The I Theory of Money

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Baseline Model

- consumption and investment

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \underbrace{\sigma dZ_t}_{\text{aggregate}} + \underbrace{\tilde{\sigma} d\tilde{Z}_t}_{\text{idiosyncratic}}$$

- capital produces output $a - \iota_t$, has price q
- price of money (infinitely divisible "coin") pK_t
- portfolio weight on money $\theta = p/(p + q)$

Risk and return

- consumption and investment

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \underbrace{\sigma dZ_t}_{\text{aggregate}} + \underbrace{\tilde{\sigma} d\tilde{Z}_t}_{\text{idiosyncratic}}$$

- capital produces output $a - \iota_t$, has price q
- price of money (infinitely divisible "coin") pK_t
- portfolio weight on money $\theta = p/(p + q)$
- return on capital:

$$\frac{a - \iota_t}{q} + \underbrace{(\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t}_{\frac{d(qk_t)}{qk_t}}$$

- return on money

$$\underbrace{(\Phi(\iota_t) - \delta)dt + \sigma dZ_t}_{\frac{d(pK_t)}{pK_t}}$$

Equilibrium

- pricing for log utility $E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t)$

$$\underbrace{\frac{a-\iota_t}{q}}_{\text{diff. in return}} = \underbrace{(\sigma - \sigma)}_{\text{diff. in agg. risk}} \sigma + \underbrace{(\tilde{\sigma} - 0)}_{\text{diff. in idios. risk}} (1-\theta)\tilde{\sigma}$$

- mkt. clearing for log $\underbrace{\rho \frac{q_t}{1-\theta} K_t}_{\text{total wealth}} = \underbrace{(a-\iota_t)K_t}_{\text{total consumption}}$

- return on capital:

portfolio shares

$$\frac{a-\iota_t}{q} + \underbrace{(\Phi(\iota_t)-\delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t}_{\frac{d(qk_t)}{qk_t}}$$

1 – θ

- return on money

$$\underbrace{(\Phi(\iota_t)-\delta)dt + \sigma dZ_t}_{\frac{d(pK_t)}{pK_t}}$$

θ

Equilibrium

- pricing for log utility $E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t)$

$$\underbrace{\frac{a-\iota_t}{q}}_{\text{diff. in return}} = \underbrace{(\sigma - \sigma)}_{\text{diff. in agg. risk}} \sigma + \underbrace{(\tilde{\sigma} - 0)}_{\text{diff. in idios. risk}} (1-\theta)\tilde{\sigma}$$

- mkt. clearing for log $\underbrace{\rho \frac{q_t}{1-\theta} K_t}_{\text{total wealth}} = \underbrace{(a-\iota_t)K_t}_{\text{total consumption}}$
- Hence,

$$\frac{\rho}{1-\theta} = (1-\theta)\tilde{\sigma}^2 \quad \Rightarrow \quad \theta = 1 - \sqrt{\rho}/\tilde{\sigma}$$

money has value only if: $\tilde{\sigma}^2 > \rho$

- **Suppose** Φ(ι) = log(κι + 1)/κ, so ι = (q – 1)/κ

$$a - \iota = \rho \frac{q}{1-\theta} \Rightarrow q = \frac{a + 1/\kappa}{\rho/(1-\theta) + 1/\kappa}, \quad \text{declines in } \theta$$

Policy / Controlling the Value of Money

- suppose policy maker taxes capital, so # of coins follows

$$\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t$$

  - fiscal backing of currency
  - more on different types of taxes later…
- If pK_t is the value of all M_t coins held, value of one coin is pK_t/M_t and return on money is

$$\frac{d(pK_t / M_t)}{pK_t / M_t} = \underbrace{(\Phi(\iota_t) - \delta)dt + \sigma dZ_t}_{\frac{d(qk_t)}{qk_t}} \underbrace{-\mu^M dt - \sigma^M dZ_t}_{-\frac{dM_t}{M_t}} + \underbrace{\sigma^M(\sigma^M - \sigma)dt}_{\text{Ito term}}$$

Equilibrium with Policy

- # of coins follows $$\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t$$
- return on money is

$$\frac{d(pK_t/M_t)}{pK_t/M_t} = \underbrace{(\Phi(\iota_t)-\delta)dt + \sigma dZ_t}_{\frac{d(qk_t)}{qk_t}} \underbrace{-\mu^M dt - \sigma^M dZ_t}_{-\frac{dM_t}{M_t}} + \underbrace{\sigma^M(\sigma^M - \sigma)dt}_{\text{Ito term}}$$

- return on portfolio of capital and money

$$\underbrace{\rho\, dt}_{\substack{\text{dividend yield} \\ \text{(cons. mkt clearing)}}} + (\Phi(\iota_t)-\delta)\; dt + \sigma dZ_t + (1-\theta)\tilde{\sigma} d\tilde{Z}_t$$

- pricing $E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t)$

$$\rho + \mu^M - \sigma^M(\sigma^M - \sigma) = \underbrace{(\sigma - \sigma + \sigma^M)}_{\text{diff. in agg. risk}}\sigma + \underbrace{((1-\theta)\tilde{\sigma} - 0)}_{\text{diff. in idios. risk}}(1-\theta)\tilde{\sigma}$$

Policy and Money Value

- # of coins follows $$\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t$$
- money value $$\rho + \mu^M - (\sigma^M)^2 = (1-\theta)^2 \tilde{\sigma}^2$$

1. Money value increases in fiscal backing (when $\mu^M < 0$).
2. Planner can make money risk-free by setting $\sigma^M = \sigma$.
   Then θ goes up further.
3. Price level follows

$$\frac{d(M_t/(pK_t))}{M_t/(pK_t)} = \mu^M dt + \sigma^M dZ_t - (\Phi(\iota_t) - \delta)dt - \sigma dZ_t + \sigma(\sigma - \sigma^M)dt$$

4. Planner can create inflation by paying interest on money, so

$$\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t + i\, dt$$

Remarks

- Taxes / transfers

1) proportionately to money holdings: i
2) proportionately to capital holdings: $\mu^M dt + \sigma^M dZ_t$
3) proportionately to net worth
4) per capita

Remarks

- Taxes / transfers

1) proportionately to money holdings: i
   - no real effect, affects price level
2) proportionately to capital holdings: $\mu^M dt + \sigma^M dZ_t$
   - μ^M pushes down money return
   - capital return goes up
   - pushes people to hold less money, invest more
3) proportionately to net worth
   - only transfers to capital matter, effect less by 1 - θ
4) per capita
   - no real effect – people simply borrow against the transfers they expect to receive

Model with Intermediaries

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \underbrace{\sigma dZ_t}_{\text{aggregate}} + \underbrace{\tilde{\sigma} d\tilde{Z}_t}_{\text{idiosyncratic}}$$

- Intermediaries can hold equity share up to $\bar{\chi}$
- can diversify some idiosyncratic risk, reduce it to $\phi\tilde{\sigma}$
- Intermediaries' wealth share $\eta_t = N_t / ((p_t + q_t)K_t)$
- No policy: risk of capital $\sigma + \sigma^q$, money $\sigma + \sigma^p$, incremental risk of capital ($\theta = p/(p + q)$)

$$\sigma_t^q - \sigma_t^p = -\frac{\sigma^\theta}{1-\theta}$$

- Policy: risk of money $\sigma + \sigma^p - \sigma^M$, capital, $\sigma + \sigma^q + \theta\sigma^M/(1- \theta)$, incremental

$$\frac{-\sigma^\theta + \sigma^M}{1-\theta}$$

Allocation

- Minimize weighted average cost of financing
- Price of risk = volatility of wealth. Can use any numeraire, e.g. world wealth

$$\text{price of aggregate risk:} \quad \underbrace{\sigma_t^\eta,}_{\text{intermediaries}} \qquad \underbrace{\frac{-\eta\sigma_t^\eta}{1-\eta}}_{\text{households}}$$

$$\text{price of idiosyncratic risk:} \quad \underbrace{(1-\theta)\frac{\chi_t\phi\tilde{\sigma}}{\eta_t}}_{\text{intermediaries}} \qquad \underbrace{(1-\theta)\frac{(1-\chi_t)\tilde{\sigma}}{1-\eta_t}}_{\text{households}}$$

FOC (equality if $\chi = \bar{\chi}$)

$$\underbrace{\frac{-\sigma^\theta + \sigma^M}{1-\theta}}_{\text{incremental aggregate risk}} \quad \sigma_t^\eta + \phi\tilde{\sigma}(1-\theta)\frac{\chi_t\phi\tilde{\sigma}}{\eta_t} \le \underbrace{\frac{-\sigma^\theta + \sigma^M}{1-\theta}}_{\text{incremental aggregate risk}} \quad \frac{(-\eta\sigma_t^\eta)}{1-\eta} + \tilde{\sigma}(1-\theta)\frac{(1-\chi_t)\tilde{\sigma}}{1-\eta_t}$$

Risk of η and amplification

- σ^η is the risk of intermediaries' wealth, with world wealth as numeraire
- In this numeraire the risk of money is σ^θ without policy… $\sigma^\theta - \sigma^M$ with policy
- Then

$$\sigma_t^\eta = \sigma^\theta - \sigma^M + \underbrace{\frac{(1-\theta)\chi}{\eta}}_{\text{capital portfolio weight}} \underbrace{\frac{-\sigma^\theta + \sigma^M}{1-\theta}}_{\text{incremental risk of capital}} = \frac{\chi-\eta}{\eta}(-\underbrace{\sigma^\theta}_{\frac{\theta'(\eta)}{\theta(\eta)}\eta\sigma_t^\eta} + \sigma^M)$$

$$\Rightarrow \quad \eta\sigma_t^\eta = \frac{(\chi-\eta)\sigma^M}{1+(\chi-\eta)\dfrac{\theta'(\eta)}{\theta(\eta)}}$$

- Without policy $\sigma^\eta = 0$. Otherwise (e.g. if policy maker wants money to be risk-free), σ^M is amplified…
- We'll see the shape of θ in a moment…

Capital allocation

- Suppose $\sigma^M = 0$, so $\sigma^\eta, \sigma^\theta = 0$
- Then χ is given by the FOC

$$\phi\tilde{\sigma}(1-\theta)\frac{\chi_t\phi\tilde{\sigma}}{\eta_t} \le \tilde{\sigma}(1-\theta)\frac{(1-\chi_t)\tilde{\sigma}}{1-\eta_t} \Rightarrow \chi = \min\left(\frac{\eta}{\eta+(1-\eta)\phi^2}, \bar{\chi}\right)$$

Money return

- Return on money w/o policy (numeraire = world wealth)

$$\frac{d\theta_t}{\theta_t} = \mu_t^\theta dt + \sigma_t^\theta dZ_t$$

- W/ policy (outside money is M_t coins), one coin is θ_t/M_t of world wealth, money return is

$$\frac{d(\theta_t / M_t)}{(\theta_t / M_t)} = \mu_t^\theta dt + \sigma_t^\theta dZ_t - \mu_t^M dt - \sigma_t^M dZ_t + \sigma_t^M (\sigma_t^M - \sigma_t^\theta) dt = \mu_t^\hbar dt + \sigma_t^\hbar dZ_t$$

- Next, the law of motion of η and money valuation…

Pricing individuals' portfolios

$$E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t)$$

- Holds regardless of numeraire. Let's use global wealth
- Suppose η^i is wealth share of some agent group i.
  Asset A: their portfolio, B: money

$$\mu_t^{\eta,i} + \rho^i - \mu_t^{\hbar} = (\sigma_t^{\eta,i} - \sigma_t^{\hbar})\sigma_t^{\eta,i} + (\tilde{\sigma}_t^i)^2$$

drift and volatility of various variables, as in

idiosyncratic risk exposure in group i

where $$\frac{d\eta^i}{\eta^i} = \mu_t^{\eta,i}dt + \sigma_t^{\eta,i}dZ_t$$

Money valuation equation

$$\mu_t^{\eta,i} + \rho^i - \mu_t^{\hbar} = (\sigma_t^{\eta,i} - \sigma_t^{\hbar})\sigma_t^{\eta,i} + (\tilde{\sigma}_t^i)^2$$

- Add across agents with weights η^i

$$\sum_i \eta^i \mu_t^{\eta,i} + \sum_i \eta^i \rho^i - \mu_t^{\hbar} = \sum_i \eta^i (\sigma_t^{\eta,i})^2 - \sigma_t^{\hbar} \sum_i \eta^i \sigma_t^{\eta,i} + \sum_i \eta^i (\tilde{\sigma}_t^i)^2$$

$$\Rightarrow \sum_i \eta^i \rho^i - \underbrace{(\mu_t^{\theta} - \mu_t^M + \sigma_t^M(\sigma_t^M - \sigma_t^{\theta}))}_{\mu_t^{\hbar}} = \sum_i \eta^i \left((\sigma_t^{\eta,i})^2 + (\tilde{\sigma}_t^i)^2 \right)$$

- This generalizes $\rho + \mu^M - (\sigma^M)^2 = (1-\theta)^2 \tilde{\sigma}^2$ (baseline model)
- Average risk exposure > ave. discount rate when money is depreciating
- Planner can attain any function θ(η) and any risk of money (by choosing μ^M that satisfies the money value equation)

Law of motion of wealth shares

$$\mu_t^{\eta,i} + \rho^i - \mu_t^{\hbar} = (\sigma_t^{\eta,i} - \sigma_t^{\hbar})\sigma_t^{\eta,i} + (\tilde{\sigma}_t^i)^2$$

- Subtract money valuation

$$\sum_i \eta^i \rho^i - \mu_t^{\hbar} = \sum_i \eta^i \left((\sigma_t^{\eta,i})^2 + (\tilde{\sigma}_t^i)^2 \right)$$

$$\mu_t^{\eta,i} = (\sigma_t^{\eta,i} - \sigma_t^{\hbar})\sigma_t^{\eta,i} + (\tilde{\sigma}_t^i)^2 - \sum_j \eta^j \rho^j - \sum_j \eta^j \left((\sigma_t^{\eta,j})^2 + (\tilde{\sigma}_t^j)^2 \right) - \rho^i$$

- Drift of η determined by
  - consumption rate of group i relative to others
  - risk exposure (agg. + idiosyncratic) of group i relative to others
  - covariance of the risk of group i and money

Let's apply this to our model

$$\sigma^{\eta} = 0, \quad \tilde{\sigma}^{I} = (1-\theta)\frac{\chi\phi\tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^{H} = (1-\theta)\frac{(1-\chi)\tilde{\sigma}}{1-\eta}$$

$$\mu_t^{\eta} = (\tilde{\sigma}_t^{I})^2 - \eta(\tilde{\sigma}_t^{I})^2 - (1-\eta)(\tilde{\sigma}_t^{H})^2 = (1-\eta)(1-\theta)^2\left(\frac{\chi^2\phi^2}{\eta^2} - \frac{(1-\chi)^2}{(1-\eta)^2}\right)\tilde{\sigma}^2$$

$$\rho - \mu_t^{\theta} = \underbrace{(1-\theta)^2\left(\eta\frac{\chi^2\phi^2}{\eta^2} + (1-\eta)\frac{(1-\chi)^2}{(1-\eta)^2}\right)\tilde{\sigma}^2}_{\eta(\tilde{\sigma}_t^{I})^2 + (1-\eta)(\tilde{\sigma}_t^{H})^2}$$

$$\text{where } \chi = \min\left(\frac{\eta}{\eta + (1-\eta)\phi^2}, \bar{\chi}\right)$$

θ minimized when $\mu^\eta = 0$

$$\sigma^\eta = 0, \quad \tilde{\sigma}^I = (1-\theta)\frac{\chi\phi\tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = (1-\theta)\frac{(1-\chi)\tilde{\sigma}}{1-\eta}$$

$$\mu_t^\eta = (\tilde{\sigma}_t^I)^2 - \eta(\tilde{\sigma}_t^I)^2 - (1-\eta)(\tilde{\sigma}_t^H)^2 = (1-\eta)(1-\theta)^2 \underbrace{\left(\frac{\chi^2\phi^2}{\eta^2} - \frac{(1-\chi)^2}{(1-\eta)^2}\right)\tilde{\sigma}^2}_{-dX/d\eta}$$

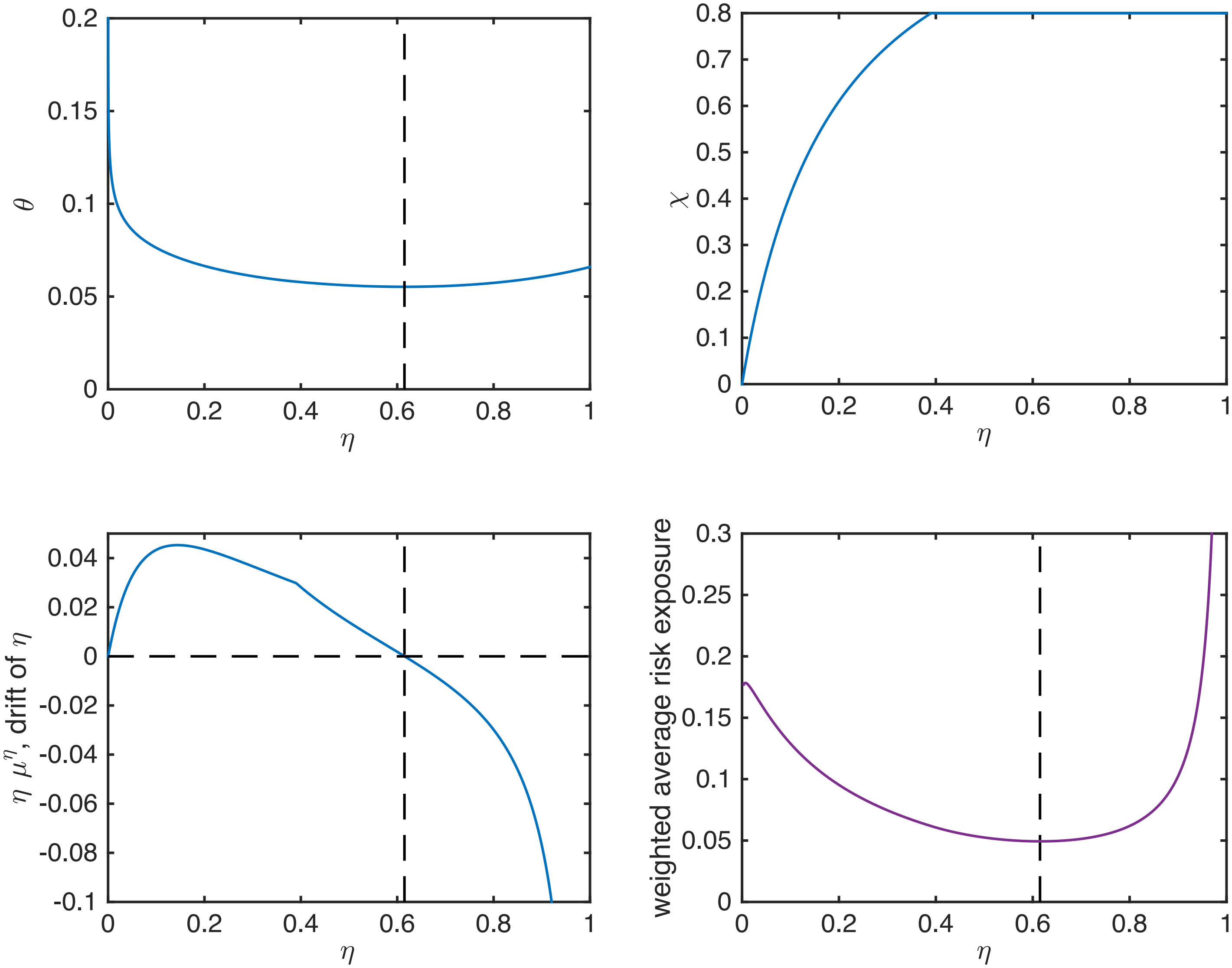
$$\rho - \mu_t^\theta = \underbrace{(1-\theta)^2 \overbrace{\left(\eta\frac{\chi^2\phi^2}{\eta^2} + (1-\eta)\frac{(1-\chi)^2}{(1-\eta)^2}\right)\tilde{\sigma}^2}^{X(\eta)}}_{\eta(\tilde{\sigma}_t^I)^2 + (1-\eta)(\tilde{\sigma}_t^H)^2}$$

$$\text{where } \chi = \min\left(\frac{\eta}{\eta + (1-\eta)\phi^2}, \bar{\chi}\right)$$

- Ave. idiosyncratic risk exposure (before the effect of money) is minimized at the steady state of η

Example

$$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\chi} = 0.8$$



Aggregate risk on I's balance sheets

- I get exposed to aggregate risk if their investments ≠ average in the economy, i.e. they specialize
- Easiest way to capture this: assume that fraction $\bar{\psi}$ of capital has to be in technology b and $1-\bar{\psi}$ in *a*
- I can invest only in technology b, diversify idios. risk to $\phi\tilde{\sigma}$

$$\underbrace{\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \underbrace{\tilde{\sigma} d\tilde{Z}_t}_{\text{idiosyncratic}}}_{\text{technology a}} \qquad \underbrace{\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \underbrace{\sigma dZ_t}_{\text{aggregate}} + \underbrace{\tilde{\sigma} d\tilde{Z}_t}_{\text{idiosyncratic}}}_{\text{technology b}}$$

- Fundamental risk of money / economy: $\bar{\psi}\sigma$
- Incremental risk of technologies a and b

$$-\bar{\psi}\sigma + \frac{-\sigma^\theta + \sigma^M}{1-\theta}, \qquad (1-\bar{\psi})\sigma + \frac{-\sigma^\theta + \sigma^M}{1-\theta}$$

Allocation

- Minimize weighted average cost of financing

$$\left((1-\bar{\psi})\sigma - \frac{\sigma^{\theta}-\sigma^{M}}{1-\theta}\right)\sigma_t^{\eta} + \phi\tilde{\sigma}(1-\theta)\frac{\psi\phi\tilde{\sigma}}{\eta} \le \left((1-\bar{\psi})\sigma - \frac{\sigma^{\theta}-\sigma^{M}}{1-\theta}\right)\frac{(-\eta\sigma_t^{\eta})}{1-\eta} + \tilde{\sigma}(1-\theta)\frac{(1-\psi)\tilde{\sigma}}{1-\eta}$$

(equality if $\chi = \bar{\chi}$)

$$\sigma_t^{\eta} = \sigma^{\theta} - \sigma^{M} + \underbrace{\frac{(1-\theta)\psi}{\eta}}_{\text{capital portfolio weight}} \underbrace{\left((1-\bar{\psi})\sigma - \frac{\sigma^{\theta}-\sigma^{M}}{1-\theta}\right)}_{\text{incremental risk of capital}} \Rightarrow$$

$$\eta\sigma_t^{\eta} = \frac{(1-\theta)\psi(1-\bar{\psi})\sigma + (\psi-\eta)\sigma^{M}}{1+(\psi-\eta)\frac{\theta'(\eta)}{\theta(\eta)}} \qquad \left(= (1-\theta)\psi(1-\bar{\psi})\sigma \quad \text{if } \sigma^{M} = \sigma^{\theta}\right)$$

policy removes endogenous risk / amplification

Law of motion of η and money valuation

- Money valuation

$$\rho - \mu_t^{\hbar} = \eta\left((\sigma_t^{\eta})^2 + (\tilde{\sigma}_t^{I})^2\right) + (1-\eta)\left(\left(\frac{\eta\sigma_t^{\eta}}{1-\eta}\right)^2 + (\tilde{\sigma}_t^{H})^2\right)$$

$$\mu_t^{\eta} = (1-\eta)\left((\sigma_t^{\eta})^2 + (\tilde{\sigma}_t^{I})^2 - \left(\frac{\eta\sigma_t^{\eta}}{1-\eta}\right)^2 - (\tilde{\sigma}_t^{H})^2\right) - \sigma_t^{\eta}\underbrace{\sigma_t^{\hbar}}_{\sigma^{\theta}-\sigma^{M}}$$

If the policy removes endogenous risk…

$$\sigma^M = \sigma^\theta$$

$$\psi = \min\left(\frac{\eta}{\eta + (1-\eta)\phi^2 + (1-\bar{\psi})^2 \sigma^2 / \tilde{\sigma}^2}, \bar{\psi} \right)$$

closed form

$$\sigma^\eta = (1-\theta)\frac{\psi}{\eta}(1-\bar{\psi})\sigma$$

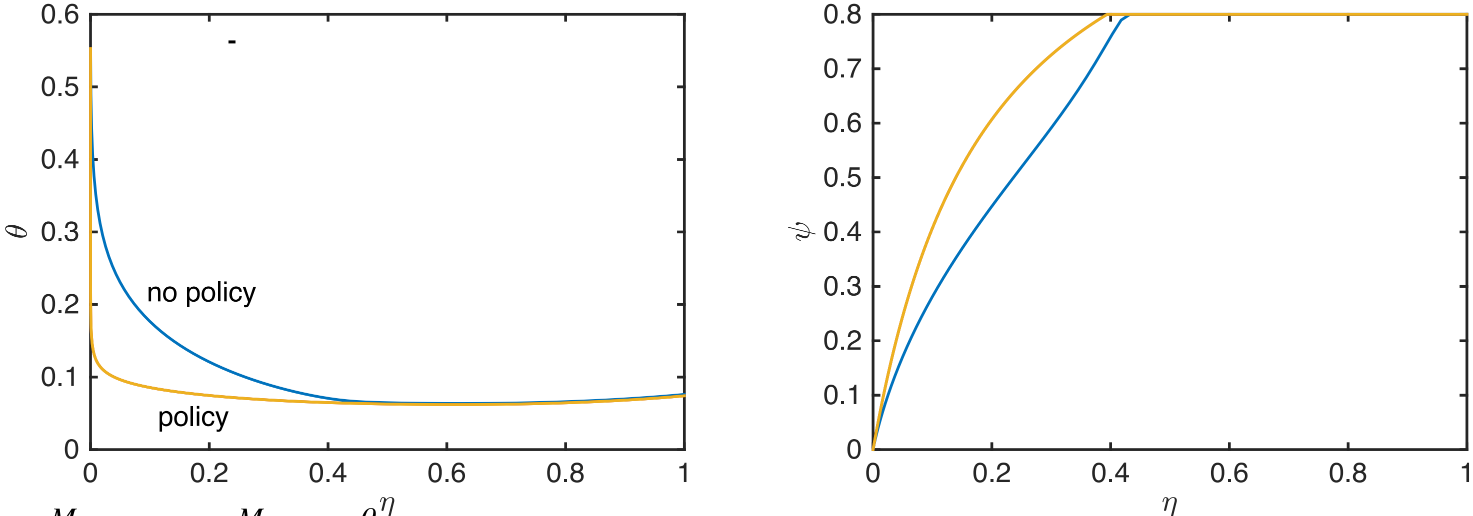
closed form up to θ (choice of planner)

$$\eta\mu_t^\eta = \eta(1-\eta)(1-\theta)^2 \left(\frac{1-2\eta}{(1-\eta)^2} \frac{\psi^2}{\eta^2}(1-\bar{\psi})^2 \sigma^2 + \frac{\psi^2 \phi^2 \tilde{\sigma}^2}{\eta^2} - \frac{(1-\psi)^2 \tilde{\sigma}^2}{(1-\eta)^2} \right)$$

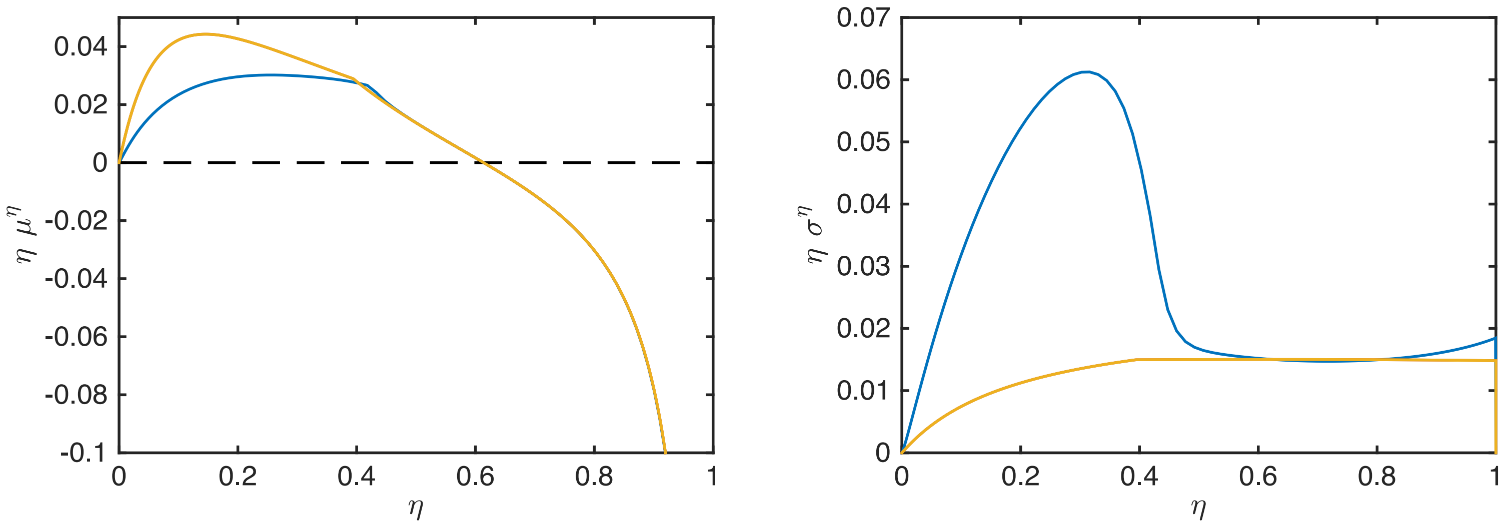
$$\psi = \frac{\eta}{(1-\bar{\psi})^2 \sigma^2 / \tilde{\sigma}^2 + \phi^2 (1-\eta) + \eta}$$

Example

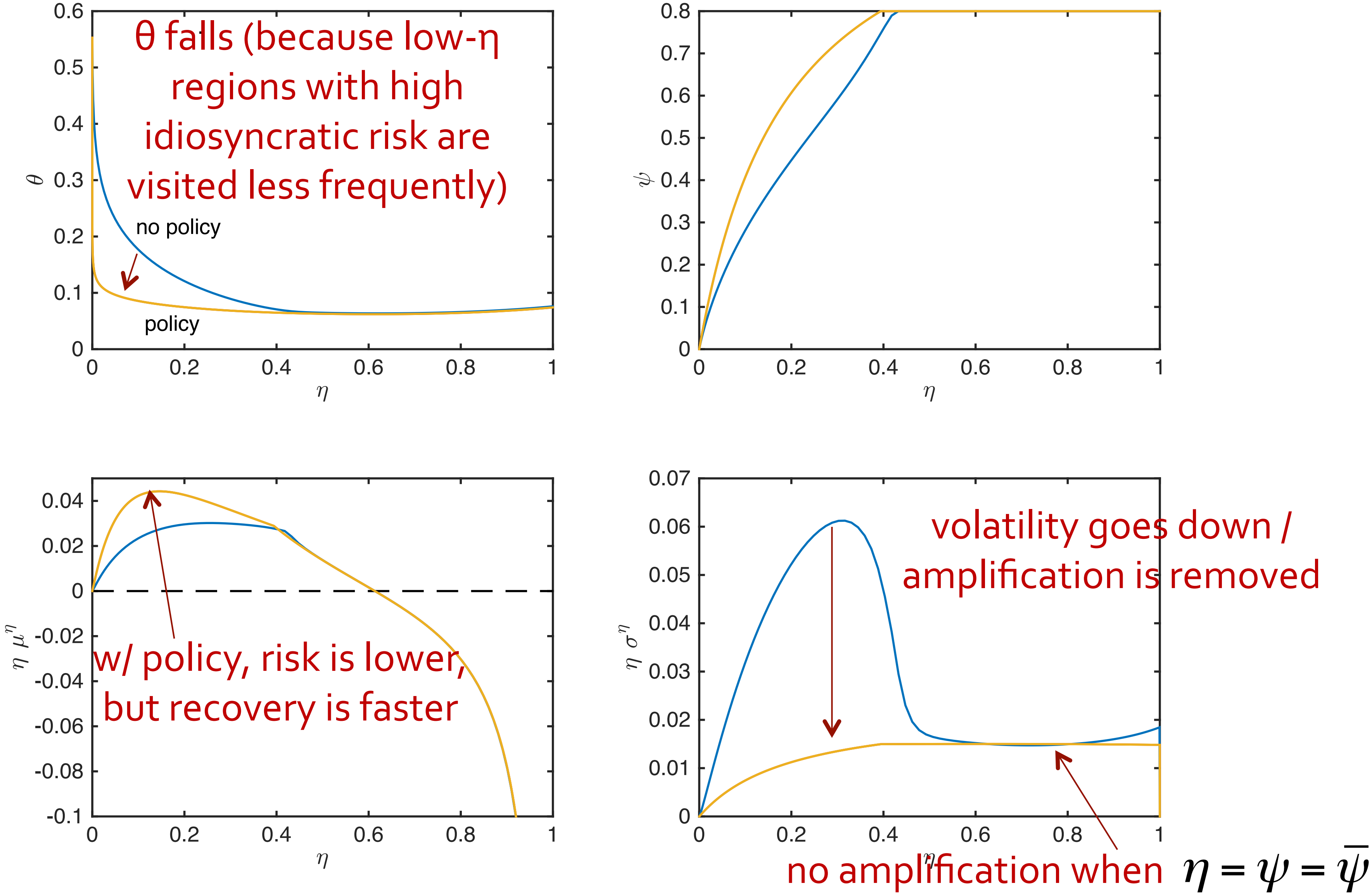
$$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\chi} = 0.8, \sigma = 0.1$$



policy: $\mu^M = 0,\ \sigma^M = \sigma^{\theta^\eta}$



Example $\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\psi} = 0.8, \sigma = 0.1$



Optimal Policy?

- Generally hard question: need a precise definition of the policy space / analytical tools to characterize the optimum
- One side: inefficiencies / tradeoffs
  - insurance vs. investment
  - allocation of assets / risk
- Other side: policy space
  - (1) controlling money growth rate
  - (2) macroprudential tools / wealth redistribution
  - (3) risk redistribution
- Many moving parts, but we can get clear answers in some simple settings

Optimal policy? Welfare with log utility:

- Class of models: price capital q_t, value of money $p_t K_t$, two types of agents I and H, have wealth shares η_t and $1 - \eta_t$, idiosyncratic risk exposures

$$\tilde{\sigma}_t^I \quad \text{and} \quad \tilde{\sigma}_t^H$$

- Then the welfare of I is (similar formula for H)

$$E\left[\int_0^\infty e^{-\rho t} \log(c_t^I)\, dt\right] = E\left[\int_0^\infty e^{-\rho t} \log(\eta_t (a(\psi_t) - \iota_t) K_t \tilde{\eta}_t^I)\, dt\right],$$

$$\tilde{\eta}_0^I = 1, \quad \frac{d\tilde{\eta}_t^I}{\tilde{\eta}_t^I} = \tilde{\sigma}_t^I d\tilde{Z}_t$$

Welfare with log utility

- The welfare of I is

$$E\left[\int_0^\infty e^{-\rho t}\log(\eta_t(a(\psi_t)-\iota_t)K_t\tilde{\eta}_t)\,dt\right] = \underbrace{E\left[\int_0^\infty e^{-\rho t}\log\eta_t\,dt\right]}_{\frac{\log\eta_0}{\rho}+E\left[\int_0^\infty e^{-\rho t}\left(\frac{\mu_t^\eta}{\rho}-\frac{|\sigma_t^\eta|^2}{2\rho}\right)dt\right]} +$$

$$E\left[\int_0^\infty e^{-\rho t}\log(a(\psi_t)-\iota_t)\,dt\right] + \underbrace{E\left[\int_0^\infty e^{-\rho t}\log K_t\,dt\right]}_{\frac{\log K_0}{\rho}+E\left[\int_0^\infty e^{-\rho t}\left(\frac{\Phi(\iota_t)-\delta}{\rho}-\frac{|\sigma_t^K|^2}{2\rho}\right)dt\right]} + \underbrace{E\left[\int_0^\infty e^{-\rho t}\log\tilde{\eta}_t\,dt\right]}_{-E\left[\int_0^\infty e^{-\rho t}\frac{(\tilde{\sigma}_t^I)^2}{2\rho}dt\right]}$$

$$\tilde{\eta}_0^I = 1, \quad \frac{d\tilde{\eta}_t^I}{\tilde{\eta}_t^I} = \tilde{\sigma}_t^I\,d\tilde{Z}_t$$

Welfare

- We see that policy can affect welfare in several ways

$$E\left[\int_0^\infty e^{-\rho t}\log\eta_t\,dt\right]+E\left[\int_0^\infty e^{-\rho t}\log(a(\psi_t)-\iota_t)\,dt\right]+$$

$$E\left[\int_0^\infty e^{-\rho t}\left(\frac{\Phi(\iota_t)-\delta}{\rho}-\frac{|\sigma_t^K|^2}{2\rho}\right)dt\right]-E\left[\int_0^\infty e^{-\rho t}\frac{(\tilde{\sigma}_t^I)^2}{2\rho}\,dt\right]$$

- investment vs. consumption
- allocation of capital – idiosyncratic risk, total output
- η – the distribution of consumption and risk absorption capacity

One at a time: policy tools and equilibrium features

- Generally, idiosyncratic risk exposures $\tilde{\sigma}_t^I$ and $\tilde{\sigma}_t^H$ are stochastic (depend on η, risk absorption capacity, allocation)
- If intermediaries help reduce idiosyncratic risk, these may rise when η declines (or goes away from the middle)
- Let's see, how this matters with a simple model

Stochastic idiosyncratic risk

- One type of agents H, idiosyncratic risk of capital is stochastic (hence it is a state variable)

$$d\tilde{\sigma}_t = \tilde{\mu}(\tilde{\sigma}_t)dt + \tilde{v}(\tilde{\sigma}_t)dZ_t$$

e.g. as in Di Tella, CIR process

$$d\tilde{\sigma}_t = \lambda(\bar{\sigma} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}\,dZ_t$$

- Global wealth as numeraire, agents' entire portfolio has return ρ (just the consumption rate)
- Money has return

$$\mu_t^{\theta}dt + \sigma_t^{\theta}dZ_t - \mu_t^{M}dt$$

rate of money printing, which is distributed to capital

Stochastic idiosyncratic risk

- Global wealth as numeraire, wealth has return ρ
  Money has return

$$\mu_t^\theta dt + \sigma_t^\theta dZ_t - \mu_t^M dt$$

- Money valuation equation

$$\rho - (\mu_t^\theta - \mu_t^M) = \underbrace{(1-\theta_t)\tilde{\sigma}_t}_{\text{idiosync. risk of wealth}} \quad \underbrace{(1-\theta_t)\tilde{\sigma}_t}_{\text{price of idiosync. risk}} \quad \underbrace{-\sigma_t^\theta}_{\text{agg. risk of wealth rel. to money}} \quad \underbrace{0}_{\text{price of agg. risk}}$$

- Without policy, equation

$$\rho - \mu_t^\theta = (1-\theta_t)^2 \tilde{\sigma}_t^2$$

has a unique solution in $\vartheta(\tilde{\sigma}_t) \in (0,1)$ (if idiosyncratic risk is sufficiently large)

Optimal Policy

- Market-clearing for output

$$a-\iota(q)=\rho\frac{q}{\underbrace{1-\theta}_{p+q}},\quad \text{if } \Phi(\iota)=\frac{\log(\kappa\iota+1)}{\kappa}, \iota(q)=\frac{q-1}{\kappa},\quad q=\frac{(a\kappa+1)(1-\theta)}{\rho\kappa+1-\theta}$$

- welfare is $\dfrac{\log K_0}{\rho}-\dfrac{\delta}{\rho^2}+$

$$\underbrace{E\left[\int_0^\infty e^{-\rho t}\log(a-\iota_t)\,dt\right]}_{E\left[\int_0^\infty e^{-\rho t}\log\left(\rho\frac{a\kappa+1}{\rho\kappa+1-\theta_t}\right)dt\right]}+\underbrace{E\left[\int_0^\infty e^{-\rho t}\frac{\Phi(\iota_t)}{\rho}\,dt\right]}_{\frac{1}{\rho\kappa}E\left[\int_0^\infty e^{-\rho t}\log\left(\frac{(a\kappa+1)(1-\theta_t)}{\rho\kappa+1-\theta_t}\right)dt\right]}-E\left[\int_0^\infty e^{-\rho t}\frac{(1-\theta_t)^2\tilde{\sigma}_t^2}{2\rho}\,dt\right]$$

- let $\vartheta^*(\tilde{\sigma}^2)$ be the maximizer of (optimal baseline policy)

$$\frac{1}{\rho\kappa}\log(1-\theta)-\frac{\rho\kappa+1}{\rho\kappa}\log(\rho\kappa+1-\theta)-\frac{(1-\theta)^2\tilde{\sigma}^2}{2\rho}$$

Optimal policy

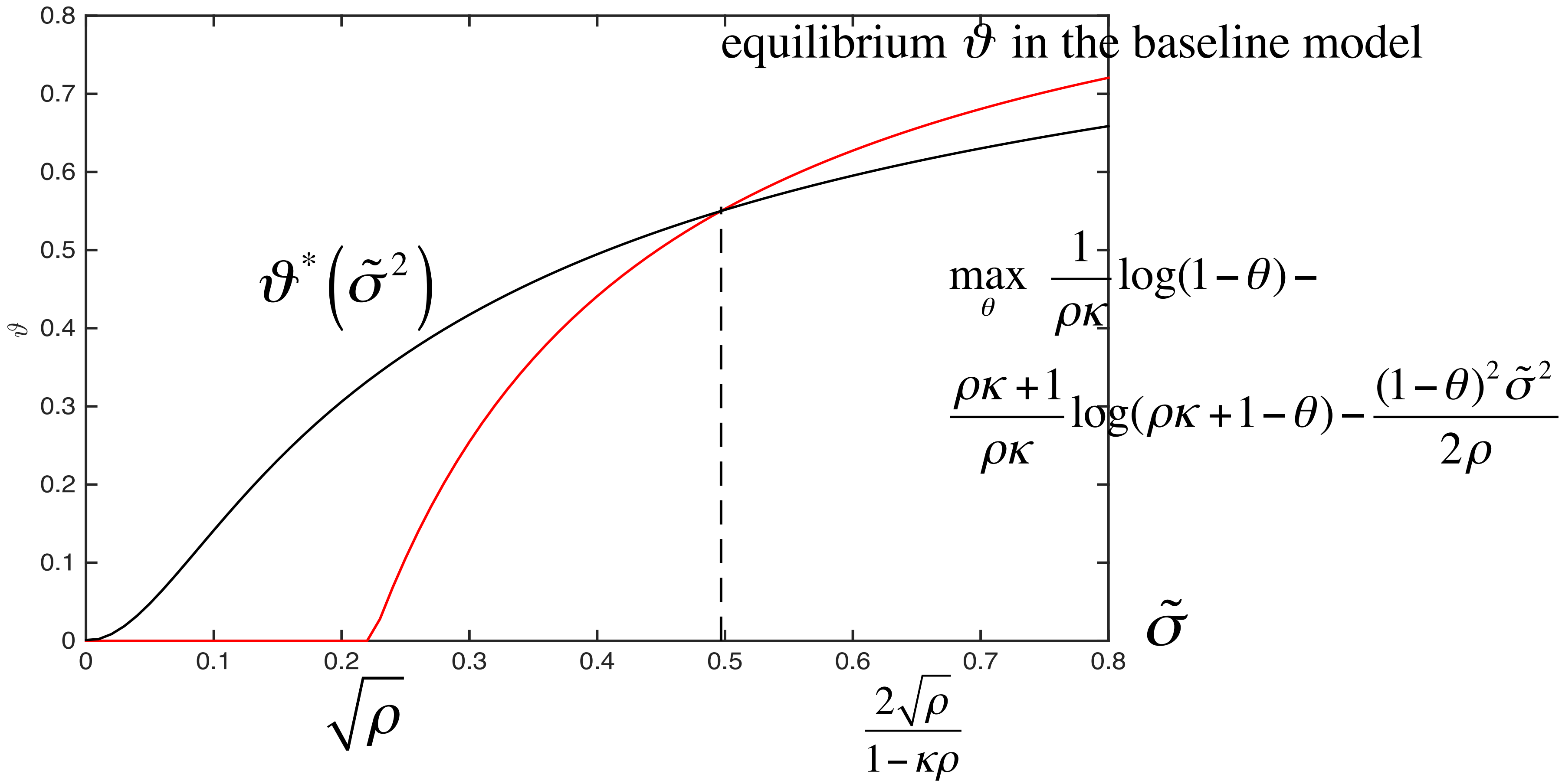
- If the planner could control θ_t directly, she would set $\theta_t = \vartheta^*(\tilde{\sigma}_t^2)$
- Controlling indirectly by choosing μ_t^M the planner can achieve any function - including $\vartheta^*(\tilde{\sigma}_t^2)$ - by solving

$$\rho - (\mu_t^\theta - \mu_t^M) = (1-\theta_t)^2 \tilde{\sigma}_t^2$$

for μ_t^M

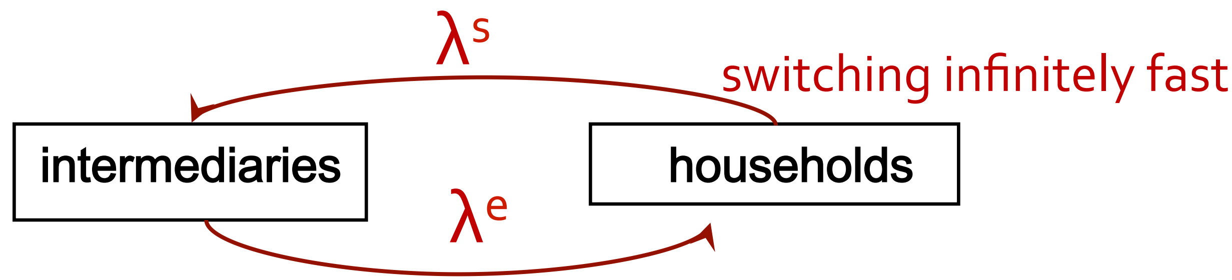
- Optimal policy is easier to find than even the equilibrium outcome (differentiation vs. integration)
- Risk-free rate $\Phi(\iota_t) - \delta + \mu_t^\theta - \mu_t^M = \rho - (1-\theta_t)^2 \tilde{\sigma}_t^2 + \Phi(\iota_t) - \delta$
  declines as $\tilde{\sigma}_t^2$ increases
- Nice relationship b/w baseline and dynamic model

- The relationship between idiosyncratic risk level $\tilde{\sigma}_t$ and optimal insurance $\vartheta^*(\tilde{\sigma}_t^2)$ pops up everywhere



- Let's consider another model, with heterogeneous agents but with exogenous wealth distribution

Switching types



share of agents also wealth share	η	$1-\eta$
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$ diversification	$\tilde{\sigma}$
output per unit of capital	the same, independently of the allocation	

Policy maker can choose the money growth rate μ_t^M

Remarks

- Policy-maker cannot affect the wealth shares (exogenously fixed by the switching process)
- Welfare weights on intermediaries and households are η and $1 - \eta$ from the setup
- Optimal monetary (with or w/o macroprudential policy – controlling capital allocation)

Equilibrium capital allocation

- Fraction ψ of capital is held by the intermediaries
- Capital allocation must be such that

$$\underbrace{\phi\tilde{\sigma}}_{\text{idiosync. risk of I}} \underbrace{\frac{(1-\theta)\psi\phi\tilde{\sigma}}{\eta}}_{\text{I's price of idiosync. risk}} = \underbrace{\tilde{\sigma}}_{\text{idiosync. risk of H}} \underbrace{\frac{(1-\theta)(1-\psi)\tilde{\sigma}}{1-\eta}}_{\text{H's price of idiosync. risk}}$$

$$\Rightarrow \quad \psi = \frac{\eta}{\phi^2(1-\eta)+\eta}$$

- Policy maker may try to affect ψ…

Welfare

- Law of large numbers: switching risk does not matter. Everyone's wealth growth averages out to $\Phi(\iota_t) - \delta$ and idiosyncratic risk exposure, to

$$\eta(\tilde{\sigma}^I)^2 + (1-\eta)(\tilde{\sigma}^H)^2 = (1-\theta)^2 \underbrace{\tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta} + \frac{(1-\psi)^2}{1-\eta} \right)}_{(\tilde{\sigma}^A)^2}$$

$$\tilde{\sigma}^I = \frac{(1-\theta)\psi\phi\tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = \frac{(1-\theta)(1-\psi)\tilde{\sigma}}{1-\eta}$$

- Welfare

$$E\left[\int_0^\infty e^{-\rho t} \log(a - \iota(\theta))\, dt\right] + E\left[\int_0^\infty e^{-\rho t} \frac{\Phi(\iota(\theta)) - \delta}{\rho}\, dt\right] - E\left[\int_0^\infty e^{-\rho t} \frac{(1-\theta)^2 (\tilde{\sigma}^A)^2}{2\rho}\, dt\right]$$

- Given $\tilde{\sigma}^A$, optimal to set $\theta = \vartheta^*((\tilde{\sigma}^A)^2)$

Money valuation

$$\rho - (\mu_t^\theta - \mu_t^M) = \underbrace{\eta(\tilde{\sigma}^I)^2 + (1-\eta)(\tilde{\sigma}^H)^2}_{(1-\theta)^2(\tilde{\sigma}^A)^2}$$

- Without policy,

$$\rho = (1-\theta)^2(\tilde{\sigma}^A)^2$$

Macroprudential tools

- Average idiosyncratic risk of capital

$$(\tilde{\sigma}^A)^2 = \tilde{\sigma}^2 \left(\frac{\psi^2 \phi^2}{\eta} + \frac{(1-\psi)^2}{1-\eta} \right)$$

is minimized when

$$\frac{\psi \phi^2}{\eta} = \frac{1-\psi}{1-\eta} \Rightarrow \psi = \frac{\eta}{\phi^2 (1-\eta) + \eta}$$

This is the equilibrium allocation! Optimal not to use macroprudential tools.

Remarks

- Same trade-off between insurance and investment
- Equilibrium allocation is efficient, minimizes the cost of risk exposure
- Policy space (1) money growth and (1) + (2) (also macroprudential tools) leads to the same outcome

Endogenous law of motion of η

- Wealth distribution can change endogenously with (i) risk exposure of intermediaries and households (ii) risk premia (iii) consumption rates
- Consider the following model

Fixed types (no switching)

	intermediaries	households	types fixed (no switching)
wealth shares	η	1-η	
welfare weights	λ	1-λ	You have already seen this model except here $\bar{\psi} = 1$
idiosyncratic risk of capital	$\phi\tilde{\sigma}, \phi \in (0,1)$	$\tilde{\sigma}$	
aggregate risk	σ	σ	
output per unit of capital	the same, independently of the allocation		

Two policy classes:

(1) choose the money growth rate μ_t^M

(1) + (2) also choose allocation (macroprudential) and transfer wealth between groups (why / how?)

Welfare of I and H

- Intermediaries (weight λ)

$$E\left[\int_0^\infty e^{-\rho t}\left(\log\eta_t + \log(a-\iota_t) + \frac{\Phi(\iota_t)-\delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1-\theta_t)^2}{2\rho}\frac{\psi^2\phi^2\tilde{\sigma}^2}{\eta^2}\right)dt\right]$$

- households (weight 1 – λ)

$$E\left[\int_0^\infty e^{-\rho t}\left(\log(1-\eta_t) + \log(a-\iota_t) + \frac{\Phi(\iota_t)-\delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1-\theta)^2}{2\rho}\frac{(1-\psi)^2\tilde{\sigma}^2}{(1-\eta)^2}\right)dt\right]$$

Optimal policy, (1) + (2)

- Planner chooses θ, ψ and η to maximize the disc. integral of

$$\lambda \log \eta_t + (1-\lambda)\log(1-\eta_t) + \log(a - \iota(\theta)) + \frac{\Phi(\iota(\theta)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$-\frac{(1-\theta)^2 \tilde{\sigma}^2}{2\rho} \underbrace{\left(\lambda \frac{\psi^2 \phi^2}{\eta^2} + (1-\lambda)\frac{(1-\psi)^2}{(1-\eta)^2}\right)}_{\frac{\lambda(1-\lambda)\phi^2}{\lambda\phi^2(1-\eta)^2+(1-\lambda)\eta^2},}$$

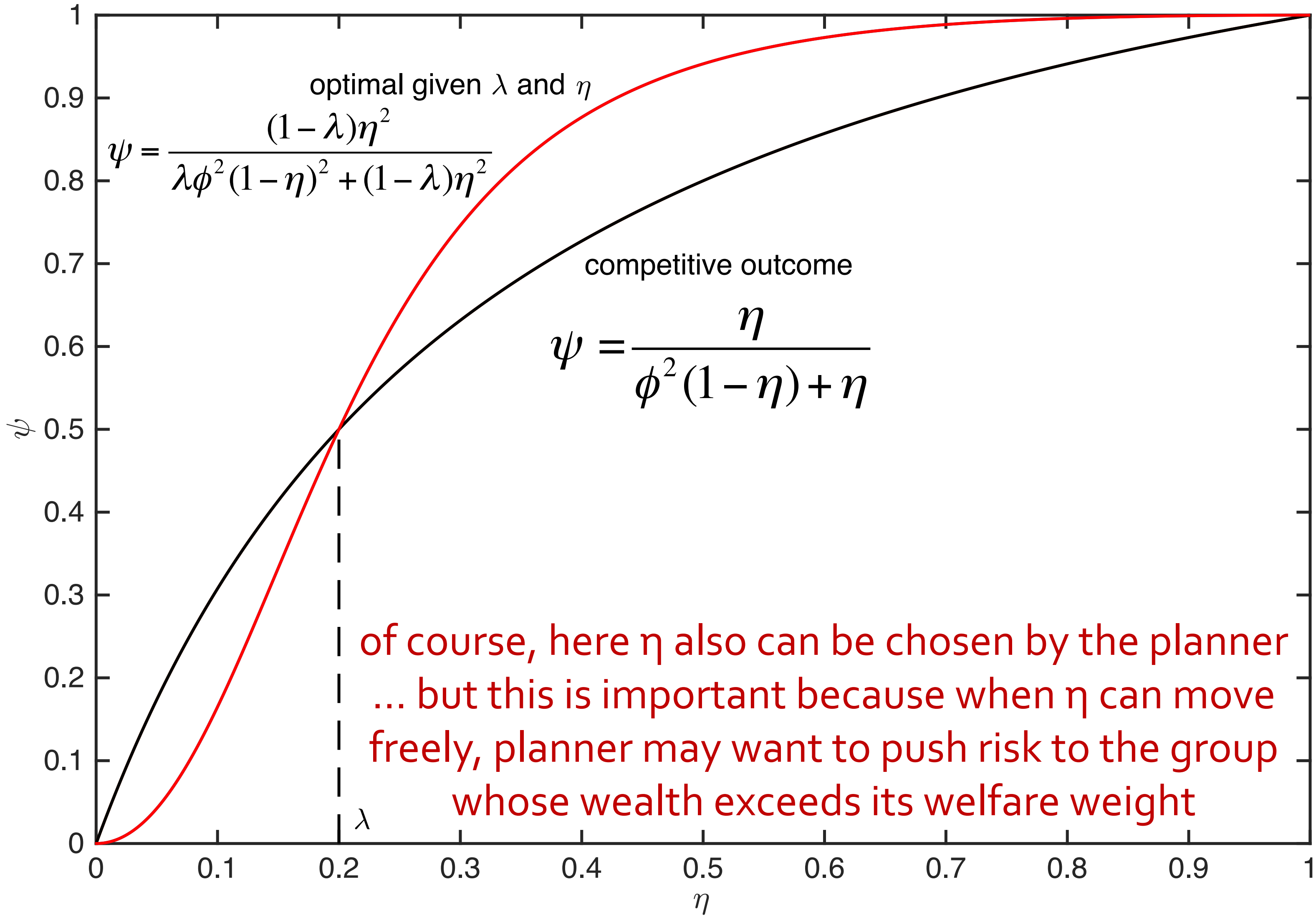
given the optimal choice of $\psi = \frac{(1-\lambda)\eta^2}{\lambda\phi^2(1-\eta)^2 + (1-\lambda)\eta^2}$

not the competitive allocation (unless η = λ)

- given ψ and η, optimal to set θ to

$$\theta = \vartheta^* \underbrace{\left(\tilde{\sigma}^2 \frac{\lambda(1-\lambda)\phi^2}{\lambda\phi^2(1-\eta)^2 + (1-\lambda)\eta^2}\right)}_{\text{welfare weighted average risk exposure}}$$

Competitive ψ vs. minimizing cost of risk



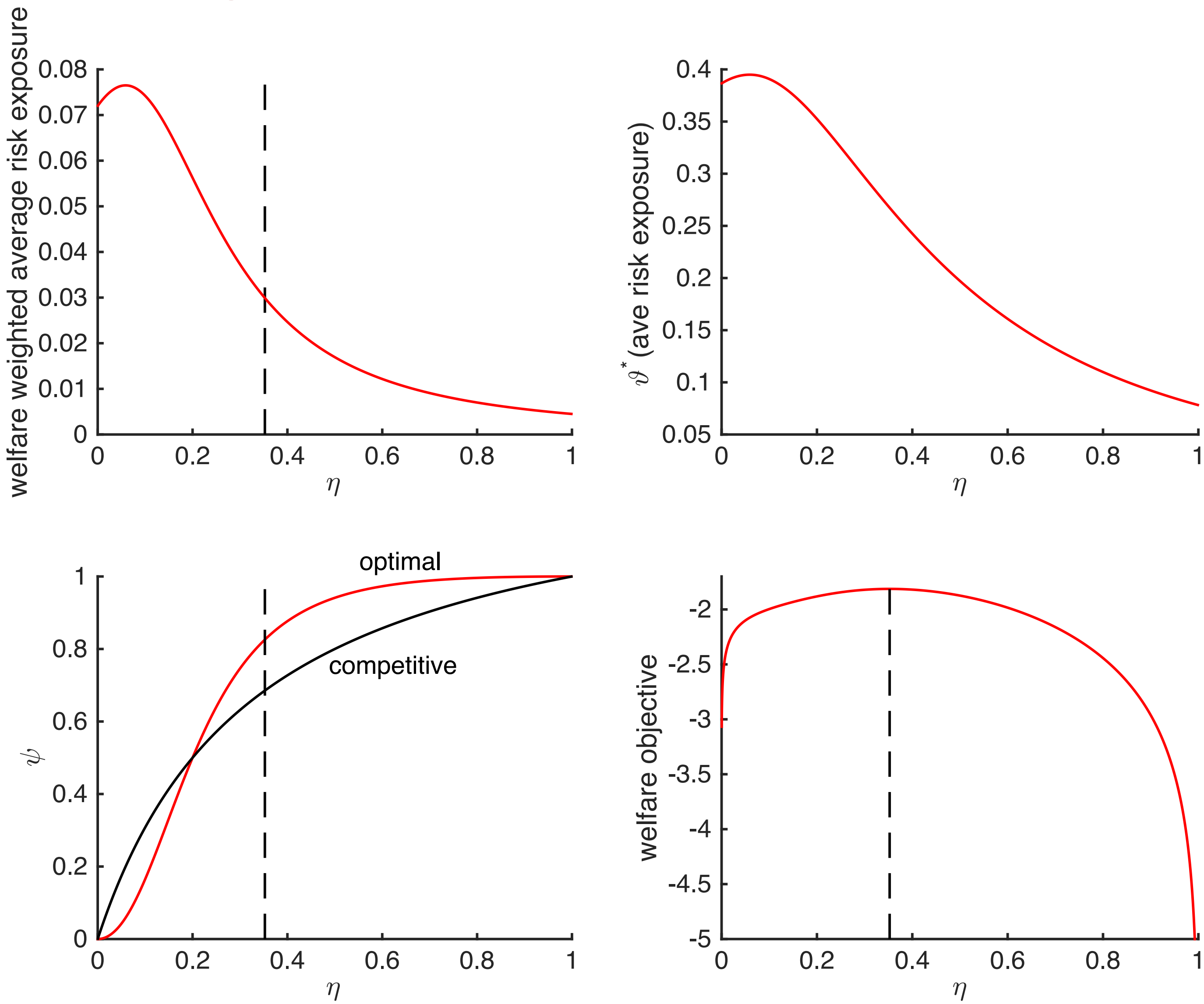
Optimal policy, (1) + (2)

- Finally, optimal η (given ϑ) – let's look at terms containing η

$$\max_{\eta} \underbrace{\lambda \log \eta + (1-\lambda)\log(1-\eta)}_{\text{concave, max at } \eta = \lambda \text{, goes to -}\infty \text{ at 0 and 1}} - \frac{(1-\vartheta)^2 \tilde{\sigma}^2}{2\rho} \frac{\lambda(1-\lambda)\phi^2}{\underbrace{\lambda\phi^2(1-\eta)^2 + (1-\lambda)\eta^2}_{\text{concave (!) also, max at } \frac{\lambda\phi^2}{\lambda\phi^2+1-\lambda} < \lambda}}$$

- hence, it is optimal to set η > λ (unfortunately I could not get a closed-form expression for the optimal η)
- push more risk to intermediaries than they'd take under competitive outcome
- relative to previous infinite switching model
  - it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  - overall risk is reduced and the value of money is lower (more intermediation)

Optimizing over η $\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2$



Optimal policy, (1) only

- What about monetary policy alone?
- Planner cannot alter the comp. allocation, $\psi = \dfrac{\eta}{\phi^2(1-\eta)+\eta}$
- Welfare is the disc. integral of

$$\lambda \log \eta + (1-\lambda)\log(1-\eta) + \log(a-\iota(\theta)) + \frac{\Phi(\iota(\theta))-\delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$-\frac{(1-\theta)^2\tilde{\sigma}^2}{2\rho}\underbrace{\left(\lambda\frac{\psi^2\phi^2}{\eta^2}+(1-\lambda)\frac{(1-\psi)^2}{(1-\eta)^2}\right)}_{\frac{\lambda\phi^2+(1-\lambda)\phi^4}{(\phi^2(1-\eta)+\eta)^2}}$$

- s.t.

$$\frac{d\eta}{\eta} = (1-\eta)((\tilde{\sigma}_t^I)^2 - (\tilde{\sigma}_t^H)^2)dt = (1-\eta)\frac{(1-\theta)^2\tilde{\sigma}^2\phi^2(1-\phi^2)}{(\phi^2(1-\eta)+\eta)^2}dt$$

- planner cannot choose ψ or η but has some control over μ^η

Optimal monetary policy

Payoff flow

$$f(\eta,\theta)=\lambda\log\eta+(1-\lambda)\log(1-\eta)+\frac{\log(1-\theta)}{\rho\kappa}-\frac{\rho\kappa+1}{\rho\kappa}\log(\rho\kappa+1-\theta)$$

$$-\frac{(1-\theta)^2\tilde{\sigma}^2}{2\rho}\left(\lambda\frac{\psi^2\phi^2}{\eta^2}+(1-\lambda)\frac{(1-\psi)^2}{(1-\eta)^2}\right),\quad \psi=\frac{\eta}{\phi^2(1-\eta)+\eta}$$

HJB equation

$$\rho V(\eta)=\max_{\theta} f(\eta,\theta)+V'(\eta)\mu^{\eta}\eta+\frac{1}{2}V''(\eta)(\sigma^{\eta}\eta)^2$$

Law of motion of η

$$\frac{d\eta}{\eta}=(1-\eta)\frac{(1-\theta)^2\tilde{\sigma}^2\phi^2(1-\phi^2)}{(\phi^2(1-\eta)+\eta)^2}dt$$

Optimal ϑ

$$\max_{\theta} \frac{\log(1-\theta)}{\rho\kappa} - \frac{\rho\kappa+1}{\rho\kappa}\log(\rho\kappa+1-\theta)$$

$$-(1-\theta)^2 \frac{\tilde{\sigma}^2}{2\rho}\left(\lambda\frac{\psi^2\phi^2}{\eta^2} + (1-\lambda)\frac{(1-\psi)^2}{(1-\eta)^2}\right) + V'(\eta)\underbrace{(1-\theta)^2\frac{\eta(1-\eta)\tilde{\sigma}^2\phi^2(1-\phi^2)}{(\phi^2(1-\eta)+\eta)^2}}_{\mu^\eta \eta}$$

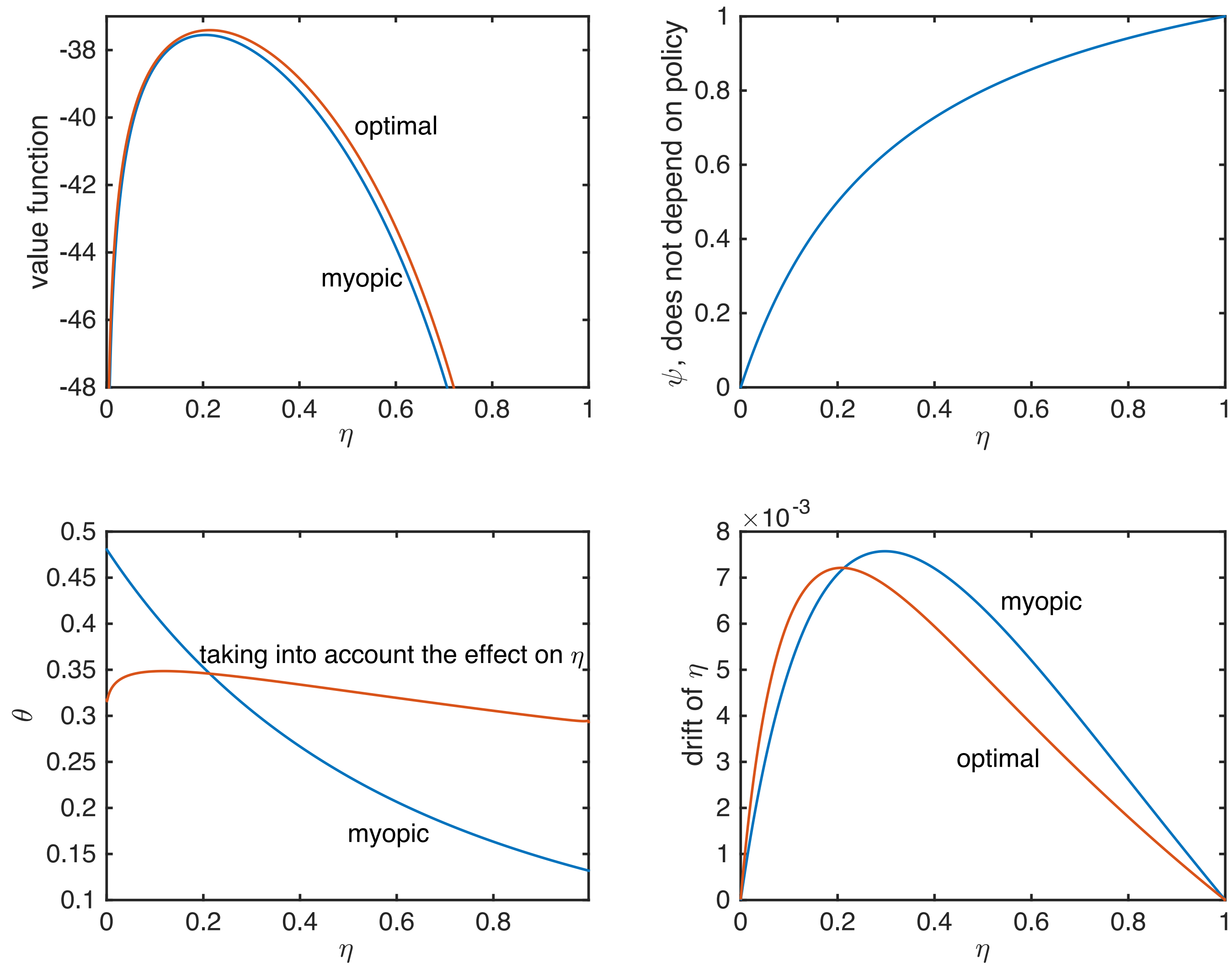
- θ affects the drift of η. It is optimal to choose

$$\vartheta^*\left(\tilde{\sigma}^2\left(\lambda\frac{\psi^2\phi^2}{\eta^2} + (1-\lambda)\frac{(1-\psi)^2}{(1-\eta)^2}\right) - 2\rho V'(\eta)\frac{\eta(1-\eta)\tilde{\sigma}^2\phi^2(1-\phi^2)}{(\phi^2(1-\eta)+\eta)^2}\right)$$

- Speed up η when V' > 0, slow down when V' < 0

Example: using ϑ to push η

$$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2$$



Thank you!