Problem Set for the 2015 Princeton Initiative
Based on Yuliy's and Markus's lectures

This is an optional problem set. You are welcome to do it. If you e-mail your solutions to sannikov@gmail.com by Monday September 28, then I will e-mail you my own solutions. Please $\boxed{BOX}$ your answers.

Problem 1. This problem is based on the I-Theory of Money. Consider the following idiosyncratic-shock economy with identical agents (i.e. there are no intermediaries) who have CRRA preferences with risk aversion coefficient γ and discount rate ρ. Agents choose portfolios of capital and money. The goal is to characterize equilibrium prices of capital and money, q and p. That is, if the aggregate amount of capital is K_t, then the value of all capital is qK_t and the value of all money is pK_t.

Assume that capital held by any agent follows

$$\frac{dk_t}{k_t} = (\Phi(\iota) - \delta)\, dt + \tilde{\sigma}\, d\tilde{Z}_t,$$

where $\tilde{Z}_t$ denotes the agent's idiosyncratic Brownian motion (these shocks are independent across agents and cancel out in the aggregate). Capital k_t produces output at rate $(a - \iota)k_t$.

(a) Write down an expression for the return on capital, given price q. What is the first-order condition for the investment rate ι that maximizes return?

(b) Write down an expression for the return on money, assuming that money has positive value in equilibrium. What is the real risk-free rate in this economy?

(c) Given these returns, and given q, what is the optimal portfolio allocation to capital of an agent with constant relative risk aversion coefficient γ? Hint: With constant investment opportunities, the volatility of consumption is proportional to the volatility of net worth.

(d) With these investment opportunities, what is the optimal consumption rate of an agent with net worth n_t? Hint: To answer this question, you need to derive or look up the ratio of consumption to net worth for an agent with CRRA preferences and constant investment opportunities.

(e) Now, assume that function $\Phi(\iota)$ is fully inelastic, i.e. agents must invest $\iota \in (0, a)$, resulting in the growth of $\Phi(\iota)$. Assume that

$$\rho + (\gamma - 1)(\Phi(\iota) - \delta) > 0.$$

Under what condition can money have positive value in equilibrium? If this condition holds, derive the equilibrium prices of money and capital.

Problem 2. The goal of this problem is solving a variation of the model of Di Tella (2014), allowing for firesales. Suppose that there are two types of agents, experts and households. When managed by any agent, capital follows

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t + \phi\nu_t\, d\tilde{Z}_t, \tag{1}$$

where Z_t is aggregate risk common to all and $\tilde{Z}_t$ is idiosyncratic agent-specific risk (and $\phi\nu_t d\tilde{Z}_t$ is the portion of idiosyncratic risk that cannot be traded due to agency frictions). Idiosyncratic uncertainty ν_t follows a Markov process, e.g.

$$d\nu_t = \lambda(\bar{\nu} - \nu_t)\, dt + \sigma_\nu \sqrt{\nu_t}\, dZ_t,$$

where Z_t is the same aggregate Brownian motion as that in (1). Capital k_t produces output $(a - \iota_t)k_t$ when held by experts. For now, assume that capital cannot be held by households - later on we relax this assumption.

As in Di Tella (2014), assume that aggregate risk dZ_t *can* be traded between experts and households and carries the equilibrium price of risk of π_t. Then the return on capital when held by experts is

$$dr_t^k = \left(\frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + (\sigma + \sigma_t^q)\, dZ_t + \phi\nu_t\, d\tilde{Z}_t,$$

and so any individual expert earns the premium of

$$\tilde{\pi}_t = \frac{E[dr_t^k]/dt - \pi_t(\sigma + \sigma_t^q) - r_t^F}{\phi\nu_t},$$

where r_t^F is the risk-free rate, for each unit of idiosyncratic risk that this expert chooses to get exposed to.

Hence, the net worth of an individual expert follows

$$\frac{dn_t}{n_t} = \left(r_t^F + \sigma_t^n\pi_t + \tilde{\pi}_t\tilde{\sigma}_t^n\right)\, dt - \frac{c_t}{n_t}\, dt + \sigma_t^n\, dZ_t + \tilde{\sigma}_t^n\, d\tilde{Z}_t,$$

where c_t is the expert's consumption.

The net worth of an individual household, that cannot hold capital, follows

$$\frac{dh_t}{h_t} = \left(r_t^F + \sigma_t^h \pi_t\right)\, dt - \frac{c_t}{h_t}\, dt + \sigma_t^h\, dZ_t.$$

Suppose that all agents have logarithmic utility, experts have discount rate ρ and households, $r \leq \rho$.

(a) Given π_t, what is the expert's choice σ_t^n of their exposure to aggregate risk? How about that of households, σ_t^h?

(b) Denote the aggregate net worth of experts by N_t and the aggregate net worth of households by $q_t K_t - N_t$. If experts hold all the capital, then what is the exposure $\tilde{\sigma}_t^n$ to idiosyncratic risk of any individual expert? What is the value of $\tilde{\pi}_t$ in equilibrium, as a function of ν_t and $\eta_t = N_t/(q_t K_t)$?

(c) Given your answers in parts (a) and (b), aggregate net worth and derive the law of motion of η_t in equilibrium.

(d) Assuming that $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$, what is the equilibrium price of capital q_t?

For the rest of the problem, assume that households can also buy capital, but if they do so, they get a lower output rate of $\underline{a} < a$ than experts. We want to understand the equilibrium allocation of capital as well as dynamics, i.e. the joint law of motion of the state variables (ν_t, η_t).

Denote by $\underline{\tilde{\pi}}_t$ the premium that households earn per unit of idiosyncratic risk, if they hold capital directly. Thus, the net worth of an individual household follows

$$\frac{dh_t}{h_t} = \left(r_t^F + \sigma_t^h \pi_t + \tilde{\sigma}_t^h \underline{\tilde{\pi}}_t\right)\, dt - \frac{c_t}{h_t}\, dt + \sigma_t^h\, dZ_t + \tilde{\sigma}_t^h\, d\tilde{Z}_t.$$

(e) Since households are less productive, $\underline{\tilde{\pi}}_t < \tilde{\pi}_t$. What is the difference $\tilde{\pi}_t - \underline{\tilde{\pi}}_t$, expressed in terms of ν_t, q_t and model parameters?

(f) Under what conditions on ν_t, η_t and q_t will households want to hold a positive amount of capital? In this case, express ψ_t, the fraction of capital that experts hold, as a function of ν_t, η_t, q_t and model parameters. Hint: Consider what happens when all capital is held by experts - under what conditions is the premium on household idiosyncratic risk high enough for households to want to buy some capital?

(g) Assuming that $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$, for what values of (η_t, ν_t) will households hold a positive amount of capital? In this region, find the equilibrium allocation $\psi(\eta, \nu)$ and price $q(\eta, \nu)$.

(h) Derive the law of motion of η_t in the region where $\psi_t < 1$. Please do not plug in your expression for $\psi(\eta, \nu)$ to express your answer (it'll be too messy), rather leave $\psi(\eta, \nu)$ as a function.