Problem Set: Solving Heterogeneous-Agent Economies with Financial Frictions.

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Problem. The goal of this problem is to derive a system of equations and compute equilibria (using Matlab) in an economy with financial frictions in which two agent types have general CRRA utility functions.

There are two types of agents: experts and households. When held by either agent type, capital evolves according to

$$dk_t/k_t = (\Phi(\iota_t) - \delta)\, dt + \sigma\, dZ_t$$

where $\Phi(\iota)$ is an investment function.[1] For the computational part of this problem, you may assume that investment of $\iota = \Phi + \kappa\Phi^2/2$ generates capital at rate Φ.

Experts are more productive than households: when held by experts, capital k_t produces net output at rate $(a - \iota_t)k_t$, and when held by households, only $(\underline{a} - \iota_t)k_t$, where $\underline{a} < a$.

Both agent types have CRRA utility with the coefficient of risk aversion γ,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

Experts have the discount rate ρ and households, r, with $\rho > r$.

Denote the aggregate net worth of experts by N_t and the aggregate amount of capital by K_t. We postulate that the price of capital by q_t follows

$$dq_t/q_t = \mu_t^q\, dt + \sigma_t^q\, dZ_t.$$

Denote the aggregate consumption of experts by C_t, and households, D_t, where we postulate that

$$dC_t/C_t = \mu_t^C\, dt + \sigma_t^C\, dZ_t \quad \text{and} \quad dD_t/D_t = \mu_t^D\, dt + \sigma_t^D\, dZ_t \tag{1}$$

Let $c_t = C_t/K_t$ and $d_t = D_t/K_t$ be the consumption rates of experts and households per unit of total capital in the economy.

$$dc_t/c_t = \mu_t^c\, dt + \sigma_t^c\, dZ_t \quad \text{and} \quad dd_t/d_t = \mu_t^d\, dt + \sigma_t^d\, dZ_t \tag{2}$$

[1] Note that δ is the same for households and experts.

Part I. The goal of this part is to write down the equilibrium conditions. This part should be (relatively) easy, given the lecture.

(a) Write down the market-clearing condition for consumption goods, using the following variables: the allocation of capital to experts, $\psi_t \in [0, 1]$ as well as a, $\underline{a}$, ι_t, q_t, c_t and d_t.

(b) Write down the expression for dr_t^k, the return that experts earn from investing in capital.

(c) Write down the law of motion of aggregate capital K_t in the economy. Use the law of motion of K_t to express μ_t^C and σ_t^C from μ_t^c and σ_t^c.

(d) Write down the asset-pricing condition for capital held by experts. You may use the expression for the return on capital you derived in part (b), as well as μ_t^C, σ_t^C and γ.

(e) Write down an analogous condition for households. This time, it is an inequality condition, with equality if $\psi_t < 1$.

(f) Write down two expressions for the risk-free rate: one from the consumption process of experts and another one from that of households.

Part II. The goal of the second part of the problem is to derive the law of motion of the state variable $\eta_t = N_t/(q_t K_t)$. This should also be (relatively) easy, given the lecture.

(g) Write down the law of motion of the total wealth in the economy $q_t K_t$, using dr_t^k, a, ι_t and q_t.

(h) Write down the law of motion of aggregate expert net worth N_t, in terms of dr_t^k, r_t^F, ψ_t, q_t, K_t and C_t, where r_t^F is the risk-free rate.

(i) From your answers to parts (g) and (h), derive the law of motion of $\eta_t = N_t/(q_t K_t)$. Then, use the combination of asset-pricing conditions (for experts) from parts (d) and (f) to remove all drift terms (such as μ_t^q or μ_t^c) from the expression for the law of motion of η_t. Your final answer should be written in terms of ψ, η, γ, σ, σ_t^C, σ_t^q, c, q_t a and ι_t.

(j) Provide an expression for σ_t^η in terms of ψ, η, σ, $q(\eta)$ and $q'(\eta)$.

Part III. This part is harder. The goal of this part is to write code that solves the relevant system of differential equations in the region $[\eta^\psi, 1]$, where experts hold all the capital (i.e. $\psi_t = 1$). For reasons that will become clear later, it will be convenient to express c and d in the form $c = \chi f$ and $d = (1-\chi)f$, where χ is the consumption share of experts and $f = c + d$ is the total consumption per unit of capital.

(k) The goal will be to write down a system of ordinary differential equations for $q(\eta)$ and $\chi(\eta)$. With this goal, one has to express f and its derivatives in terms of q. From the market-clearing condition for consumption goods (part (a)), derive an expression for $f(\eta)$ in terms of $q(\eta)$. Also, express the first and second-order derivatives of f in terms of those derivatives of q.

(l) From part (f) (the two equivalent expressions for the risk-free rate), write an expression that will let you calculate χ'' from the lower-order derivatives of χ and q.

(m) Use the remaining conditions to devise a procedure to compute q'' from the lower-order derivatives of χ and q. You may also use χ'' in your expressions, as you know how to compute it from part (l).

(n) What conditions should the functions $\chi(\eta)$ and $q(\eta)$ satisfy at the boundary $\eta = 1$? Write a Matlab program that, starting from any quadruple of boundary conditions $q(\eta)$, $q'(\eta)$, $\chi(\eta)$ and $\chi'(\eta)$ computes the solution over the interval $[\eta, 1]$ (or until a blow-up point).

Part IV. The goal of this part is to write code that can solve the relevant system of equations in the region $[0, \eta^{\psi})$, where $\psi_t < 1$. Here the system of equations turns out to be more complicated. To make things easier, I recommend using the implicit Matlab ODE solver, *ode*15*i*. This solver takes in a function (such as $fnct(eta, y, yp)$ that you'll be asked to write in part (o)), which provides a set of conditions that a vector y has to satisfy jointly together with its derivative yp.

(o) Take vector y of the form $(q(\eta), q'(\eta), c(\eta), c'(\eta), d(\eta), d'(\eta), \psi)$. Write down function $fnct$ whose output is a vector F with seven entries, such that each entry is zero if and only if all the equilibrium conditions are satisfied. The first three components of F can be $y(2) - yp(1)$, $y(4) - yp(3)$ and $y(6) - yp(5)$. The fourth component can be the market-clearing condition. The last three components should be the asset-pricing conditions from parts (d), (e) and (f). In this function, since the second derivatives of q, c and d are known (i.e. $yp(2) = q''(\eta)$, etc.) you can express the drift terms (such as μ_t^q) directly using Ito's lemma, and plug them into the equilibrium conditions.

(p) What appropriate boundary conditions should the functions q, c, d and ψ satisfy at $\eta = 0$? Given the boundary conditions, how many degrees of freedom do you have to perturb the vector y at $\eta = 0$, in order to match the boundary conditions at $\eta = 1$?

(q) Write code that, starting from any *consistent* boundary conditions near $\eta = 0$, solves the system of relevant equations until ψ reaches 1. To test the code, you may use the function *decic* to find consistent boundary conditions (or to fix some of the conditions, and find the remaining conditions consistent with the ones you chose).

(r) Write code that perturbs the boundary conditions near $\eta = 0$ to solve for the equilibrium functions q, c, d and ψ over the entire interval $[0, 1]$. After ψ reaches 1, you'll have to switch from the code you wrote for Part IV to the code you wrote for Part III. At the switching point η^ψ, you'll have to use the fact that functions $\theta = c/(c + d)$ and $(q'(\eta)/q(\eta) - \gamma f'(\eta)/f(\eta))\sigma_t^\eta$ have to be continuous and differentiable (can you justify these requirements from the equilibrium conditions?). Note that the latter requirement implies that the function $q(\eta)$ will have a kink at η^ψ when $\gamma > 0$, since function ψ is not differentiable at η^ψ. You should use these two requirements to compute the left derivatives of functions q and θ at point η^ψ (for the code from Part III) given their right derivatives at η^ψ (from the code from Part IV).

Compute the equilibrium for $\gamma = 1.5$, $a = 0.014$, $\underline{a} = 0.04$, $\delta = 0.05$, $r = 0.05$, $\rho = 0.06$, $\sigma = 0.1$ and $\kappa = 2$.[2]

Tips: Once you have written programs to solve the ODEs from parts III and IV, you are almost done. You should then write a proper transition procedure (from conditions to the left of η^ψ to those to the right of η^ψ) and start playing with ways to perturb the solutions near $\eta = 0$ to change *both* functions q and θ above η^ψ in a meaningful way. Some of the initial conditions near $\eta = 0$ may lead to solutions that go completely astray, e.g. blow up before ψ reaches 1. It is helpful to find a relevant range of initial conditions, for which the solutions behave relatively regularly, in order to implement a search procedure. You may want to terminate integration when the solution starts behaving irregularly, to save time. It may be helpful to know that the function $q(\eta)$ is decreasing on the range $[\eta^\psi, 1]$, given that $\rho > r$.

[2]Generally, the code will not converge all the way to $\eta = 1$, because the solution there is very sensitive to the initial conditions near $\eta = 0$. The reason is that the equilibrium state variable η_t rarely gets close to $\eta = 1$. In your solutions, you may either leave things as they are or, if you are super-ambitious, find a way to extend the solution to $\eta = 1$ (e.g. by solving the ODE from part III from the right).