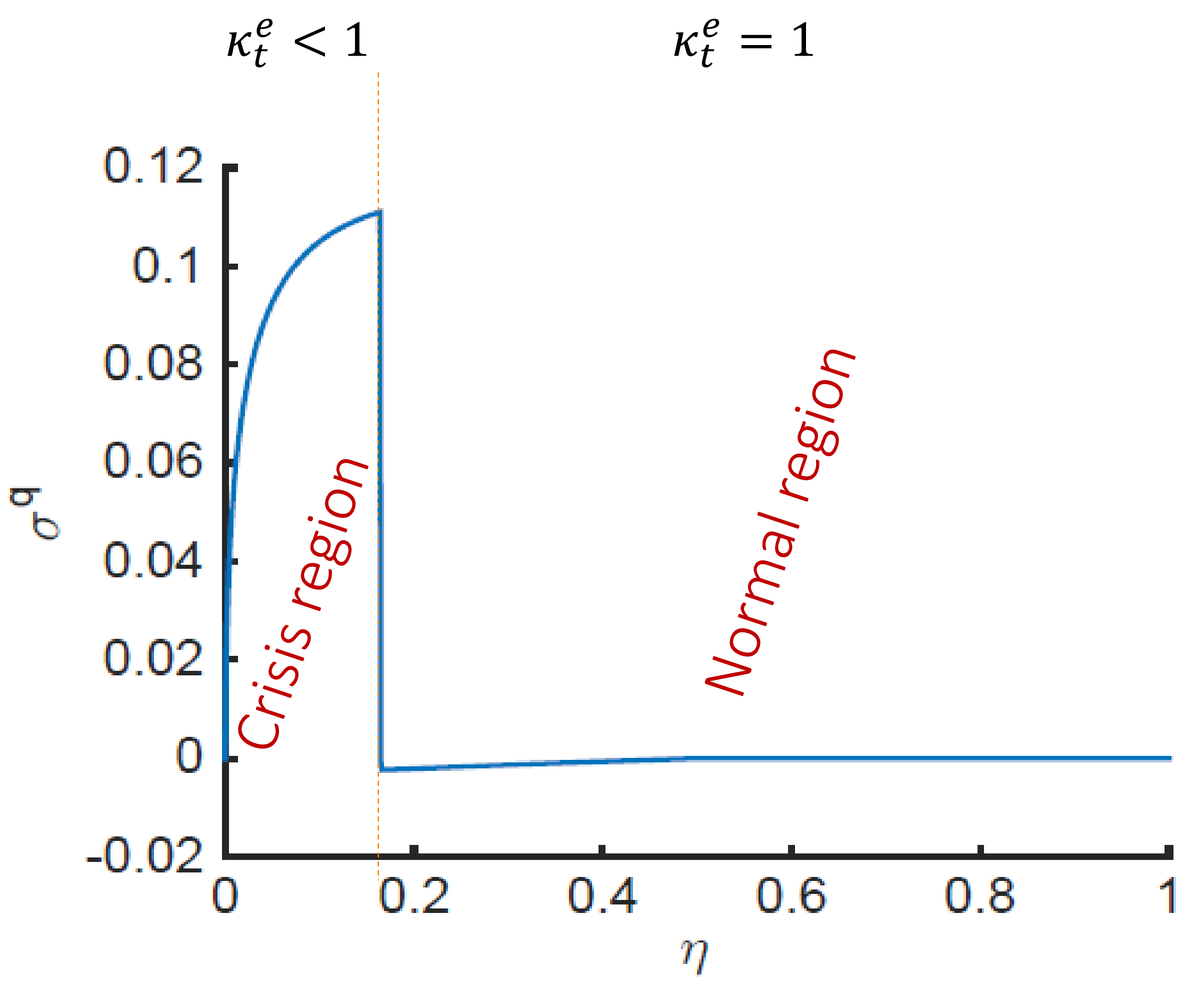
   b. Separating value fcn. $V^i(n^{\tilde{i}};\eta,K)$ into $v^i(\eta)u(K)\left(n^{\tilde{i}}/n^i\right)^{1-\gamma}$
   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5. KFE: Stationary distribution, Fan charts

Solution

- Price of capital



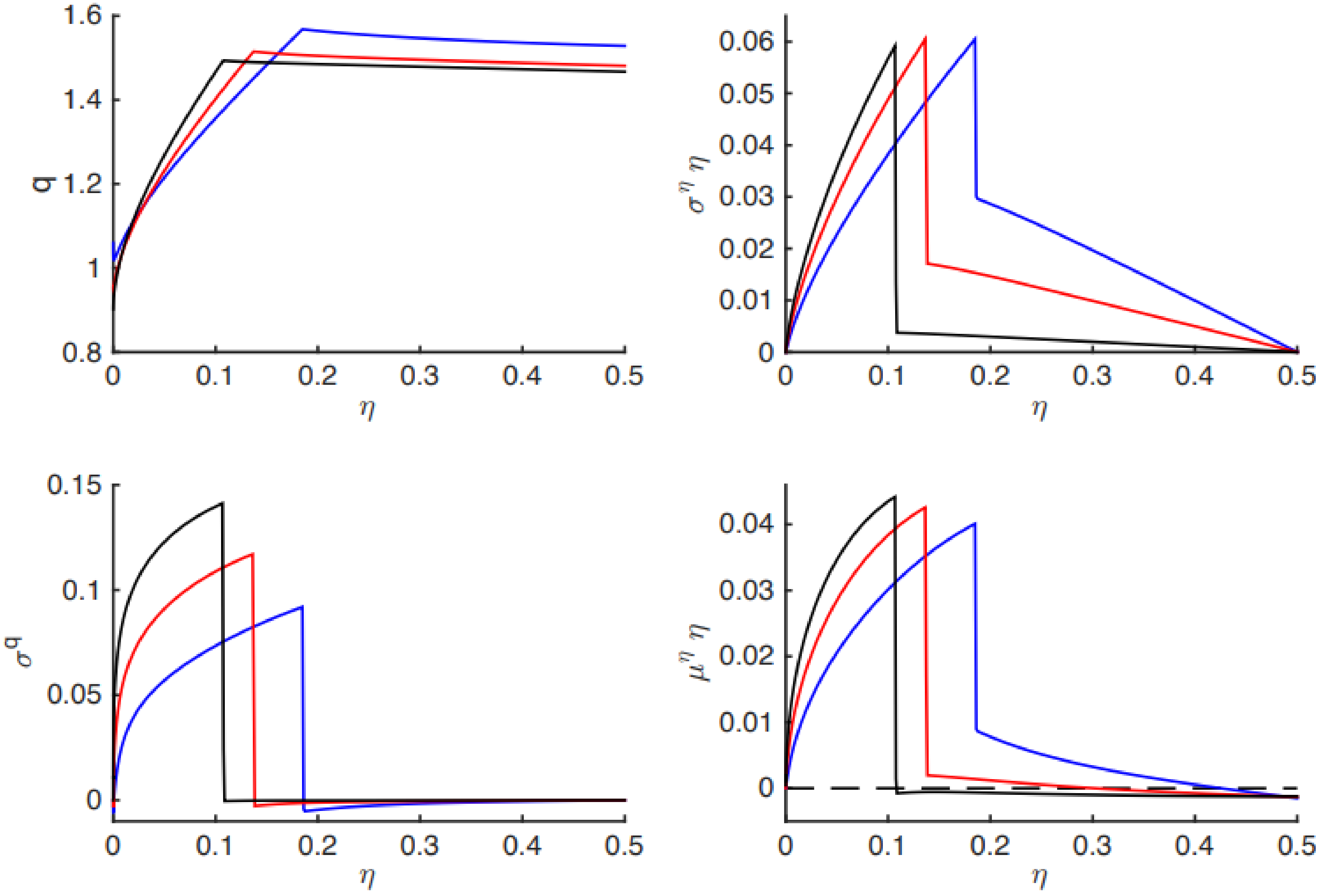
Amplification



Parameters: $\rho^e = .06, \rho^h = .05, a^e = .11, a^h = .03,$
$\delta = .05, \sigma = .01, \alpha = .50, \gamma = 2, \phi = 10$

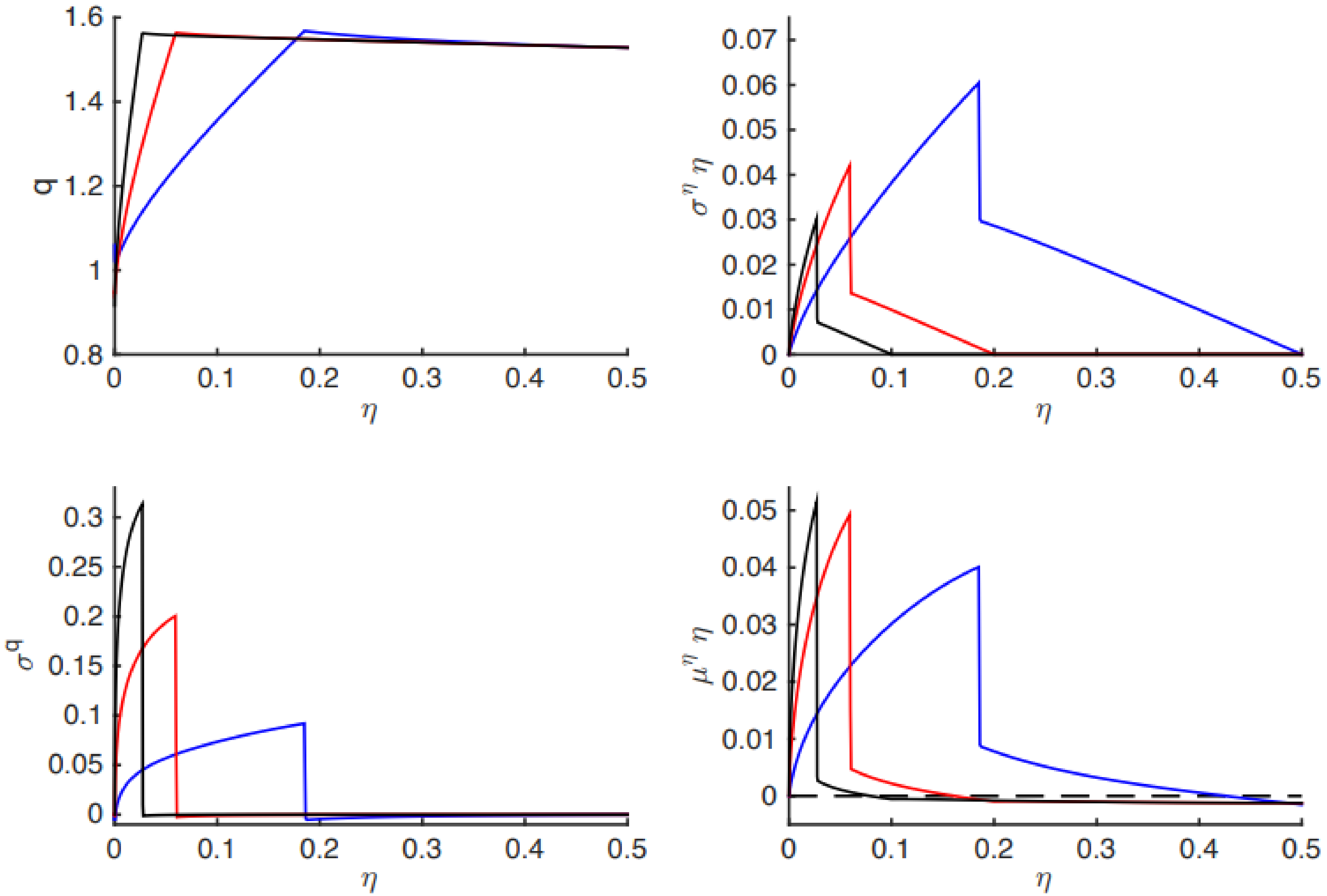
Volatility Paradox

- Comparative Static w.r.t. $\sigma = .01, .05, .1$



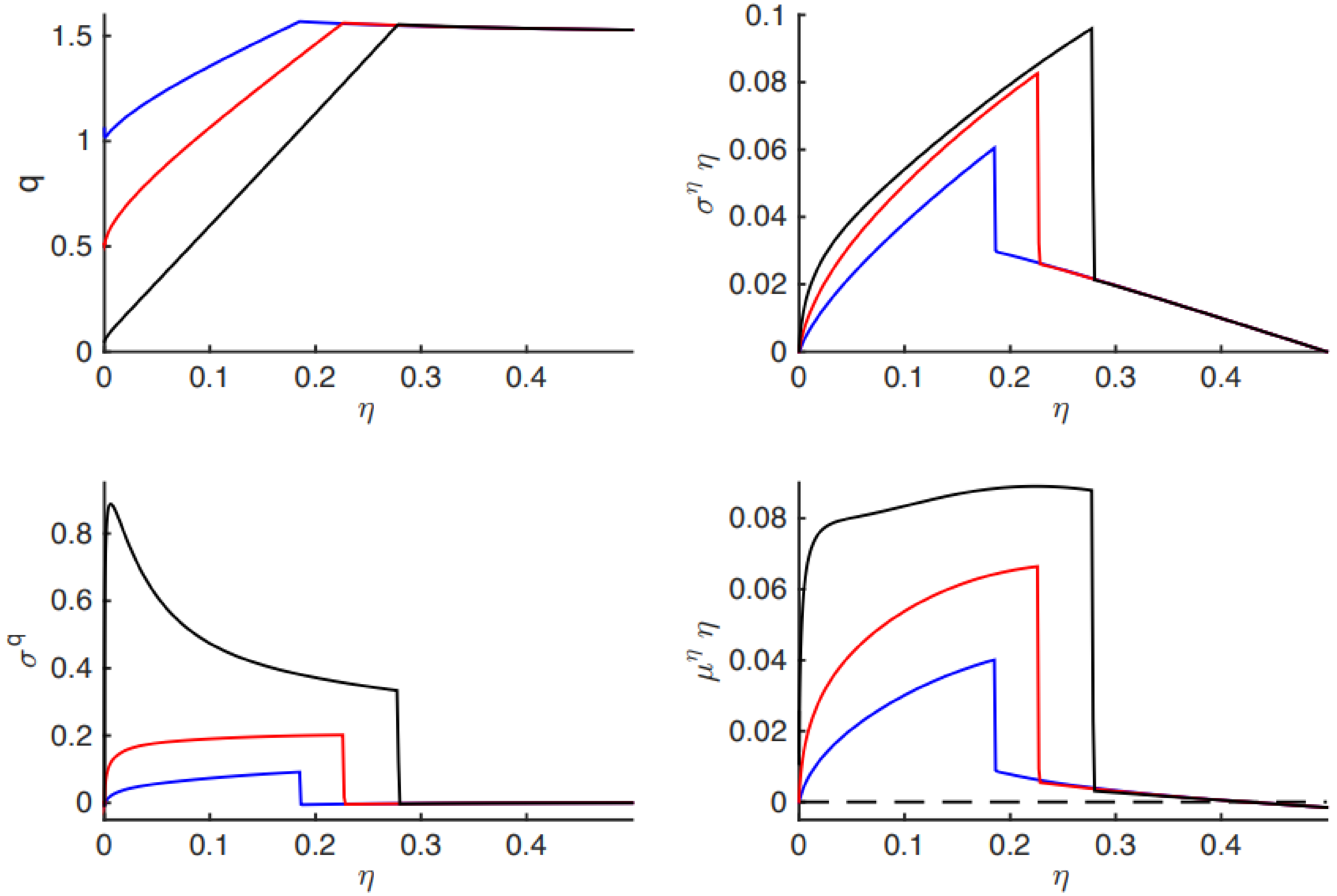
Risk Sharing via Outside Equity

- Comparative Static w.r.t. Risk sharing $\alpha = .1, .2, .5$ (skin the game constraint)



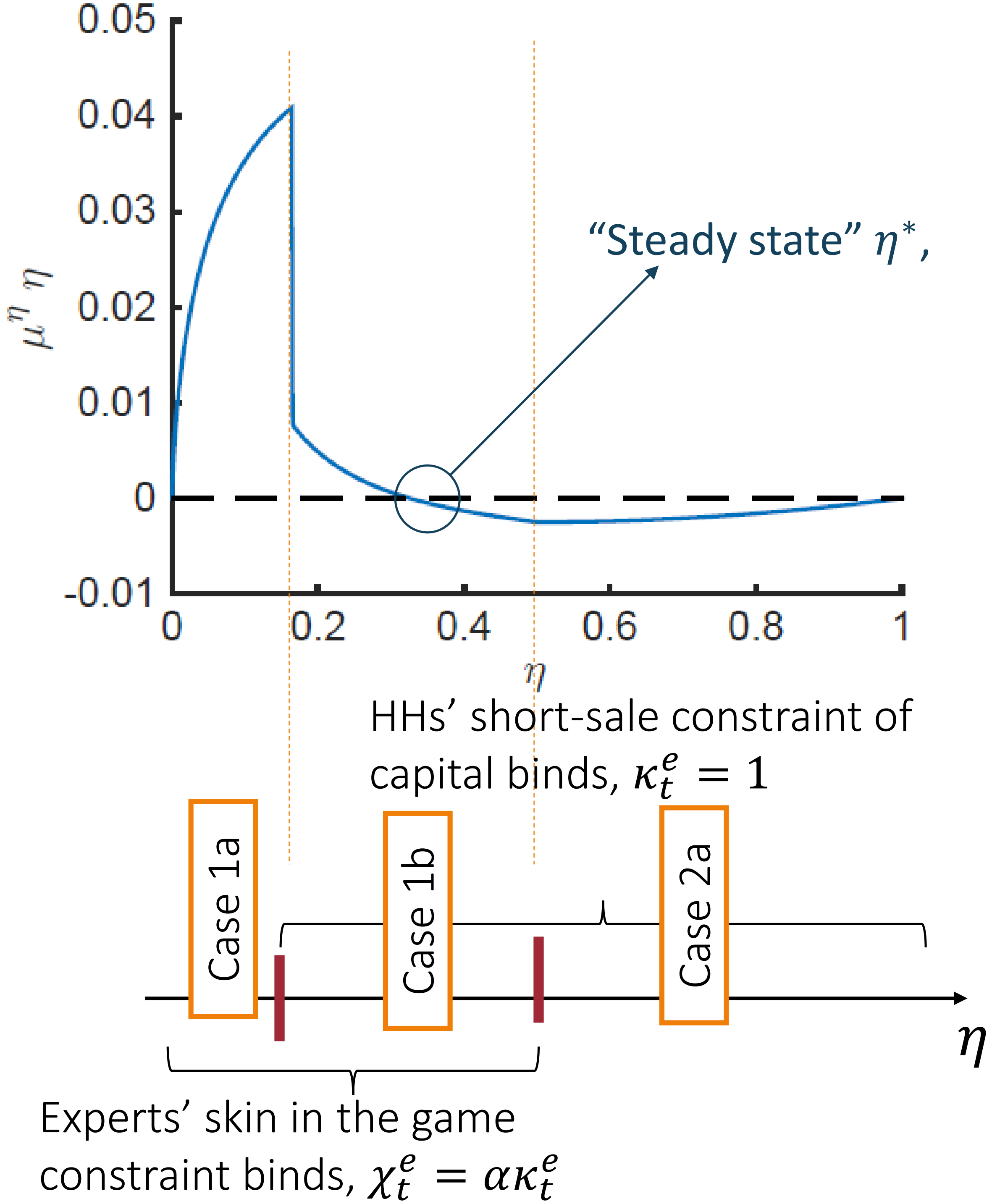
Market Liquidity

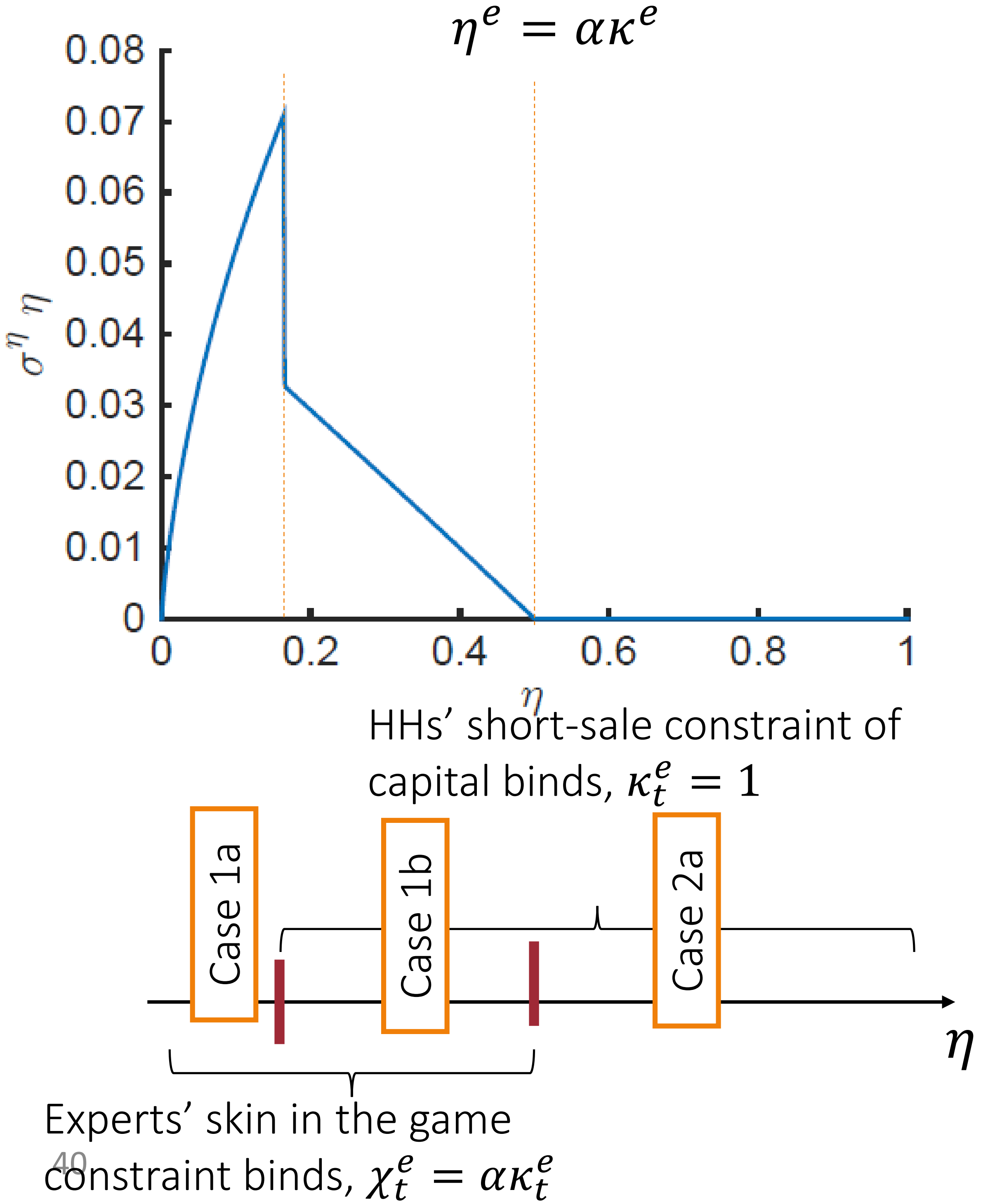
- Comparative static w.r.t. $a^h = .03, -.03, -.09$



From $\mu^{\eta^e}(\eta^e)$ & $\sigma^{\eta^e}(\eta^e)$ to Stationary Distribution

- Drift and Volatility of η^e





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   c. Derive C/N-ratio and ς price of risk
4. Numerical model solution
   a. Transform BSDE for separated value fcn. $v^i(\eta)$ into PDE
   b. Solve PDE via value function iteration
5.

5. Kolmogorov Forward Equation

- Given an initial distribution $f(\eta, 0) = f_0(\eta)$, the density diffusion follows PDE

$$\frac{\partial f(\eta,t)}{\partial t} = \frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

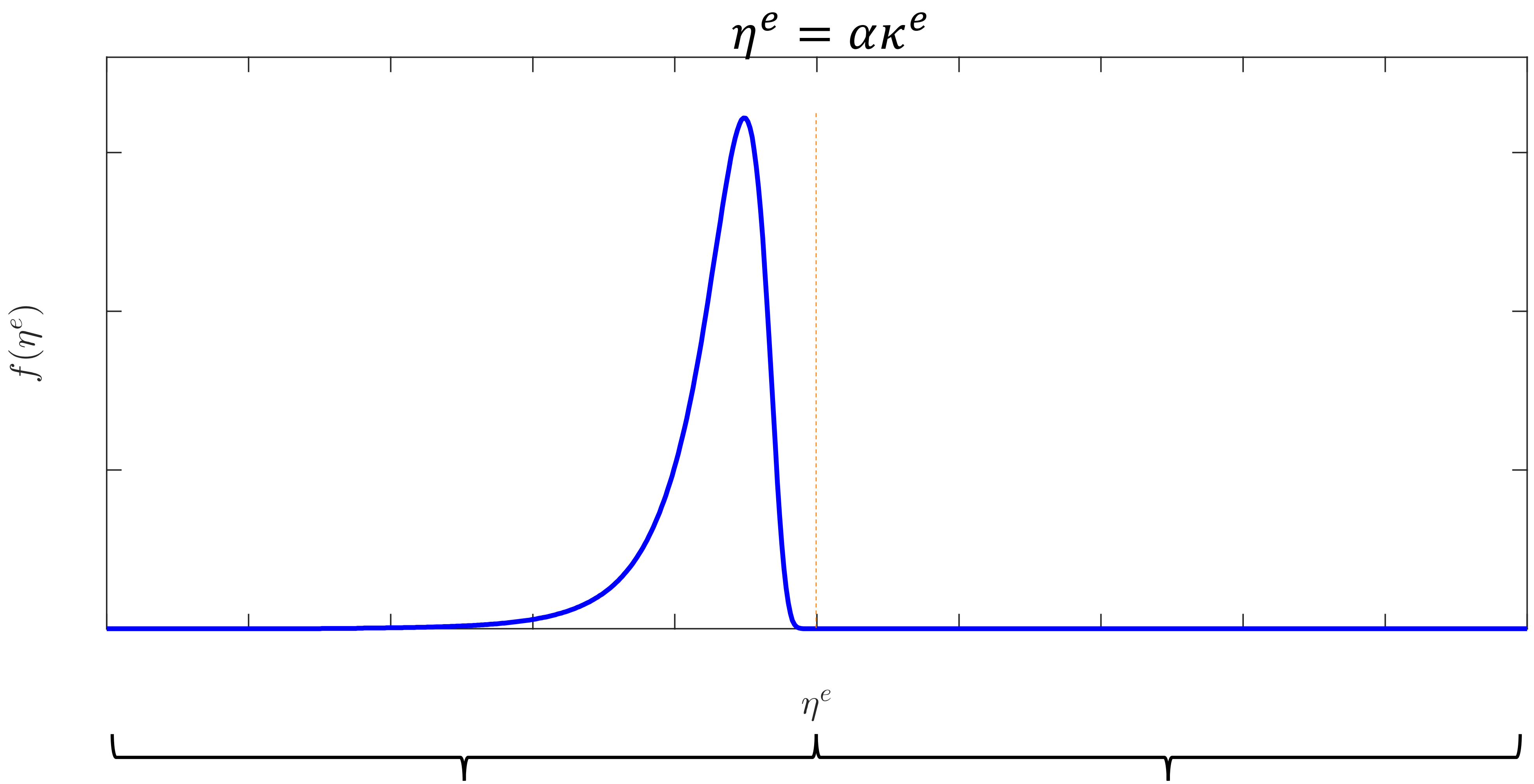
- "Kolmogorov Forward Equation" is in physics referred to as "Fokker-Planck Equation"

- Corollary: if stationary distribution $f(\eta)$ exists, it satisfies the ODE

$$0 = \frac{\partial[f(\eta,t)\mu(\eta)]}{\partial \eta} + \frac{1}{2}\frac{\partial^2[f(\eta,t)\sigma^2(\eta)]}{\partial \eta^2}$$

5. Stationary Distribution

- Stationary distribution of η^e



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

Poll 43: Is the constraint always (not just occasionally) binding

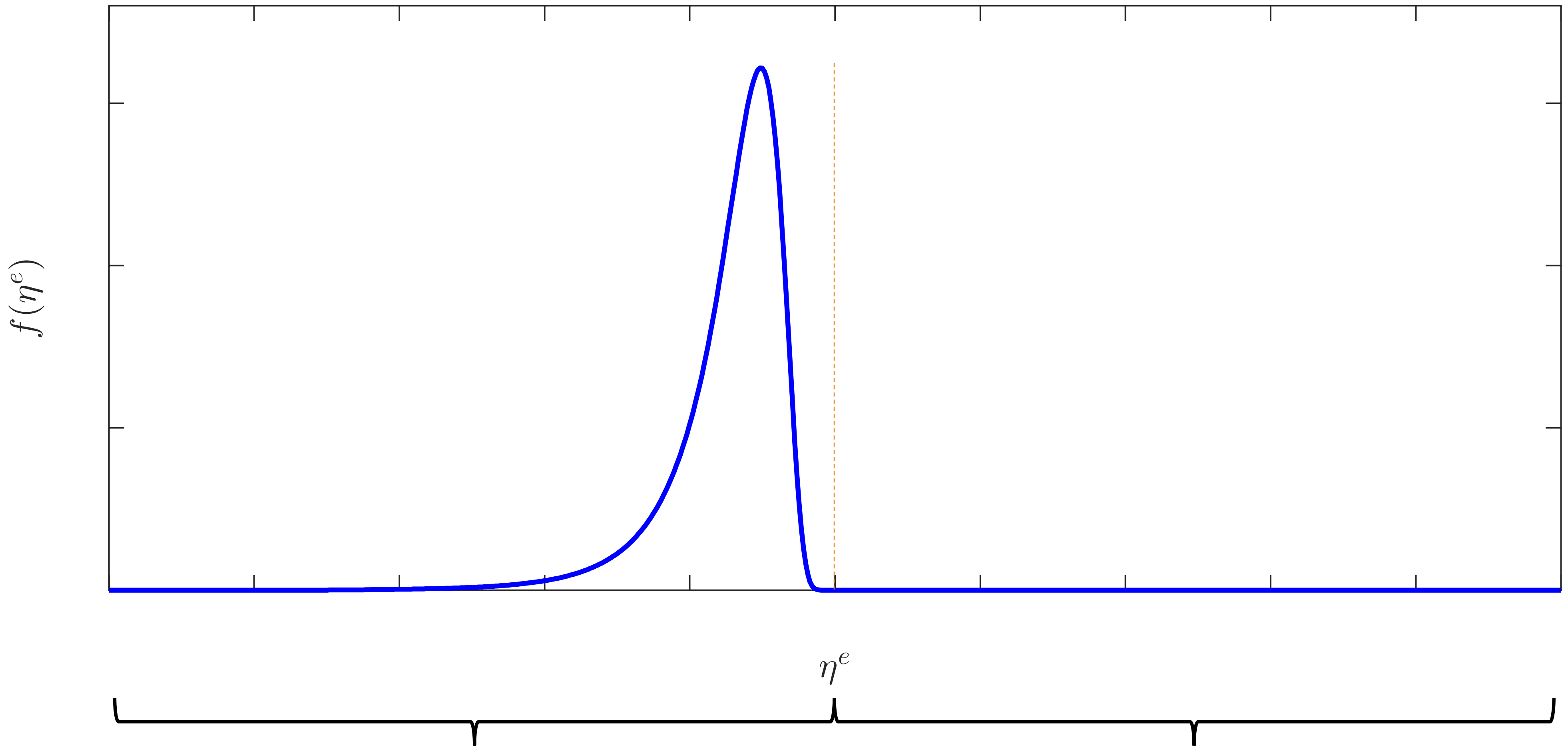
a) yes

b) no, only for some parameters $\rho^e > \rho^h$

5. Stationary Distribution

- Stationary distribution of η^e

$\eta^e = \alpha\kappa^e$



Experts' skin in the game constraint binds $\chi_t^e = \alpha\kappa_t^e$

Perfect risk-sharing region (infeasible)

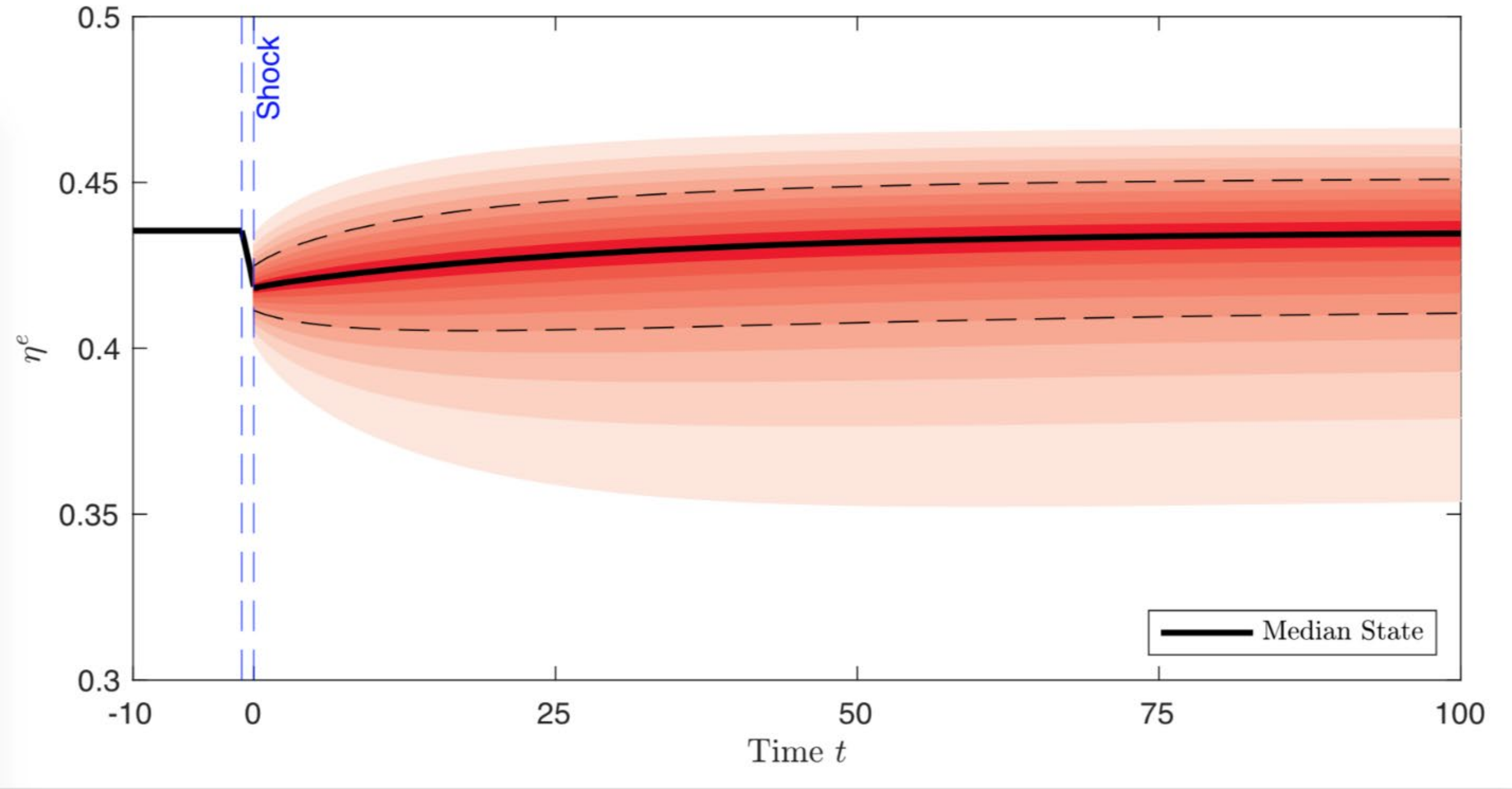
Poll 44: What happens for $\rho^e = \rho^h$

a) experts take over the economy, $\eta \to 1$

b) there is a steady state at $\eta = \alpha$

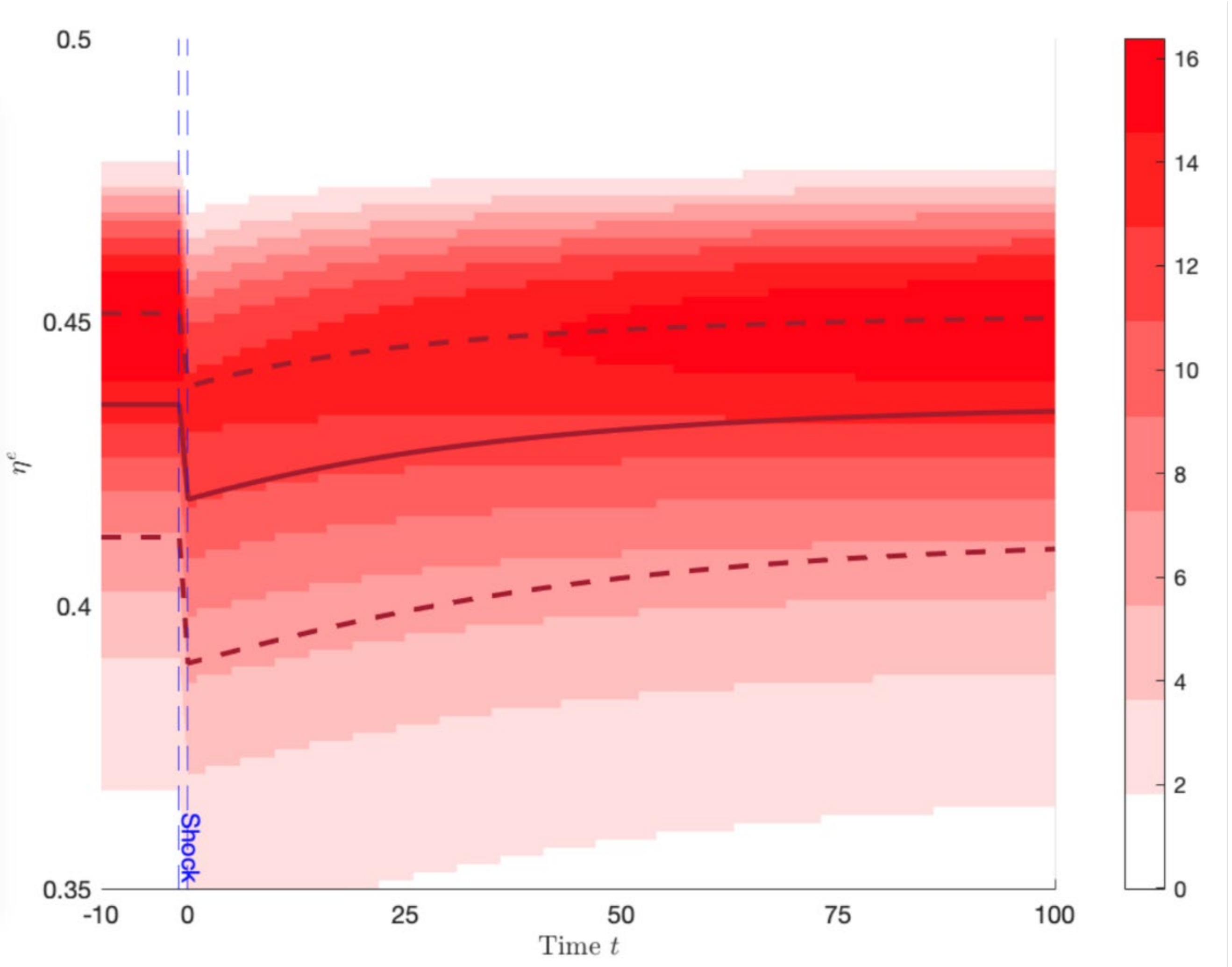
5. Fan chart and distributional impulse response

- … the theory to Bank of England's empirical fan charts
- Starts at η_0, the median of stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\, dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



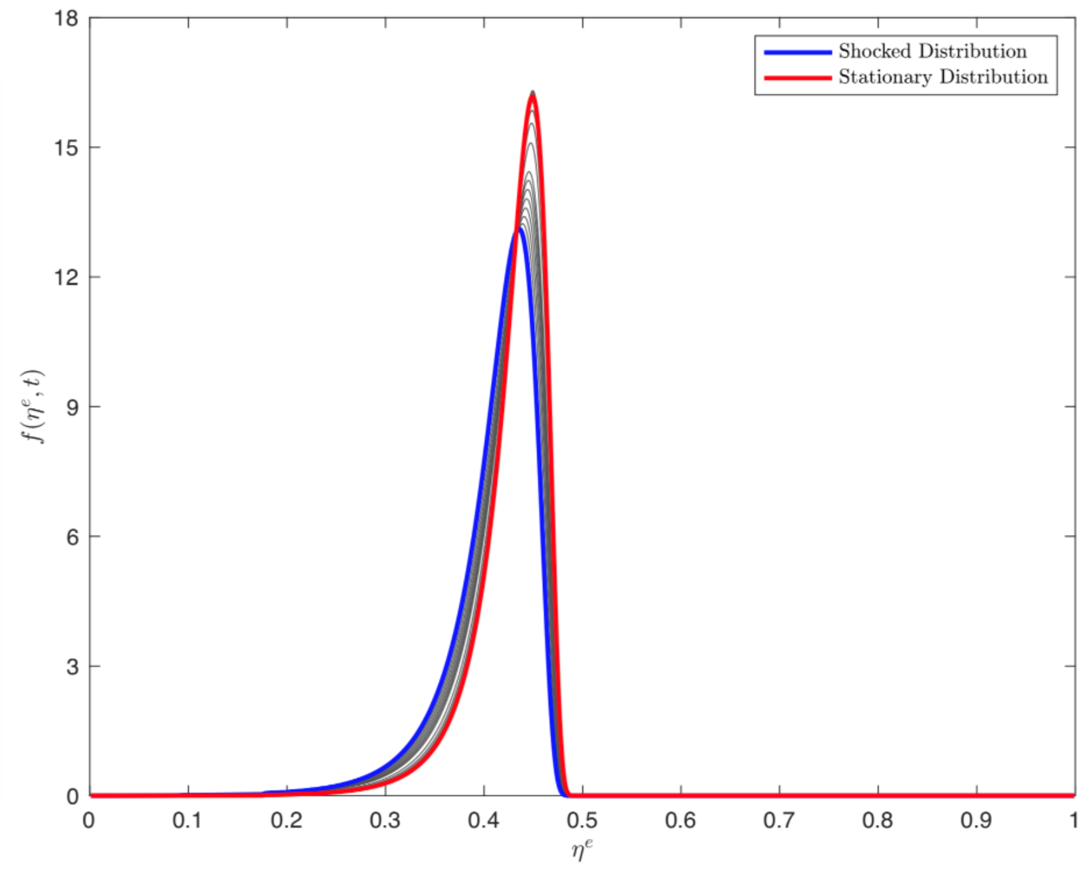
5. Fan chart and distributional impulse response

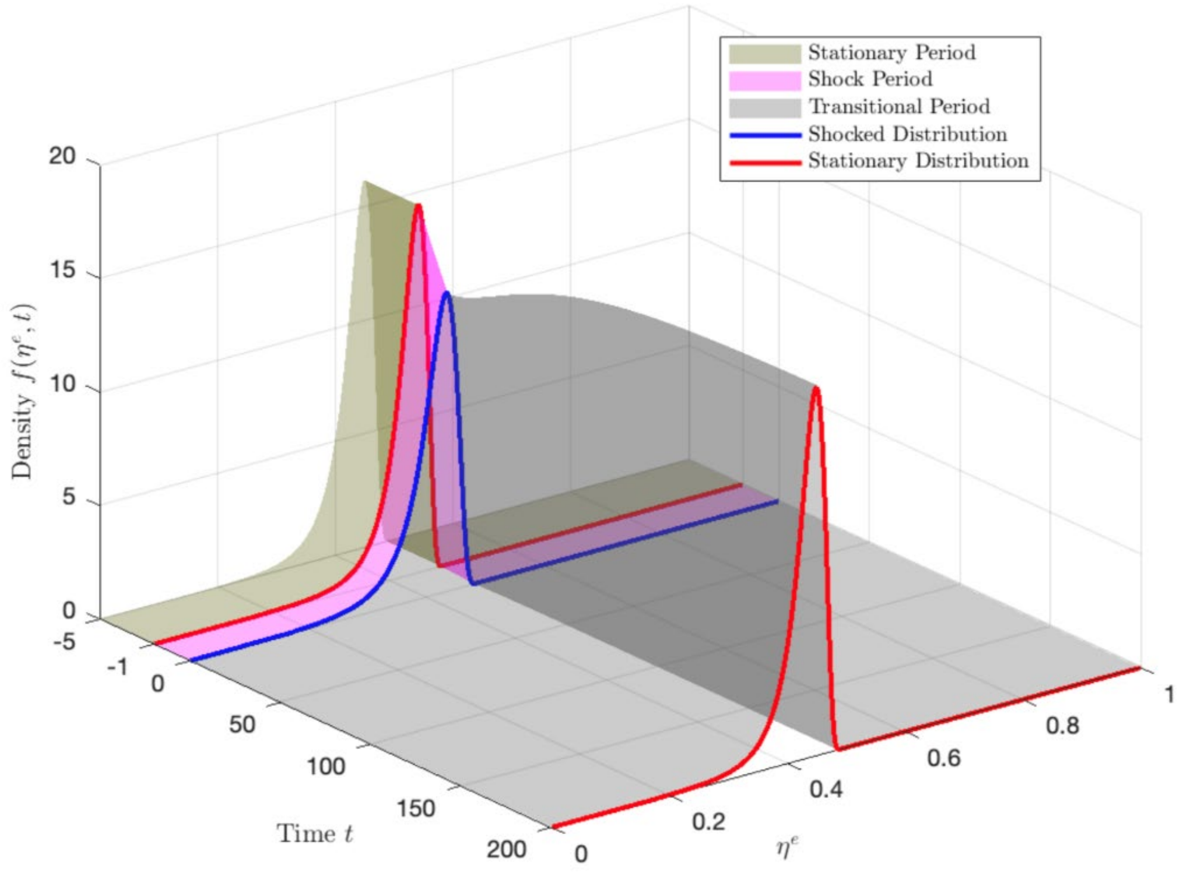
- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution



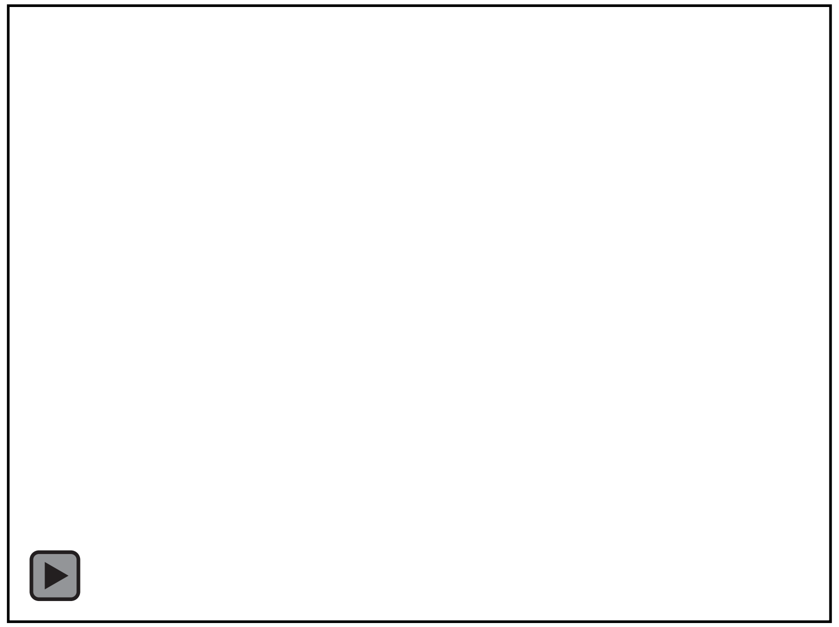
5. Density Diffusion

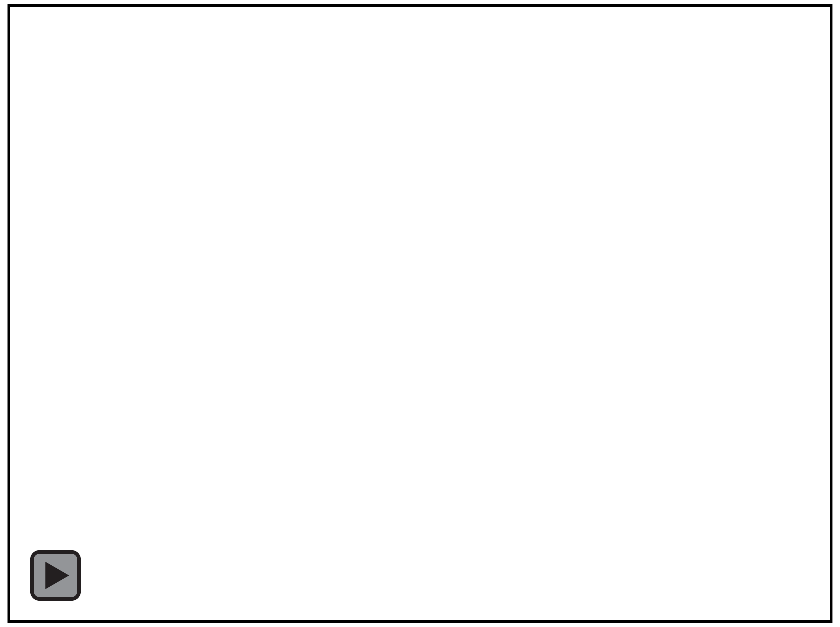
- Starts at stationary distribution
- Simulate a shock at 1% quantile of original Brownian shock ($dZ_t = -2.32\ dt$) for a period of $\Delta t = 1$.
- Converges back to stationary distribution





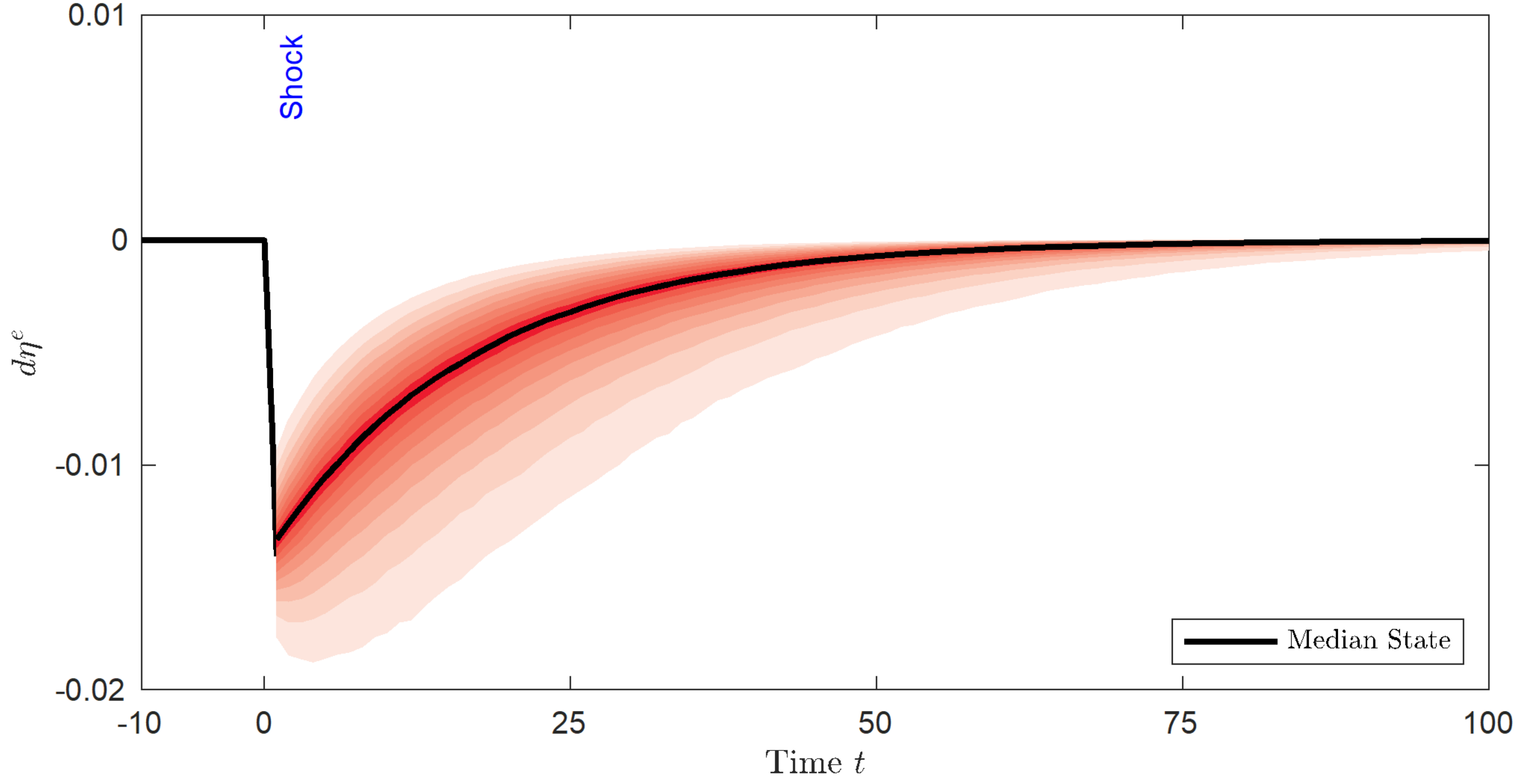
5.Density Diffusion Movies





5. Distributional Impulse Response

- Difference between path with and without shock
- Difference converges to zero in the long-run



$\sigma = 0.15$