Exercises, "Princeton Initiative: Macro, Money and Finance."

These exercises are provided to help you get comfortable with continuous-time methods of modeling heterogeneous-agent economies. They are based on my lecture at the Princeton Initiative in 2012. These are variations of the models from the lecture, and should be solvable with methods we discussed. If you would like to try them and e-mail your solutions to sannikov@gmail.com by Friday September 14, I will take a look at them and e-mail you my own solutions. Please feel free to also consult the notes posted on the Princeton Initiative website. -Yuliy

Problem 1. Consider a version of the Basak and Cuoco (1998) model described in the lecture notes, with one modification. Suppose that experts discount utility at rate $\rho > r$, while households discount utility at rate r. Please answer the following questions. You may use any expressions already derived in the notes.

(a) Using the market-clearing condition, derive the price of capital $q(\eta_t)$.

(b) Assuming that the percent volatility of q_t is σ_t^q, what is the volatility of capital? What is the volatility of expert net worth? What is the Sharpe ratio of the risky investment? Derive an expression for $dr_t^D - r_t\,dt$.

(c) Derive the law of motion of η_t in equilibrium.

(d) Derive an explicit equation for σ_t^q.

(e) Derive an equation that characterizes the steady state of the system, i.e. point η^* where the drift of η_t is zero.

(f) Derive μ_t^q, as well as the risk-free rate r_t.

Problem 2. Consider the model from Brunnermeier and Sannikov (2012) that we discussed in lecture. The goal of this problem is to understand what happens in that model if the government recapitalizes the expert sector when η_t drops to a critically low level $\underline{\eta}$. Assume that transfers are proportional to the experts' net worth, and that together transfers ensure that η_t reflects when it hits $\underline{\eta}$. The government obtains money by taxing households (through a flat per-person tax, rather than a wealth tax).

(a) In lecture, we derived the equilibrium equations of the form

$$0 = \mu_t^\theta - \rho + \underbrace{\frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q}_{E[dr_t^k]/dt} + (\sigma + \sigma_t^q)\sigma_t^\theta,$$

$$0 = \mu_t^\theta - \rho + r \;\; \text{and} \;\; \frac{\underline{a} - \iota_t}{q_t} + \Phi(\iota_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q \leq r,$$

with equality if $\psi_t < 1$. Also, the law of motion of η_t was derived to be

$$d\eta_t = (\psi_t - \eta_t)(dr_t^k - r\,dt - (\sigma + \sigma_t^q)^2\,dt) + \eta_t \frac{a - \iota_t}{q_t}\,dt + \eta_t(1 - \psi_t)(\underline{\delta} - \delta).$$

Do these relationships still hold when $\eta_t \in (\underline{\eta}, \eta^*)$? If they hold, just say "Yes." If not, please modify them as appropriate.

(b) What boundary conditions should functions $\theta(\eta)$ and $q(\eta)$ satisfy at $\eta = \underline{\eta}$ and $\eta = \eta^*$? Please give *five* boundary conditions.

Hint: the trickiest condition is for $\theta(\eta)$ at $\eta = \underline{\eta}$. When η_t is reflecting at $\underline{\eta}$, i.e. each expert's net worth n_t increases by ϵn_t and $\eta_t = \underline{\eta}$ increases by $\epsilon\underline{\eta}$, then each expert's value function $\theta(\eta)n_t$ must remain constant. What boundary condition does $\theta(\eta)$ need to satisfy in order for this to be the case.

(c) Modify the code to compute the equilibrium for the case when the government recapitalizes the expert sector at $\underline{\eta} = 0.01$. Plot the equilibrium with the policy, and without (use the 3-by-3 format for the figure, as in the code provided with this assignment).

Suppose that, instead of making direct transfers to experts at $\underline{\eta}$, the government instead decides to create a tradable "insurance security." It is perfectly divisible, and available in unit supply.

The security works as follows: as soon as η_t drops below $\underline{\eta}$, the government makes a transfer to the holders of this security in the amount that is required to boost η_t to $\underline{\eta}$.

(d) Argue informally, that in equilibrium experts will hold the entire supply of this security.

(e) Consider the equilibrium under the recapitalization policy, and denote by $N_t\,d\xi_t$ the total "recapitalization" transfers to experts. Then, in addition to the usual payoff from capital, each expert gets transfers of $d\xi_t$ per dollar invested in capital, and the same transfers $d\xi_t$ per dollar invested in the "risk-free" asset (and he has to make those transfers if he borrows). Suppose that these transfers could be separated from capital and the risk-free asset and traded separately. Argue that there would be a price for these transfers, and that separate trading of transfers would not affect the equilibrium law of motion of η_t.

(f) Argue that the law of motion of η_t and the value of a unit of experts' net worth θ_t in the equilibrium with the insurance security are identical to those in the equilibrium with the recapitalization policy.

(g) Suppose that q_t and θ_t represent the price of capital and the experts' value functions in equilibrium with the recapitalization policy. Suppose that $p_t K_t$ is the aggregate value of the (total supply of) the insurance security in equilibrium with that security. Given this what is the price of capital (per unit) in equilibrium with the insurance security.

(h) Let $dp_t/p_t = \mu_t^p \, dt + \sigma_t^p \, dZ_t$. Write down the asset-pricing equation for the insurance security (analogous to equations (16) and (17) in the notes).

(i) What boundary conditions does $p(\eta)$ have to satisfy? Add code to your program to compute and plot the value of $p(\eta)$.