The 2019 Princeton Initiative: Problem Set

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This is an optional problem set. I highly encourage you to do it - it's the best way to learn! If you e-mail your solutions to sannikov@gmail.com by Friday, September 20, then I will e-mail you my own solutions. Please $\boxed{BOX}$ your answers.

Problem 1. Consider an economy with two types of agents: entrepreneurs and households. When managed by households, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \tilde{\sigma}\, d\tilde{Z}_t,$$

where shocks dZ_t are aggregate and $d\tilde{Z}_t$ are idiosyncratic (agent-specific, and canceling out in the aggregate). When managed by entrepreneurs, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \phi\tilde{\sigma}\, d\tilde{Z}_t,$$

where $\phi \in (0, 1)$. That is, entrepreneurs are able to diversify and reduce their idiosyncratic risk exposure.

Capital k_t produces output ak_t if managed by any agent, so after investment of ι_t, the remaining output that can be consumed (or traded) is given by $(a - \iota_t)k_t$. Assume that the investment function $\Phi(\iota)$ satisfies $\Phi(0) = 0$, $\Phi > 0$ and $\Phi'' \leq 0$. Denote the aggregate supply of capital by K_t and the price of capital per unit by q_t.

Assume that aggregate risk dZ_t can be traded between the households and entrepreneurs and carries the equilibrium price of risk of π_t. Denote the price of idiosyncratic risk of entrepreneurs by $\tilde{\pi}_t$ and that of households by $\underline{\tilde{\pi}}_t$. Thus, the budget constraint/law of motion of wealth of any entrepreneur is given by

$$\frac{dn_t}{n_t} = (r_t^F + \pi_t\sigma_t^n + \tilde{\pi}_t\tilde{\sigma}_t^n)\, dt + \sigma_t^n\, dZ_t + \tilde{\sigma}_t^n\, d\tilde{Z}_t - \frac{c_t}{n_t}\, dt,$$

where r_t^F is the equilibrium risk-free rate and c_t is the entrepreneur's consumption. The budget constraint of a household is written similarly. Agents controls her idiosyncratic risk exposure $\tilde{\sigma}_t^n \geq 0$ by choosing how much capital to hold, and aggregate risk exposure σ_t^n by also trading derivatives. Agents cannot use derivatives to insure idiosyncratic risk. Aggregate-risk derivatives and the risk-free asset are in zero net supply.

Assume that households have utility

$$E\left[\int_0^\infty e^{-\rho t}\log \underline{c}_t\, dt\right],$$

where $\rho > 0$ is a discount rate and $\underline{c}_t$ is the consumption of a household. Entrepreneurs have utility

$$E\left[\int_0^\infty \xi_t \log c_t\, dt\right],$$

where ξ_t is the process of discounting and *taste shocks* that follows

$$\xi_0 = 1, \quad \frac{d\xi_t}{\xi_t} = -\rho\, dt + \varsigma\, dZ_t,$$

where ρ is the same discount rate as that of households and $\varsigma \geq 0$ is an exogenous parameter. In order to solve this problem, you can rely (without proof) on the fact that the optimal consumption rate (of entrepreneurs as well) is given by the discount rate ρ times net worth.

The goal of this problem is to characterize the equilibrium in this environment.

(a) Given π_t, what is the entrepreneur's choice aggregate risk exposure σ_t^n? How about that of households?

(b) Characterize the prices of idiosyncratic risk $\tilde{\pi}_t$ and $\underline{\tilde{\pi}}_t$ for entrepreneurs and households and determine the fraction of capital ψ_t held by entrepreneurs in this economy, as functions of the entrepreneurs' wealth share η_t.

(c) Derive the law of motion of η_t. Please express your answer in terms of model primitives (no endogenous variables).

So far, we have assumed that households and entrepreneurs get an equal output rate of a per unit of capital. For the rest of the problem, consider instead the case in which entrepreneurs get output rate of a, while households get an output rate of only $\underline{a} < a$.

(d) Characterize the fraction of capital ψ_t that entrepreneurs hold as a function of η_t, given the price of capital q_t.

(e) Suppose that $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$. Use the market-clearing condition for output to obtain another relationship between ψ and q, for any η.

(f) Derive the quadratic equation that ψ must satisfy in the region where $\psi < 1$. Under what conditions does ψ reach 1 at some level $\eta^\psi < 1$? For those parameters, give a formula for η^ψ.

Problem 2. Consider the model of the I Theory of money from the lecture. Capital has to be divided into technology a (fraction $1 - \bar{\psi}$) and b (fraction $\bar{\psi}$) to produce the consumption good. In technology a capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \tilde{\sigma}\, d\tilde{Z}_t,$$

and in technology b,

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \tilde{\sigma}\, d\tilde{Z}_t,$$

where Z is aggregate risk and $\tilde{Z}$ is the idiosyncratic risk. Households can invest in capital in both technologies, and intermediaries can invest only in technology b, but they can diversify idiosyncratic risk down to $\phi\tilde{\sigma}$.

Consider the policy which eliminates endogenous risk by selecting appropriate value of σ_t^M. Recall that this led to

$$\psi = \min\left(\frac{\eta}{\eta + (1-\eta)\phi^2 + (1-\bar{\psi})^2\sigma^2/\tilde{\sigma}^2}, \bar{\psi}\right), \quad \sigma^\eta = (1-\theta)\frac{\psi}{\eta}(1-\bar{\psi})\sigma$$

and

$$\mu^\eta = (1-\eta)(1-\theta)^2\left(\frac{1-2\eta}{(1-\eta)^2}\frac{\psi^2}{\eta^2}(1-\bar{\psi})^2\sigma^2 + \frac{\psi^2\phi^2\tilde{\sigma}^2}{\eta^2} - \frac{(1-\psi)^2\tilde{\sigma}^2}{(1-\eta)^2}\right).$$

Suppose the planner targets a C^2 portfolio weight on money $\theta(\eta)$ that satisfies $\theta(\eta) < 1$ uniformly.

(a) Under what conditions does ψ reach $\bar{\psi}$ for at some level $\eta^\psi < 1$? In this case, find the value of η^ψ.

(b) This part asks you about the stationary density $g(\eta)$ over the state variable η in equilibrium. Function $g(\eta)$ is determined by the Kolmogorov Forward Equation (KFE), which can be reduced to

$$\mu(\eta)g(\eta) = \frac{1}{2}\frac{d}{d\eta}(\sigma(\eta)^2 g(\eta))$$

when the state variable η follows

$$d\eta_t = \mu(\eta_t)\,dt + \sigma(\eta_t)\,dZ_t.$$

Stationary density exists if and only if there is a solution to the above equation that satisfies

$$\int_0^1 g(\eta)\,d\eta = 1.$$

It is useful to know that solutions to the KFE take the form

$$g(\eta) = \frac{C}{\sigma(x)^2}\exp\left(2\int^\eta \mu(x)/\sigma(x)^2\,dx\right)$$

for $C \in \mathbb{R}$.

Determine the range of parameters for which the stationary density exists. For those parameters, determine in closed form the stationary density of up to an integration constant. In your representation of the stationary density, you may use notation $z = (1-\bar{\psi})^2\sigma^2/\tilde{\sigma}^2$ to simplify some of the algebra.