The I Theory of Money

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Baseline Model

- consumption and investment

\[ \frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t \]

- capital produces output \( a - \iota_t \), has price \( q \)
- price of money (infinitely divisible “coin”) \( pK_t \)
- portfolio weight on money \( \theta = p/(p + q) \)
Risk and return

- consumption and investment

\[
\frac{dk_t}{k_t} = (\Phi(\nu_t) - \delta) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t
\]

- capital produces output \( a - i_t \), has price \( q \)
- price of money (infinitely divisible “coin”) \( pK_t \)
- portfolio weight on money \( \theta = p/(p + q) \)
- return on capital:

\[
\frac{a - i_t}{q} + (\Phi(\nu_t) - \delta) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t
\]

- return on money

\[
(\Phi(\nu_t) - \delta) dt + \sigma dZ_t
\]
Equilibrium

- pricing for log utility \( E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t) \)
  \[
  \frac{a - \iota_t}{q} = (\sigma - \sigma) \sigma + (\tilde{\sigma} - 0) (1 - \theta)\tilde{\sigma} 
  \]
  - diff. in aggr. risk
  - diff. in idios. risk

- mkt. clearing for log \( \rho \frac{q_t}{1 - \theta} K_t = (a - \iota_t)K_t \)

- return on capital:
  \[
  \frac{a - \iota_t}{q} + (\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma}d\tilde{Z}_t 
  \]

- return on money
  \[
  (\Phi(\iota_t) - \delta)dt + \sigma dZ_t 
  \]
Equilibrium

• pricing for log utility \[ E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t) \]

\[
\frac{a - \iota_t}{q} = \frac{(\sigma - \sigma)}{\text{diff. in agg. risk}} \cdot \sigma + \frac{(\tilde{\sigma} - 0)}{\text{diff. in idios. risk}} \cdot (1 - \theta)\tilde{\sigma}
\]

• mkt. clearing for log \[ \rho \frac{q_t}{1 - \theta} K_t = (a - \iota_t) K_t \]

• Hence, \[ \frac{\rho}{1 - \theta} = (1 - \theta)\tilde{\sigma}^2 \Rightarrow \theta = 1 - \sqrt{\rho / \tilde{\sigma}} \]

money has value only if: \( \tilde{\sigma}^2 > \rho \)

• Suppose \( \Phi(\iota) = \log(k\iota + 1)/\kappa \), so \( \iota = (q - 1)/\kappa \)

\[
a - \iota = \rho \frac{q}{1 - \theta} \Rightarrow q = \frac{a + 1/\kappa}{\rho / (1 - \theta) + 1/\kappa}, \text{ declines in } \theta
\]
Policy / Controlling the Value of Money

• Suppose policy maker taxes capital, so # of coins follows

\[
\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t
\]

  – Fiscal backing of currency
  – More on different types of taxes later...

• If \( pK_t \) is the value of all \( M_t \) coins held, value of one coin is \( \frac{pK_t}{M_t} \) and return on money is

\[
\frac{d(pK_t / M_t)}{pK_t / M_t} = \left( \Phi(t) - \delta \right) dt + \sigma dZ_t - \mu^M dt - \sigma^M dZ_t + \sigma^M (\sigma^M - \sigma) dt
\]

  \[\text{Ito term}\]
Equilibrium with Policy

• # of coins follows

\[
\frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t
\]

• return on money is

\[
\frac{d(pK_t / M_t)}{pK_t / M_t} = \left(\Phi(t_t) - \delta\right) dt + \sigma dZ_t - \mu^M dt - \sigma^M dZ_t + \sigma^M (\sigma^M - \sigma) dt
\]

\[
\frac{d(qk_t)}{qk_t} - \frac{dM_t}{M_t}
\]

Ito term

• return on portfolio of capital and money

\[
\rho \ dt + (\Phi(t_t) - \delta) \ dt + \sigma dZ_t + (1 - \theta) \tilde{\sigma} d\tilde{Z}_t
\]

dividend yield
(cons. mkt clearing)

• pricing

\[
E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t)
\]

\[
\rho + \mu^M - \sigma^M (\sigma^M - \sigma) = \left(\sigma - \sigma + \sigma^M\right) \sigma + ((1 - \theta) \tilde{\sigma} - 0)(1 - \theta) \tilde{\sigma}
\]

diff. in agg. risk
diff. in idios. risk
Policy and Money Value

• # of coins follows
  \[
  \frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t
  \]

• money value
  \[
  \rho + \mu^M - (\sigma^M)^2 = (1 - \theta)^2 \tilde{\sigma}^2
  \]

1. Money value increases in fiscal backing (when \(\mu^M < 0\)).
2. Planner can make money risk-free by setting \(\sigma^M = \sigma\).
   Then \(\theta\) goes up further.
3. Price level follows
  \[
  \frac{d(M_t / (pK_t))}{M_t / (pK_t)} = \mu^M dt + \sigma^M dZ_t - (\Phi(t) - \delta)dt - \sigma dZ_t + \sigma(\sigma - \sigma^M)dt
  \]
4. Planner can create inflation by paying interest on money, so
  \[
  \frac{dM_t}{M_t} = \mu^M dt + \sigma^M dZ_t + i\ dt
  \]
Remarks

• Taxes / transfers

1) proportionately to money holdings: \( i \)

2) proportionately to capital holdings: \( \mu^M dt + \sigma^M dZ_t \)

3) proportionately to net worth

4) per capita
Remarks

• Taxes / transfers

1) proportionately to money holdings: i
   • no real effect, affects price level

2) proportionately to capital holdings: $\mu_M dt + \sigma_M dZ_t$
   – $\mu_M$ pushes down money return
   – capital return goes up
   – pushes people to hold less money, invest more

3) proportionately to net worth
   • only transfers to capital matter, effect less by $1 - \theta$

4) per capita
   – no real effect – people simply borrow against the transfers they expect to receive
Model with Intermediaries

\[
\frac{d k_t}{k_t} = (\Phi(t_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t
\]

- Intermediaries can hold equity share up to \( \bar{\chi} \)
- can diversify some idiosyncratic risk, reduce it to \( \phi \tilde{\sigma} \)
- Intermediaries’ wealth share \( \eta_t = N_t / ((p_t + q_t)K_t) \)
- No policy: risk of capital \( \sigma + \sigma^q \), money \( \sigma + \sigma^p \), incremental risk of capital \( (\theta = p/(p + q)) \)

\[
\sigma^q_t - \sigma^p_t = -\frac{\sigma^\theta}{1 - \theta}
\]

- Policy: risk of money \( \sigma + \sigma^p - \sigma^M \), capital, \( \sigma + \sigma^q + \theta \sigma^M/(1 - \theta) \), incremental

\[
-\sigma^\theta + \sigma^M
\]

\[
\frac{1}{1 - \theta}
\]
Allocation

- Minimize weighted average cost of financing
- Price of risk = volatility of wealth. Can use any numeraire, e.g. world wealth

\[
\text{price of aggregate risk: } \frac{\sigma_t}{1-\eta_t}, \quad \frac{-\eta \sigma_t}{1-\eta_t} \\
\text{price of idiosyncratic risk: } (1-\theta) \frac{\chi_t \phi \tilde{\sigma}}{\eta_t}, \quad (1-\theta) \frac{(1-\chi_t)\tilde{\sigma}}{1-\eta_t}
\]

FOC (equality if \( \chi = \bar{\chi} \))

\[
\frac{-\sigma^\theta + \sigma^M}{1-\theta} \quad \frac{-\sigma^\theta + \sigma^M}{1-\eta} \quad \frac{-\eta \sigma^n_t}{1-\eta} + \tilde{\sigma}(1-\theta) \frac{(1-\chi_t)\tilde{\sigma}}{1-\eta_t}
\]

\[
\text{incremental aggregate risk} \quad \text{incremental aggregate risk}
\]
Risk of $\eta$ and amplification

- $\sigma^\eta$ is the risk of intermediaries’ wealth, with world wealth as numeraire
- In this numeraire the risk of money is $\sigma^\theta$ without policy… $\sigma^\theta - \sigma^M$ with policy
- Then

$$\sigma^\eta_t = \sigma^\theta - \sigma^M + \frac{(1 - \theta)\chi}{\eta} - \frac{\sigma^\theta + \sigma^M}{1 - \theta} = \frac{\chi - \eta}{\eta}(- \sigma^\theta + \sigma^M)$$

$$\Rightarrow \eta\sigma^\eta_t = \frac{(\chi - \eta)\sigma^M}{1 + (\chi - \eta)\frac{\theta'(\eta)}{\theta(\eta)}}$$

- Without policy $\sigma^\eta = 0$. Otherwise (e.g. if policy maker wants money to be risk-free), $\sigma^M$ is amplified…
- We’ll see the shape of $\theta$ in a moment…
Suppose $\sigma^M = 0$, so $\sigma^n, \sigma^\theta = 0$

Then $\chi$ is given by the FOC:

$$\phi \tilde{\sigma} (1 - \theta) \frac{\chi_t \phi \tilde{\sigma}}{\eta_t} \leq \tilde{\sigma} (1 - \theta) \frac{(1 - \chi_t) \tilde{\sigma}}{1 - \eta_t} \Rightarrow \chi = \min \left( \frac{\eta}{\eta + (1 - \eta) \phi^2} , \bar{\chi} \right)$$
Money return

- Return on money w/o policy (numeraire = world wealth)

\[
\frac{d\theta_t}{\theta_t} = \mu_t^\theta dt + \sigma_t^\theta dZ_t
\]

- W/ policy (outside money is $M_t$ coins), one coin is $\theta_t/M_t$ of world wealth, money return is

\[
\frac{d(\theta_t / M_t)}{(\theta_t / M_t)} = \mu_t^\theta dt + \sigma_t^\theta dZ_t - \mu_t^M dt - \sigma_t^M dZ_t + \sigma_t^M (\sigma_t^M - \sigma_t^\theta) dt = \mu_t^h dt + \sigma_t^h dZ_t
\]

- Next, the law of motion of $\eta$ and money valuation…
Pricing individuals’ portfolios

\[ E[dr_A] - E[dr_B] = \text{cov}(dr_A - dr_B, dn_t) \]

• Holds regardless of numeraire. Let’s use global wealth
• Suppose \( \eta^i \) is wealth share of some agent group \( i \).
  Asset A: their portfolio, B: money

\[
\mu_t^\eta^i + \rho^i - \mu_t^\eta^h = (\sigma_t^\eta^i - \sigma_t^\eta^h)\sigma_t^\eta^i + (\tilde{\sigma}_t^i)^2
\]

Drift and volatility of various variables, as in

\[
\frac{d\eta^i}{\eta^i} = \mu_t^\eta^i \, dt + \sigma_t^\eta^i \, dZ_t
\]

Idiosyncratic risk exposure in group \( i \)
Money valuation equation

\[
\mu_t^{\eta,i} + \rho^i - \mu^h_t = (\sigma_t^{\eta,i} - \sigma_t^h)\sigma_t^{\eta,i} + (\tilde{\sigma}_t)^2
\]

- Add across agents with weights \(\eta^i\)

\[
\sum_i \eta^i \mu_t^{\eta,i} + \sum_i \eta^i \rho^i - \mu^h_t = \sum_i \eta^i (\sigma_t^{\eta,i})^2 - \sigma_t^h \sum_i \eta^i \sigma_t^{\eta,i} + \sum_i \eta^i (\tilde{\sigma}_t)^2
\]

\[
\Rightarrow \sum_i \eta^i \rho^i - (\mu_t^{\theta} - \mu_t^M + \sigma_t^M (\sigma_t^M - \sigma_t^\theta)) = \sum_i \eta^i \left((\sigma_t^{\eta,i})^2 + (\tilde{\sigma}_t)^2\right)
\]

- This generalizes \(\rho + \mu^M - (\sigma^M)^2 = (1 - \theta)^2 \tilde{\sigma}^2\) (baseline model)
- Average risk exposure > ave. discount rate when money is depreciating
- Planner can attain any function \(\theta(\eta)\) and any risk of money (by choosing \(\mu^M\) that satisfies the money value equation)
Law of motion of wealth shares

\[ \mu^i_t + \rho^i - \mu^h_t = (\sigma_t^{\eta,i} - \sigma^h_t)\sigma_t^{\eta,i} + (\tilde{\sigma}_t)^2 \]

- Subtract money valuation

\[ \sum_i \eta^i \rho^i - \mu^h_t = \sum_i \eta^i \left((\sigma_t^{\eta,i})^2 + (\tilde{\sigma}_t)^2\right) \]

\[ \mu^{\eta,i}_t = (\sigma_t^{\eta,i} - \sigma^h_t)\sigma_t^{\eta,i} + (\tilde{\sigma}_t)^2 - \sum_j \eta^j \rho^j - \sum_j \eta^j \left((\sigma_t^{\eta,j})^2 + (\tilde{\sigma}_t)^2\right) - \rho^i \]

- Drift of \( \eta \) determined by
  - consumption rate of group \( i \) relative to others
  - risk exposure (agg. + idiosyncratic) of group \( i \) relative to others
  - covariance of the risk of group \( i \) and money
Let’s apply this to our model

\[
\begin{align*}
\sigma^n &= 0, \quad \tilde{\sigma}^I = (1-\theta) \frac{\chi \phi \tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = (1-\theta) \frac{(1-\chi)\tilde{\sigma}}{1-\eta} \\
\mu^n_t &= (\tilde{\sigma}_t^I)^2 - \eta(\tilde{\sigma}_t^I)^2 - (1-\eta)(\tilde{\sigma}_t^H)^2 = (1-\eta)(1-\theta)^2 \left( \frac{\chi^2 \phi^2}{\eta^2} \right) - \frac{(1-\chi)^2}{(1-\eta)^2} \tilde{\sigma}^2 \\
\rho - \mu^\theta_t &= (1-\theta)^2 \left( \eta \frac{\chi^2 \phi^2}{\eta^2} + (1-\eta) \frac{(1-\chi)^2}{(1-\eta)^2} \right) \tilde{\sigma}^2 \\
\text{where} \quad \chi &= \min \left( \frac{\eta}{\eta + (1-\eta)\phi^2}, \bar{\chi} \right)
\end{align*}
\]
$$\theta \text{ minimized when } \mu^\eta = 0$$

$$\sigma^\eta = 0, \quad \tilde{\sigma}^I = (1 - \theta) \frac{\chi \phi \tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = (1 - \theta) \frac{(1 - \chi) \tilde{\sigma}}{1 - \eta}$$

$$\mu^\eta_t = (\tilde{\sigma}_t^I)^2 - \eta (\tilde{\sigma}_t^I)^2 - (1 - \eta) (\tilde{\sigma}_t^H)^2 = (1 - \eta)(1 - \theta)^2 \left( \frac{\chi^2 \phi^2}{\eta^2} - \frac{(1 - \chi)^2}{(1 - \eta)^2} \right) \tilde{\sigma}^2$$

$$\rho - \mu_t^\theta = (1 - \theta)^2 \left( \eta \frac{\chi^2 \phi^2}{\eta^2} + (1 - \eta) \frac{(1 - \chi)^2}{(1 - \eta)^2} \right) \tilde{\sigma}^2$$

where $$\chi = \min \left( \frac{\eta}{\eta + (1 - \eta) \phi^2}, \bar{\chi} \right)$$

- Ave. idiosyncratic risk exposure (before the effect of money) is minimized at the steady state of $$\eta$$
Example \( \rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\chi} = 0.8 \)
Aggregate risk on I’s balance sheets

• I get exposed to aggregate risk if their investments ≠ average in the economy, i.e. they specialize
• Easiest way to capture this: assume that fraction $\bar{\psi}$ of capital has to be in technology b and $1-\bar{\psi}$ in a
• I can invest only in technology b, diversify idios. risk to $\phi\tilde{\sigma}$

\[
\frac{d k_t}{k_t} = (\Phi(t) - \delta) dt + \tilde{\sigma} d \tilde{Z}_t \\
\frac{d k_t}{k_t} = (\Phi(t) - \delta) dt + \sigma d Z_t + \tilde{\sigma} d \tilde{Z}_t
\]

technology a

technology b

• Fundamental risk of money / economy: $\bar{\psi}\sigma$
• Incremental risk of technologies a and b

\[
-\bar{\psi}\sigma + \frac{-\sigma^\theta + \sigma^M}{1-\theta}, \quad (1-\bar{\psi})\sigma + \frac{-\sigma^\theta + \sigma^M}{1-\theta}
\]
Allocation

• Minimize weighted average cost of financing

\[
\left( (1 - \bar{\psi}) \sigma - \frac{\sigma^\theta - \sigma^M}{1 - \theta} \right) \sigma^\eta_i + \phi \tilde{\sigma} (1 - \theta) \frac{\psi \phi \tilde{\sigma}}{\eta} \leq \left( (1 - \bar{\psi}) \sigma - \frac{\sigma^\theta - \sigma^M}{1 - \theta} \right) \left( -\eta \sigma^\eta_i \right) + \tilde{\sigma} (1 - \theta) \frac{(1 - \psi) \tilde{\sigma}}{1 - \eta}
\]

(equality if \( \chi = \bar{\chi} \))

\[
\sigma^\eta_i = \sigma^\theta - \sigma^M + \frac{(1 - \theta) \psi}{\eta} \left( (1 - \bar{\psi}) \sigma - \frac{\sigma^\theta - \sigma^M}{1 - \theta} \right) \Rightarrow
\]

\[
\eta \sigma^\eta_i = \frac{(1 - \theta) \psi (1 - \bar{\psi}) \sigma + (\psi - \eta) \sigma^M}{1 + (\psi - \eta) \frac{\theta'(\eta)}{\theta(\eta)}}
\]

\[
(= (1 - \theta) \psi (1 - \bar{\psi}) \sigma \text{ if } \sigma^M = \sigma^\theta)
\]

policy removes endogenous risk / amplification
Law of motion of $\eta$ and money valuation

- Money valuation

$$\rho - \mu_t^h = \eta \left( (\sigma_t^\eta)^2 + (\tilde{\sigma}_t^I)^2 \right) + (1 - \eta) \left( \left( \frac{\eta \sigma_t^\eta}{1 - \eta} \right)^2 + (\tilde{\sigma}_t^H)^2 \right)$$

$$\mu_t^\eta = (1 - \eta) \left( (\sigma_t^\eta)^2 + (\tilde{\sigma}_t^I)^2 - \left( \frac{\eta \sigma_t^\eta}{1 - \eta} \right)^2 - (\tilde{\sigma}_t^H)^2 \right) - \sigma_t^\eta \left( \sigma_t^h \right)_{\sigma^\theta - \sigma^M}$$
If the policy removes endogenous risk...

\[ \psi = \min \left( \frac{\eta}{\eta + (1 - \eta) \phi^2 + (1 - \psi)^2 \sigma^2 / \tilde{\sigma}^2}, \psi \right) \]

\[ \sigma^M = \sigma^\theta \]

\[ \sigma^n = (1 - \theta) \frac{\psi}{\eta} (1 - \tilde{\psi}) \sigma \]

\[ \eta \mu_i^n = \eta(1 - \eta)(1 - \theta)^2 \left( \frac{1 - 2\eta}{(1 - \eta)^2} \frac{\psi^2}{\eta^2} (1 - \tilde{\psi})^2 \sigma^2 + \frac{\psi^2 \phi^2 \tilde{\sigma}^2}{\eta^2} - \frac{(1 - \psi)^2 \tilde{\sigma}^2}{(1 - \eta)^2} \right) \]

\[ \psi = \frac{\eta}{(1 - \tilde{\psi})^2 \sigma^2 / \tilde{\sigma}^2 + \phi^2 (1 - \eta) + \eta} \]

\[ \text{closed form} \]

\[ \text{closed form up to } \theta \text{ (choice of planner)} \]
Example \[ \rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \bar{\chi} = 0.8, \sigma = 0.1 \]

Policy: \[ \mu^M = 0, \quad \sigma^M = \sigma^{\theta \eta} \]
Example  \( \rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.5, \phi = 0.4, \psi = 0.8, \sigma = 0.1 \)

\( \theta \) falls (because low-\( \eta \) regions with high idiosyncratic risk are visited less frequently)

w/ policy, risk is lower, but recovery is faster

volatility goes down / amplification is removed

no amplification when \( \eta = \psi = \bar{\psi} \)
Optimal Policy?

• Generally hard question: need a precise definition of the policy space / analytical tools to characterize the optimum
• One side: inefficiencies / tradeoffs
  – insurance vs. investment
  – allocation of assets / risk
• Other side: policy space
  – (1) controlling money growth rate
  – (2) macroprudential tools / wealth redistribution
  – (3) risk redistribution
• Many moving parts, but we can get clear answers in some simple settings
Optimal policy? Welfare with log utility:

- Class of models: price capital $q_t$, value of money $p_tK_t$, two types of agents I and H, have wealth shares $\eta_t$ and $1 - \eta_t$, idiosyncratic risk exposures

$$\tilde{\sigma}_t^I \text{ and } \tilde{\sigma}_t^H$$

- Then the welfare of I is (similar formula for H)

$$E \left[ \int_0^\infty e^{-\rho t} \log(c_t^I) \, dt \right] = E \left[ \int_0^\infty e^{-\rho t} \log(\eta_t(a(\psi_t) - \iota_t)K_t\tilde{\eta}_t^I) \, dt \right],$$

$$\tilde{\eta}_0^I = 1, \quad \frac{d\tilde{\eta}_t^I}{\tilde{\eta}_t^I} = \tilde{\sigma}_t^I d\tilde{Z}_t$$
Welfare with log utility

- The welfare of I is

\[
E \left[ \int_0^\infty e^{-\rho t} \log(\eta_t(a(\psi_t) - \iota_t) K_t \tilde{\eta}_t) dt \right] = E \left[ \int_0^\infty e^{-\rho t} \log \eta_t dt \right] + \frac{\log \eta_0}{\rho} + E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\mu_t^\eta - |\sigma_t^\eta|^2}{2 \rho} \right) dt \right]
\]

\[
E \left[ \int_0^\infty e^{-\rho t} \log(a(\psi_t) - \iota_t) dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log K_t dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log \tilde{\eta}_t dt \right] + \frac{\log K_0}{\rho} + E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\Phi(\iota_t) - \delta |\sigma_t^K|^2}{2 \rho} \right) dt \right] - E \left[ \int_0^\infty e^{-\rho t} \left( \tilde{\sigma}_t^l \right)^2 \frac{2 \rho}{dt} \right]
\]

\[
\tilde{\eta}_0^l = 1, \quad \frac{d\tilde{\eta}_t^l}{\tilde{\eta}_t^l} = \tilde{\sigma}_t^l d\tilde{Z}_t
\]
Welfare

• We see that policy can affect welfare in several ways

\[
E \left[ \int_0^\infty e^{-\rho t} \log \eta_t \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \log(a(\psi_t) - \iota_t) \, dt \right] + \\
E \left[ \int_0^\infty e^{-\rho t} \left( \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{1}{2\rho} \left| \sigma_t^K \right|^2 \right) \, dt \right] - E \left[ \int_0^\infty e^{-\rho t} \frac{(\tilde{\sigma}_t^I)^2}{2\rho} \, dt \right]
\]

• investment vs. consumption
• allocation of capital – idiosyncratic risk, total output
• \( \eta \) – the distribution of consumption and risk absorption capacity
One at a time: policy tools and equilibrium features

• Generally, idiosyncratic risk exposures $\tilde{\sigma}_t^I$ and $\tilde{\sigma}_t^H$ are stochastic (depend on $\eta$, risk absorption capacity, allocation)

• If intermediaries help reduce idiosyncratic risk, these may rise when $\eta$ declines (or goes away from the middle)

• Let’s see, how this matters with a simple model
Stochastic idiosyncratic risk

• One type of agents $H$, idiosyncratic risk of capital is stochastic (hence it is a state variable)

\[ d\tilde{\sigma}_t = \tilde{\mu}(\tilde{\sigma}_t)dt + \tilde{\nu}(\tilde{\sigma}_t)dZ_t \]

e.g. as in Di Tella, CIR process

\[ d\tilde{\sigma}_t = \lambda(\tilde{\sigma}_t - \bar{\sigma}_t)dt + \nu \sqrt{\tilde{\sigma}_t}dZ_t \]

• Global wealth as numeraire, agents’ entire portfolio has return $\rho$ (just the consumption rate)

• Money has return

\[ \mu_t^\theta dt + \sigma_t^\theta dZ_t - \mu_t^M dt \]

rate of money printing, which is distributed to capital
Stochastic idiosyncratic risk

- Global wealth as numeraire, wealth has return $\rho$
  Money has return
  \[\mu_t^\theta dt + \sigma_t^\theta dZ_t - \mu_t^M dt\]

- Money valuation equation
  \[\rho - (\mu_t^\theta - \mu_t^M) = (1 - \theta_t)\tilde{\sigma}_t + (1 - \theta_t)\tilde{\sigma}_t - \sigma_t^\theta \quad 0 \quad \text{agg. risk of wealth rel. to money}
  \text{price of agg. risk}
  \text{idiosync. risk of wealth price of idiosync. risk}\]

- Without policy, equation
  \[\rho - \mu_t^\theta = (1 - \theta_t)^2 \tilde{\sigma}_t^2\]

has a unique solution in $\mathcal{V}(\tilde{\sigma}_t) \in (0,1)$ (if idiosyncratic risk is sufficiently large)
Optimal Policy

• Market-clearing for output

\[ a - \nu(q) = \rho \frac{q}{1-\theta} \], \; \text{if} \; \Phi(\nu) = \frac{\log(\kappa \nu + 1)}{\kappa}, \nu(q) = \frac{q-1}{\kappa}, \; q = \frac{(a\kappa + 1)(1-\theta)}{\rho\kappa + 1-\theta}

• welfare is

\[ \frac{\log K_0}{\rho} - \frac{\delta}{\rho^2} + 
\]

\[ E \left[ \int_0^\infty e^{-\rho t} \log(a - \nu_t) \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \frac{\Phi(\nu_t)}{\rho} \, dt \right] - E \left[ \int_0^\infty e^{-\rho t} \frac{(1-\theta_t)^2 \tilde{\sigma}_t^2}{2\rho} \, dt \right]
\]

\[ = \frac{1}{\rho \kappa} E \left[ \int_0^\infty e^{-\rho t} \log(\rho \kappa + 1 - \theta_t) \, dt \right] \]

• let \( \vartheta^* (\tilde{\sigma}^2) \) be the maximizer of (optimal baseline policy)

\[ \frac{1}{\rho \kappa} \log(1-\theta) - \frac{\rho \kappa + 1}{\rho \kappa} \log(\rho \kappa + 1-\theta) - \frac{(1-\theta)^2 \tilde{\sigma}^2}{2\rho} \]
Optimal policy

• If the planner could control $\theta_t$ directly, she would set $\theta_t = \vartheta^*(\tilde{\sigma}_t^2)$

• Controlling indirectly by choosing $\mu_t^M$ the planner can achieve any function - including $\vartheta^*(\tilde{\sigma}_t^2)$ - by solving

$$\rho - (\mu_t^\theta - \mu_t^M) = (1 - \theta_t)^2 \tilde{\sigma}_t^2$$

for $\mu_t^M$

• Optimal policy is easier to find than even the equilibrium outcome (differentiation vs. integration)

• Risk-free rate $\Phi(\nu_t) - \delta + \mu_t^\theta - \mu_t^M = \rho - (1 - \theta_t)^2 \tilde{\sigma}_t^2 + \Phi(\nu_t) - \delta$

  declines as $\tilde{\sigma}_t^2$ increases

• Nice relationship b/w baseline and dynamic model
• The relationship between idiosyncratic risk level $\tilde{\sigma}_t$ and optimal insurance $\theta^*(\tilde{\sigma}_t^2)$ pops up everywhere everywhere.

\begin{equation}
\theta^*(\tilde{\sigma}^2) = \max_{\theta} \frac{1}{\rho K} \log(1 - \theta) - \frac{\rho K + 1}{\rho K} \log(\rho K + 1 - \theta) - \frac{(1 - \theta)^2 \tilde{\sigma}^2}{2 \rho}.
\end{equation}

• Let's consider another model, with heterogeneous agents but with exogenous wealth distribution...
Switching types

intermediaries \[ \lambda^s \]

households \[ \lambda^e \]

\[ \eta \]

1-\[ \eta \]

\[ \phi \tilde{\sigma}, \phi \in (0,1) \]

diversification

\[ \tilde{\sigma} \]

Policy maker can choose the money growth rate \[ \mu_t^M \]
Remarks

• Policy-maker cannot affect the wealth shares (exogenously fixed by the switching process)
• Welfare weights on intermediaries and households are $\eta$ and $1 - \eta$ from the setup
• Optimal monetary (with or w/o macroprudential policy – controlling capital allocation)
Equilibrium capital allocation

• Fraction $\psi$ of capital is held by the intermediaries
• Capital allocation must be such that

\[
\frac{\phi \tilde{\sigma}}{\eta} = \frac{(1 - \theta)\psi \phi \tilde{\sigma}}{\eta} = \frac{(1 - \theta)(1 - \psi) \tilde{\sigma}}{1 - \eta}
\]

\[
\Rightarrow \psi = \frac{\eta}{\phi^2 (1 - \eta) + \eta}
\]

• Policy maker may try to affect $\psi$...
Welfare

- Law of large numbers: switching risk does not matter. Everyone’s wealth growth averages out to $\Phi(t_t) - \delta$ and idiosyncratic risk exposure, to

$$
\eta(\tilde{\sigma}^I)^2 + (1 - \eta)(\tilde{\sigma}^H)^2 = (1 - \theta)^2 \tilde{\sigma}^2 \left( \frac{\psi^2 \phi^2}{\eta} + \frac{(1 - \psi)^2}{1 - \eta} \right)
$$

$$
\tilde{\sigma}^I = \frac{(1 - \theta)\psi \phi \tilde{\sigma}}{\eta}, \quad \tilde{\sigma}^H = \frac{(1 - \theta)(1 - \psi)\tilde{\sigma}}{1 - \eta}
$$

- Welfare

$$
E \left[ \int_0^\infty e^{-\rho t} \log(a - \nu(t)) \, dt \right] + E \left[ \int_0^\infty e^{-\rho t} \frac{\Phi(t(t)) - \delta}{\rho} \, dt \right] - E \left[ \int_0^\infty e^{-\rho t} \frac{(1 - \theta)^2 (\tilde{\sigma}^A)^2}{2 \rho} \, dt \right]
$$

- Given $\tilde{\sigma}^A$, optimal to set $\theta = \bar{\theta}^* \left( (\tilde{\sigma}^A)^2 \right)$
Money valuation

\[
\rho - (\mu_t^\theta - \mu_t^M) = \eta(\tilde{\sigma}^I)^2 + \underbrace{(1 - \eta)(\tilde{\sigma}^H)^2}_{(1-\theta)^2(\tilde{\sigma}^A)^2}
\]

• Without policy,

\[
\rho = (1 - \theta)^2(\tilde{\sigma}^A)^2
\]
Macroprudential tools

• Average idiosyncratic risk of capital

\[(\tilde{\sigma}^A)^2 = \tilde{\sigma}^2 \left( \frac{\psi^2 \phi^2}{\eta} + \frac{(1-\psi)^2}{1-\eta} \right)\]

is minimized when

\[
\frac{\psi \phi^2}{\eta} = \frac{1-\psi}{1-\eta} \Rightarrow \psi = \frac{\eta}{\phi^2 (1-\eta) + \eta}
\]

This is the equilibrium allocation! Optimal not to use macroprudential tools.
Remarks

• Same trade-off between insurance and investment
• Equilibrium allocation is efficient, minimizes the cost of risk exposure
• Policy space (1) money growth and (1) + (2) (also macroprudential tools) leads to the same outcome
Endogenous law of motion of $\eta$

- Wealth distribution can change endogenously with (i) risk exposure of intermediaries and households (ii) risk premia (iii) consumption rates
- Consider the following model
Fixed types (no switching)

<table>
<thead>
<tr>
<th>intermediaries</th>
<th>households</th>
</tr>
</thead>
<tbody>
<tr>
<td>wealth shares</td>
<td>$\eta$</td>
</tr>
<tr>
<td>welfare weights</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>idiosyncratic</td>
<td>$\phi \tilde{\sigma}, \phi \in (0,1)$</td>
</tr>
<tr>
<td>risk of capital</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>aggregate risk</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>output per unit</td>
<td>the same, independently of the allocation</td>
</tr>
<tr>
<td>of capital</td>
<td></td>
</tr>
</tbody>
</table>

Two policy classes:
(1) choose the money growth rate $\mu_t^M$
(1) + (2) also choose allocation (macroprudential) and transfer wealth between groups (why / how?)

You have already seen this model except here $\bar{\psi} = 1$
Welfare of I and H

- Intermediaries (weight $\lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log \eta_t + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \theta_t)^2}{2\rho} \frac{\psi^2 \phi^2 \tilde{\sigma}^2}{\eta^2} \right) dt \right]$$

- Households (weight $1 - \lambda$)

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log(1 - \eta_t) + \log(a - \iota_t) + \frac{\Phi(\iota_t) - \delta}{\rho} - \frac{\sigma^2}{2\rho} - \frac{(1 - \theta_t)^2}{2\rho} \frac{(1 - \psi)^2 \tilde{\sigma}^2}{(1 - \eta)^2} \right) dt \right]$$
Optimal policy, (1) + (2)

- Planner chooses $\theta$, $\psi$ and $\eta$ to maximize the disc. integral of

$$\lambda \log \eta_t + (1 - \lambda) \log(1 - \eta_t) + \log(a - u(\theta)) + \frac{\Phi(u(\theta)) - \delta}{\rho} - \frac{\sigma^2}{2\rho}$$

$$- \frac{(1 - \theta)^2 \tilde{\sigma}^2}{2\rho} \left( \frac{\lambda \psi^2 \phi^2}{\eta^2} + (1 - \lambda) \left( \frac{1 - \psi}{1 - \eta} \right)^2 \right)$$

given the optimal choice of

$$\psi = \frac{(1 - \lambda)\eta^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}$$

- given $\psi$ and $\eta$, optimal to set $\theta$ to

$$\theta = \vartheta^* \left( \tilde{\sigma}^2 \frac{\lambda(1 - \lambda)\phi^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2} \right)$$

welfare weighted average risk exposure

not the competitive allocation (unless $\eta = \lambda$)
Competitive $\psi$ vs. minimizing cost of risk

$$\psi = \frac{(1 - \lambda)\eta^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}$$

Of course, here $\eta$ also can be chosen by the planner... but this is important because when $\eta$ can move freely, planner may want to push risk to the group whose wealth exceeds its welfare weight.
Optimal policy, (1) + (2)

• Finally, optimal $\eta$ (given $\vartheta$) – let’s look at terms containing $\eta$

\[
\max_{\eta} \lambda \log \eta + (1 - \lambda) \log(1 - \eta) \quad \frac{(1 - \vartheta)^2 \tilde{\sigma}^2}{2 \rho} \quad \frac{\lambda(1 - \lambda)\phi^2}{\lambda \phi^2 (1 - \eta)^2 + (1 - \lambda)\eta^2}
\]

concave, max at $\eta = \lambda$, goes to $-\infty$ at 0 and 1

concave (!) also, max at $\frac{\lambda \phi^2}{\lambda \phi^2 + 1 - \lambda} < \lambda$

• hence, it is optimal to set $\eta > \lambda$ (unfortunately I could not get a closed-form expression for the optimal $\eta$)

• push more risk to intermediaries than they’d take under competitive outcome

• relative to previous infinite switching model
  – it is optimal to give intermediaries more wealth, because they are more efficient at absorbing risk
  – overall risk is reduced and the value of money is lower (more intermediation)
Optimizing over $\eta$  \( \rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2 \)
Optimal policy, (1) only

- What about monetary policy alone?
- Planner cannot alter the comp. allocation, \( \psi = \frac{\eta}{\phi^2 (1 - \eta) + \eta} \)
- Welfare is the disc. integral of

\[
\begin{align*}
\lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \log(a - \nu(\theta)) + & \frac{\Phi(\nu(\theta)) - \delta}{\rho} - \frac{\sigma^2}{2\rho} \\
- \frac{(1 - \theta)^2 \bar{\sigma}^2}{2\rho} \left( \frac{\lambda \psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right) & \frac{\lambda \phi^2 + (1 - \lambda) \phi^4}{(\phi^2 (1 - \eta) + \eta)^2}
\end{align*}
\]

- s.t.

\[
\frac{d\eta}{\eta} = (1 - \eta)((\bar{\sigma}_t^I)^2 - (\bar{\sigma}_t^H)^2) dt = (1 - \eta) \frac{(1 - \theta)^2 \bar{\sigma}^2 \phi^2 (1 - \phi^2)}{(\phi^2 (1 - \eta) + \eta)^2} dt
\]

- planner cannot choose \( \psi \) or \( \eta \) but has some control over \( \mu^\eta \)
Optimal monetary policy

Payoff flow

\[ f(\eta, \theta) = \lambda \log \eta + (1 - \lambda) \log(1 - \eta) + \frac{\log(1 - \theta)}{\rho \kappa} - \frac{\rho \kappa + 1}{\rho \kappa} \log(\rho \kappa + 1 - \theta) \]

\[-\frac{(1 - \theta)^2 \tilde{\sigma}^2}{2 \rho} \left( \lambda \frac{\psi^2 \phi^2}{\eta^2} + (1 - \lambda) \frac{(1 - \psi)^2}{(1 - \eta)^2} \right), \quad \psi = \frac{\eta}{\phi^2 (1 - \eta) + \eta} \]

HJB equation

\[ \rho V(\eta) = \max_{\theta} f(\eta, \theta) + V'(\eta) \mu^\eta \eta + \frac{1}{2} V''(\eta) (\sigma^\eta \eta)^2 \]

Law of motion of \( \eta \)

\[ \frac{d\eta}{\eta} = (1 - \eta) \frac{(1 - \theta)^2 \tilde{\sigma}^2 \phi^2 (1 - \phi^2)}{\phi^2 (1 - \eta) + \eta^2} \, dt \]
Optimal $\vartheta$

\[
\max_{\vartheta} \frac{\log(1-\vartheta)}{\rho \kappa} - \frac{\rho \kappa + 1}{\rho \kappa} \log(\rho \kappa + 1 - \vartheta) \\
-(1-\vartheta)^2 \frac{\tilde{\sigma}^2}{2 \rho} \left( \frac{\lambda \psi^2 \phi^2}{\eta^2} + (1-\lambda) \frac{(1-\psi)^2}{(1-\eta)^2} \right) + V'(\eta)(1-\vartheta)^2 \frac{\eta(1-\eta)\tilde{\sigma}^2 \phi^2 (1-\phi^2)}{(\phi^2 (1-\eta) + \eta)^2}
\]

- $\vartheta$ affects the drift of $\eta$. It is optimal to choose

\[
\vartheta^* \left( \frac{\tilde{\sigma}^2}{2 \rho} \left( \frac{\lambda \psi^2 \phi^2}{\eta^2} + (1-\lambda) \frac{(1-\psi)^2}{(1-\eta)^2} \right) - 2 \rho V'(\eta) \frac{\eta(1-\eta)\tilde{\sigma}^2 \phi^2 (1-\phi^2)}{(\phi^2 (1-\eta) + \eta)^2} \right)
\]

- Speed up $\eta$ when $V' > 0$, slow down when $V' < 0$
Example: using $\vartheta$ to push $\eta$

$\rho = 0.05, \kappa = 2, \tilde{\sigma} = 0.3, \phi = 0.5, \lambda = 0.2$
Thank you!