

Productivity Losses from Financial Frictions

a.k.a. Tricks for constructing tractable dynamic heterogeneous agent models

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Princeton Initiative: Macro, Money and Finance

Two Parts: Substance and Tools

- Substance:
 - (1) Financial frictions \Rightarrow capital misallocation \Rightarrow TFP losses.
 - (2) Stochastic process of idiosyncratic productivity shocks is key for quantitative importance (reason: internal financing).
 - But different effects on steady state and transitions
 - (3) Financial frictions do not necessarily show up as an “investment wedge.”
 - Model has **undistorted** Euler equation for aggregate of firm owners.
 - Finance matters for macro because it affects the **allocation** of capital across heterogenous firms.

Two Parts: Substance and Tools

- Tools:
 - (1) Very powerful trick in heterogenous agent models: individual production functions with constant returns to scale (Angeletos, 2007; Kiyotaki and Moore, 2008; Kiyotaki, 1998; Krebs, 2003; Kocherlakota, 2009). Log-utility also helps.
 - (2) Also powerful: continuous time stochastic processes, particularly when persistent shocks needed.
 - (3) If time left: more general approach to solving heterogeneous agent models in continuous time

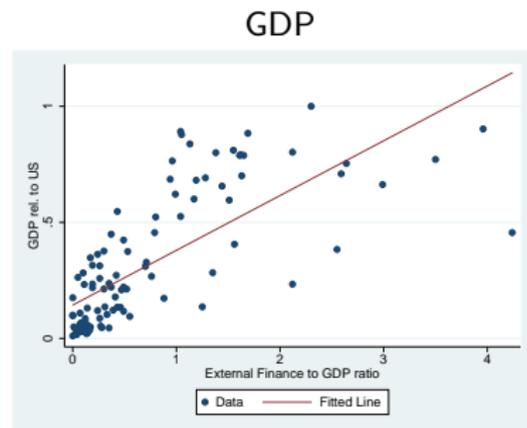
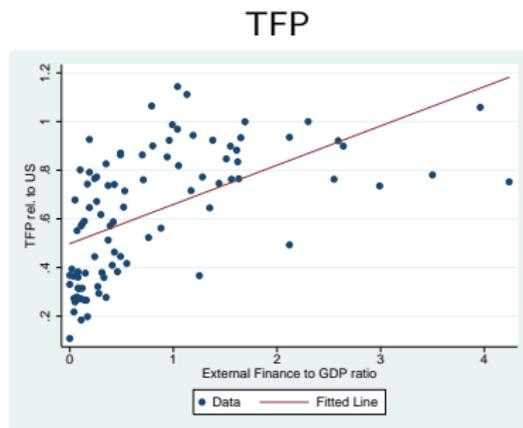
Based on Three Papers

- (1) “Productivity Losses from Financial Frictions: Can Self-financing Undo Capital Misallocation”
- (2) “Aggregate Implications of a Credit Crunch”, with Paco Buera
- (3) “Heterogeneous Agent Models in Continuous Time”, with Yves Achdou, Jean-Michel Lasry and Pierre-Louis Lions

Motivation

- Financial frictions \Rightarrow capital misallocation \Rightarrow TFP losses.
- TFP differences \Rightarrow income differences.
(Klenow and Rodríguez-Clare, 1997; Hall and Jones, 1999; Caselli, 2005)
- resource misallocation \Rightarrow TFP differences.
(Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009)
- Financial frictions \Rightarrow capital misallocation
(Banerjee and Duflo, 2005, and references therein)

Financially more developed countries have higher TFP/GDP



Note: External finance = Private credit + private bond mkt. cap. + stock mkt. cap.;
Data source: Beck, Demirguc-Kunt and Levine (2000), Caselli (2005)

Outline

- (1) Static Model.
- (2) Dynamic Model, iid Shocks.
- (3) Dynamic Model, Persistent Shocks.

Static Model

- Continuum of entrepreneurs: wealth a , productivity/ability z , distribution $g(a, z)$.
- Mass L of workers.
- Entrepreneurial technologies

$$y = f(z, k, \ell) = (zk)^\alpha \ell^{1-\alpha}$$

- Rent capital k in a rental market at rate $R = r + \delta$.
- Collateral constraints

$$k \leq \lambda a, \quad \lambda \geq 1.$$

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Profits

- Profit function

$$\Pi(a, z) = \max_{k, \ell} \{f(z, k, \ell) - w\ell - (r + \delta)k \quad \text{s.t.} \quad k \leq \lambda a\}.$$

Linearity & Cutoff

Lemma

Profits and factor demands are *linear* in wealth, and there is a productivity *cutoff* for being active \underline{z} .

$$k(a, z) = \begin{cases} \lambda a, & z \geq \underline{z} \\ 0, & z < \underline{z} \end{cases}, \quad \ell(a, z) = \left(\frac{1 - \alpha}{w} \right)^{1/\alpha} z k(a, z)$$

$$\Pi(a, z) = \max\{z\pi - r - \delta, 0\} \lambda a, \quad \pi = \alpha \left(\frac{1 - \alpha}{w} \right)^{(1-\alpha)/\alpha}.$$

The productivity cutoff is defined by

$$\underline{z}\pi = r + \delta.$$

Comment: Better credit markets (λ) \Rightarrow higher demand for credit drives up $r \Rightarrow$ higher \underline{z} .

Equilibrium

- Prices r and w , and corresponding quantities such that:
 - (i) Entrepreneurs maximize profits, taking as given r and w .
 - (ii) Markets clear

$$\int k(a, z) dG(a, z) = \int a dG(a, z), \quad \int \ell(a, z) dG(a, z) = L.$$

Aggregation

- Define wealth shares

$$\omega(z) \equiv \frac{1}{K} \int_0^\infty ag(a, z) da, \quad \Omega(z) \equiv \int_0^z \omega(x) dx.$$

Proposition

Aggregate GDP is

$$Y = ZK^\alpha L^{1-\alpha}$$

where

$$Z = \left(\frac{\int_{\underline{z}}^\infty z\omega(z) dz}{1 - \Omega(\underline{z})} \right)^\alpha = \mathbb{E}_\omega[z|z \geq \underline{z}]^\alpha$$

is measured TFP, and $\omega(z)$ is the wealth share of entrepreneurs with productivity type z . The cutoff is defined by

$$\lambda(1 - \Omega(\underline{z})) = 1.$$

Implications

- Looks like standard aggregate production function.
- TFP endogenous and increasing in quality of credit markets.

What Determines Size of TFP Losses? A Pareto Example

- Purpose: show that shape of productivity distribution matters.
- Assume: $a \perp\!\!\!\perp z$.
- $z \sim$ Pareto on $[1, \infty)$ with tail parameter $\eta > 1$.
- Can show:

$$\underline{z} = \lambda^{1/\eta} \quad \Rightarrow \quad Z = \left(\frac{\eta}{\eta - 1} \lambda^{1/\eta} \right)^\alpha.$$

- Elasticity with respect to λ is

$$\frac{\alpha}{\eta}.$$

- Fatter tail (lower η) \Rightarrow higher elasticity \Rightarrow larger TFP losses.

Dynamic Model, iid Shocks

Dynamic Model

- Discrete time.
- z drawn from $\psi(z)$, iid across entrepreneurs and *over time*.
- Preferences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t.$$

- Budgets:

$$a_{t+1} = f(z_t, k_t, \ell_t) - w_t \ell_t - (r_t + \delta)k_t + (1 + r_t)a_t - c_t$$

- Constraints:

$$k_t \leq \lambda a_t.$$

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Equilibrium

- Time paths for prices $\{r_t, w_t\}_{t=0}^{\infty}$, and corresponding quantities such that:

(i) Entrepreneurs maximize, taking as given $\{r_t, w_t\}_{t=0}^{\infty}$.

(ii) Markets clear for all t :

$$\int k_t(a, z) dG_t(a, z) = \int a dG_t(a, z), \quad \int \ell_t(a, z) dG_t(a, z) = L.$$

Where does $R_t = r_t + \delta$ & cap. mkt. clearing come from?

- Rep. capital producing firm owns and accumulates capital, rents to entrepreneurs:

$$V_0 = \max_{\{D_t, I_t, B_{t+1}, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{D_t}{(1+r)^t} \quad \text{s.t.}$$

$$B_{t+1} + I_t + D_t = R_t K_t + (1+r_t)B_t, \quad K_{t+1} = I_t + (1-\delta)K_t$$

- FOC \Rightarrow no arbitrage: $R_t = r_t + \delta$.
- Bond market clearing (zero net supply)

$$B_t + \int adG_t(a, z) = 0, \quad \text{all } t$$

- Can show: PDV of profits is

$$V_t = (1+r_t)(K_t + B_t), \quad \text{all } t$$

- Zero profits ($V_t = 0$) + bond market clearing \Rightarrow

$$K_t = \int adG_t(a, z), \quad \text{all } t.$$

Optimal Savings

- Entrepreneurs solve static problem period-by-period \Rightarrow profits linear in wealth

$$\Pi(a, z) = \max\{z\pi - r - \delta, 0\}\lambda a.$$

- Implies linear law of motion

$$a_{t+1} = [\lambda \max\{z_t\pi - r - \delta, 0\} + 1 + r] a_t - c_t.$$

Optimal Savings

Lemma

Savings are linear in wealth.

$$a_{t+1} = \beta [\lambda \max\{z_t \pi_t - r_t - \delta, 0\} + 1 + r_t] a_t$$

► Proof

- Everything linear \Rightarrow Economy aggregates nicely.
- Also key: Productivity iid over time $\Rightarrow a_t$ and z_t indep. in cross-section, i.e. $g_t(a, z) = \psi(z)\varphi_t(a)$. Entrepreneur chooses a_t at time $t - 1$ when he doesn't know z_t .

$$\Rightarrow \omega_t(z) = \frac{1}{K_t} \int_0^\infty a g_t(a, z) da = \psi(z)$$

Aggregate Dynamics

Proposition

Aggregate quantities satisfy

$$Y_t = ZK_t^\alpha L_t^{1-\alpha}$$

$$K_{t+1} = \beta[\alpha Y_t + (1 - \delta)K_t]$$

where

$$Z = \left(\frac{\int_{\underline{z}}^{\infty} z\psi(z)dz}{1 - \Psi(\underline{z})} \right)^\alpha = \mathbb{E}[z|z \geq \underline{z}]^\alpha$$

is measured TFP. The cutoff \underline{z} is defined by

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Implications

- Undistorted Euler equation for aggregate of entrepreneurs:

$$\frac{C_{t+1}^E}{C_t^E} = \beta \left[\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \right]$$

- Intuition: “investment wedge” in PE but not in GE.
- suppose credit markets worsen, $\lambda \downarrow$.
- **PE**: individuals borrow less, lend more \Rightarrow returns to wealth \downarrow .
Agg. return to wealth $<$ agg. MPK, i.e. “investment wedge”!
Also, agg. borrowing \downarrow
- **GE**: No borrowing in aggregate (capital market clears)
 \Rightarrow agg. return to wealth = agg. MPK.
- **Investment wedges from financial frictions are an artifact of PE reasoning!**

Implications

- Steady state capital-output ratio is

$$\frac{K}{Y} = \frac{\alpha}{\rho + \delta}$$

- **same** as in neoclassical growth model.
- does **not** depend on λ .

Dynamic Model, Persistent Shocks

Why Persistence?

- So far: iid shock $\Rightarrow a_t \perp z_t \Leftrightarrow$ no scope for internal financing \Rightarrow large TFP losses.
- Other extreme: fixed productivities.

$$a_{t+1} = \beta [\lambda \max\{z\pi_t - r_t - \delta, 0\} + 1 + r_t] a_t$$

- (1) Entrepreneur with highest z always accumulates faster than everyone else; holds all the wealth as $t \rightarrow \infty$
- (2) Self-financing undoes all capital misallocation, i.e. no TFP losses as $t \rightarrow \infty$.
 - (2) is well-known and much more general result (e.g. Banerjee and Moll, 2010): also holds with decreasing returns and general utility function (though (1) will not hold then).
 - Constrained \Leftrightarrow higher MPK \Leftrightarrow save out of constraint.
 - Hard to break, except with “dirty” fixes, e.g. het. β .

Intermediate Case: Persistent Shocks

- Continuous time.
- $z(t)$ follows *some* Markov process.
- Preferences:

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \log c(t) dt.$$

- Budgets:

$$\dot{a} = f(z, k, \ell) - w\ell - (r + \delta)k + ra - c.$$

- Constraints:

$$k \leq \lambda a.$$

- Still get linear savings rule [▶ Proof](#)

$$\dot{a} = [\lambda \max\{z\pi - r - \delta, 0\} + r - \rho] a$$

Aggregate Dynamics

Proposition

Aggregate quantities satisfy

$$Y = ZK^\alpha L^{1-\alpha}$$

$$\dot{K} = \alpha Y - (\rho + \delta)K \quad \Leftrightarrow \quad I = \alpha Y - \rho K$$

where

$$Z(t) = \left(\frac{\int_{\underline{z}}^{\infty} z\omega(z, t) dz}{1 - \Omega(\underline{z}, t)} \right)^\alpha = \mathbb{E}_{\omega, t}[z | z \geq \underline{z}]^\alpha$$

is measured TFP. The cutoff \underline{z} is defined by

$$\lambda(1 - \Omega(\underline{z}, t)) = 1$$

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How to characterize $\omega(z, t)$?

- Wealth shares \Leftrightarrow self-financing.
- Let productivity follow a diffusion

$$dz = \mu(z)dt + \sigma(z)dW.$$

- Recall optimal savings

$$\dot{a} = s(z)a, \quad s(z) = \lambda \max\{z\pi - r - \delta, 0\} + r - \rho.$$

How to characterize $\omega(z, t)$?

Proposition

The wealth shares $\omega(z, t)$ obey the second order PDE

$$\frac{\partial \omega(z, t)}{\partial t} = \left[s(z, t) - \frac{\dot{K}(t)}{K(t)} \right] \omega(z, t) - \frac{\partial}{\partial z} [\mu(z) \omega(z, t)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z) \omega(z, t)].$$

The wealth shares must also be non-negative and bounded everywhere, integrate to one for all t , $\int_0^\infty \omega(z, t) dz = 1$, and satisfy the initial condition $\omega(z, 0) = \omega_0(z)$ for all z .

The **stationary** wealth shares $\omega(z)$ obey the second order ODE

$$0 = s(z) \omega(z) - \frac{d}{dz} [\mu(z) \omega(z)] + \frac{1}{2} \frac{d^2}{dz^2} [\sigma^2(z) \omega(z)].$$

The stationary wealth shares must also be non-negative and bounded everywhere, and integrate to one, $\int_0^\infty \omega(z) dz = 1$.

Example with Closed Form Solution

- Purpose: illustrate importance of persistence.
- Assume $\lambda = 1$, no capital markets.
- specialize to Feller square root (Cox-Ingersoll-Ross) process

$$dz = (1/\theta)(1 - z)dt + \sigma\sqrt{z/\theta} dW$$

- $\theta =$ persistence, $1/\theta =$ speed of mean reversion
- Mean and autocorrelation are

$$\mathbb{E}[z] = 1, \quad \text{Corr}[z(t), z(t + s)] = e^{-(1/\theta)s}.$$

- Stationary distribution is Gamma:

$$\psi(z) \propto z^{\gamma-1} e^{-\gamma z}, \quad \gamma = \frac{2}{\sigma^2}$$

Example with Closed Form Solution

- Stationary distribution is Gamma:

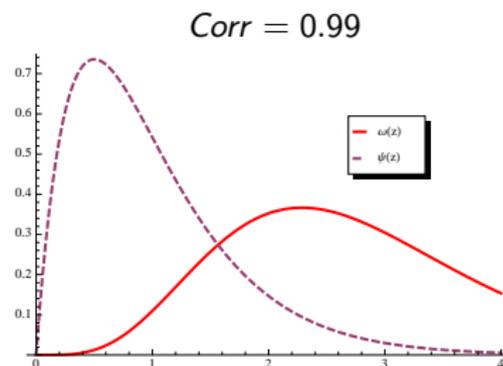
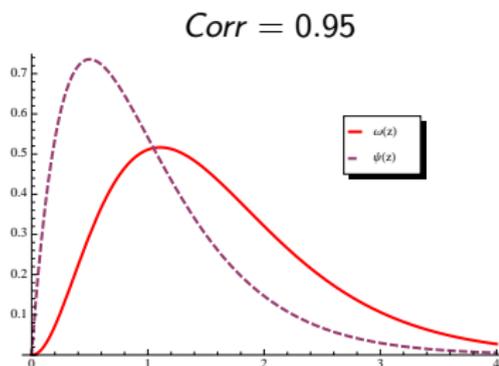
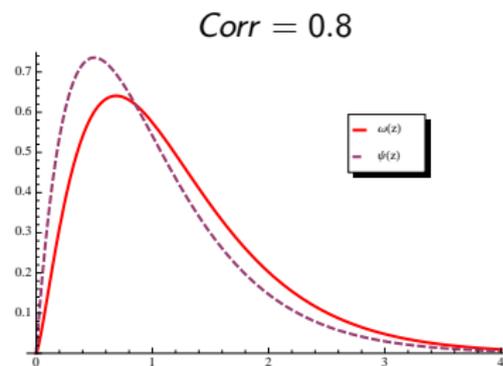
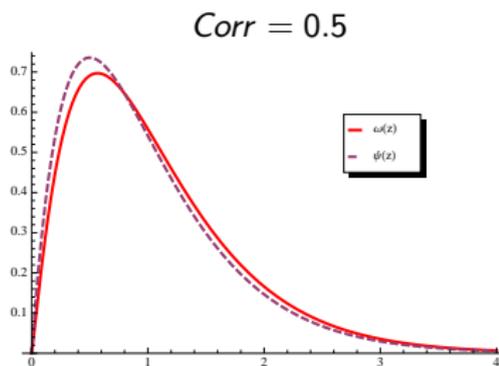
$$\psi(z) \propto z^{\gamma-1} e^{-\gamma z}, \quad \gamma = \frac{2}{\sigma^2}$$

- \Rightarrow wealth shares are also Gamma.

$$\omega(z) \propto e^{-\beta z} z^{\gamma-1}, \quad \beta = \gamma - \theta(\rho + \delta), \quad \gamma = \frac{2}{\sigma^2}$$

- Experiment: Hold stationary distribution, $\psi(z)$, fixed and vary persistence, θ .
- Higher autocorrelation \Rightarrow high z types hold more wealth.

Higher autocorrelation \Rightarrow high z types hold more wealth

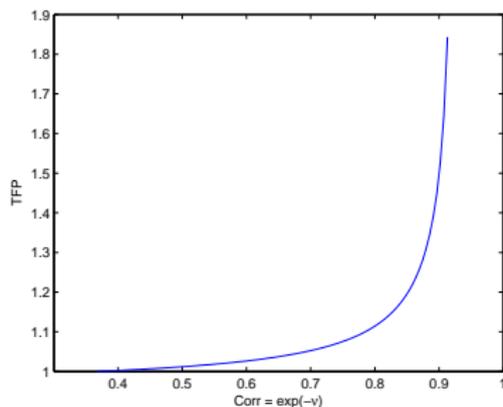


Higher autocorrelation \Rightarrow higher TFP

- TFP is

$$Z = \left(\frac{1}{1 - \theta(\rho + \delta)/\gamma} \right)^\alpha$$

Figure: TFP and Autocorrelation



Results for $\lambda \geq 1$ and Other Processes

- Same logic also applies in case $\lambda > 1$.
- And much more general class of stochastic processes:

$$dz = (1/\theta)\tilde{\mu}(z)dt + \sqrt{1/\theta} \tilde{\sigma}(z)dW. \quad (*)$$

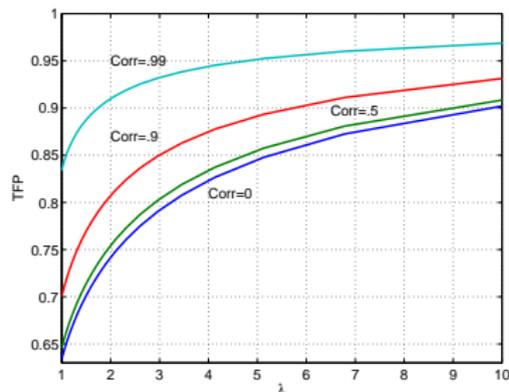
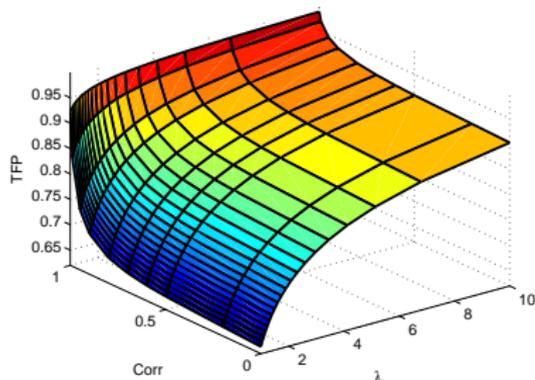
Proposition

For any ergodic productivity process () satisfying $0 < \tilde{\sigma}(z) < \infty$ for all $z \in (0, \bar{z})$, and for any $1 \leq \lambda < \infty$, stationary total factor productivity $Z(\theta) = \mathbb{E}_{\omega(\cdot, \theta)}[z|z \geq \underline{z}]^\alpha$ is continuous and strictly increasing in persistence θ , with*

$$Z(0) = \mathbb{E}[z|z \geq \underline{z}]^\alpha \quad \text{and} \quad \lim_{\theta \rightarrow \infty} Z(\theta) = \bar{z}^\alpha,$$

where $\mathbb{E}[\cdot]$ is the simple expectation taken over the stationary productivity distribution ψ , and \bar{z}^α is the first-best TFP level.

TFP and Autocorrelation

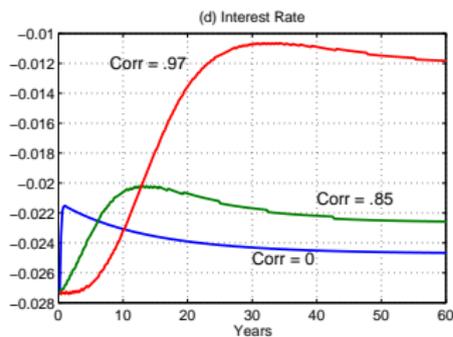
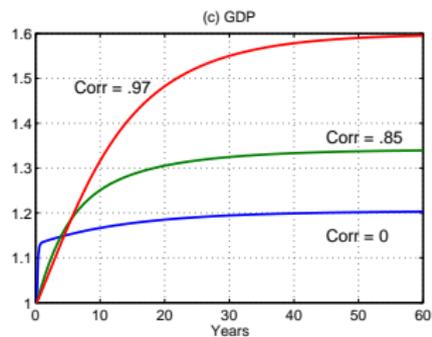
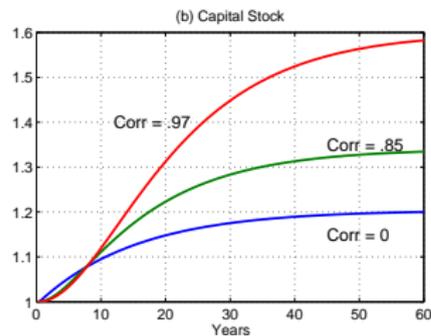
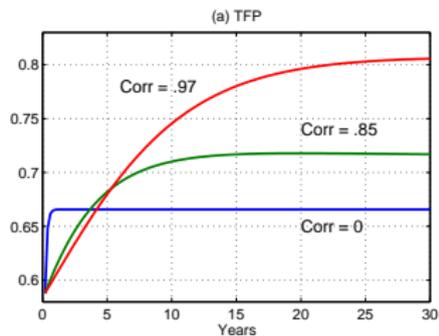


- Debate in literature: Midrigan and Xu (2010) vs. Buera and Shin (2010), Buera, Kaboski and Shin (2011).
- Much of disagreement can be attributed to different specifications and parameterizations of productivity process, particularly persistence.

Transition Dynamics

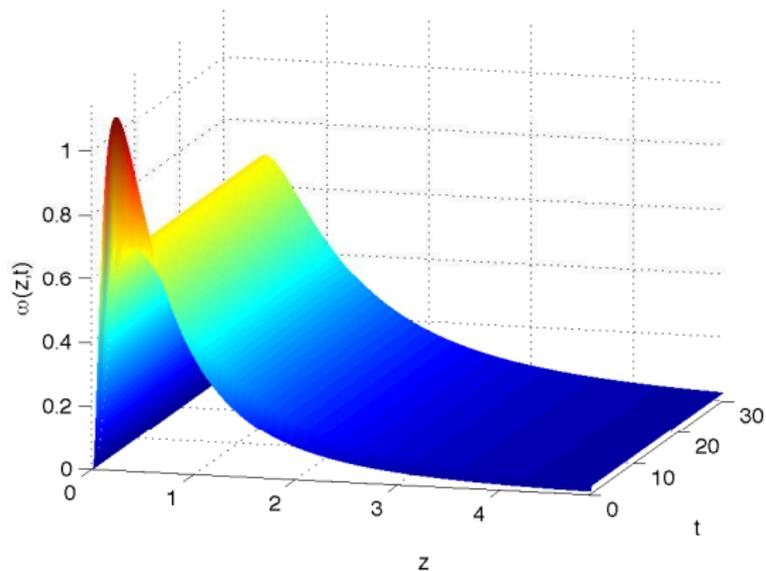
- So far: steady state. If shocks are persistent, steady state losses are small.
- But what about transitions? Show: persistent shocks \Rightarrow slow transitions.
- Even if financial frictions are unimportant in the long-run, they tend to matter in the short-run.
- Analyzing steady states only can be misleading!

Transition Dynamics from Distorted Initial Wealth Distribution



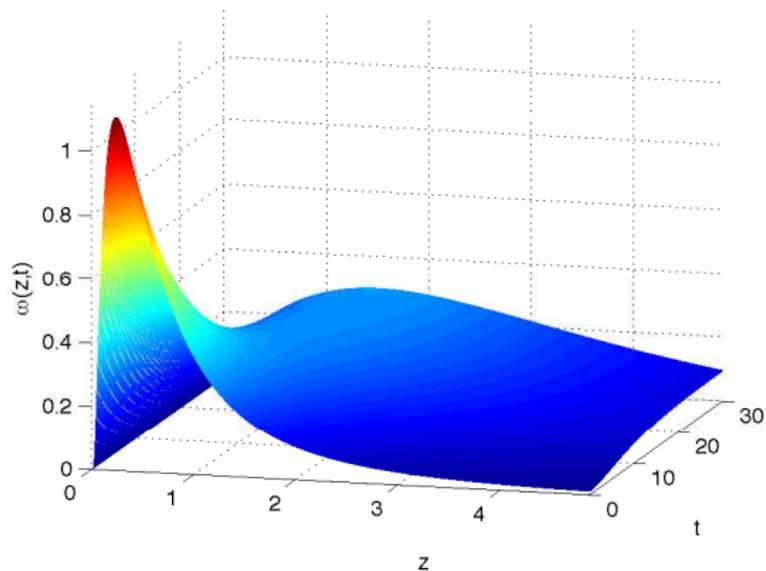
Transition Dynamics for Wealth Shares

Wealth Shares, Corr = 0



Transition Dynamics for Wealth Shares

Wealth Shares, Corr = 0.97



Relevance?

- With persistent shocks: theory of endogenous TFP dynamics.
Relevance?
- Chen, Imrohoroglu and Imrohoroglu (2006, 2007) and others: neoclassical growth model with time-varying TFP can explain transition experiences of post-war miracle economies.
- Empirically relevant range for autocorrelation: 0.75 to 0.97 (Gourio, 2008; Collard-Wexler, Asker and DeLoecker, 2011).
- in this range, financial frictions can matter **both** in the short-
and in the long-run!

Extension: Inequality and Financial Development

- Based on note on my website: “Inequality and Financial Development: A Power-Law Kuznets Curve”
- Extend model so as to feature stationary wealth distribution (death shocks)
- Results:

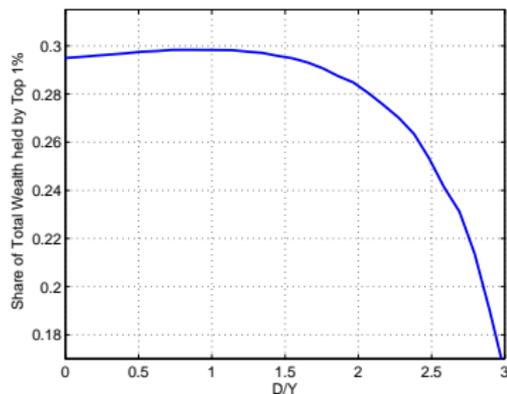
(1) wealth distribution has Pareto tail:

$$\lim_{a \rightarrow \infty} a^\zeta \Pr(\tilde{a} > a) = C$$

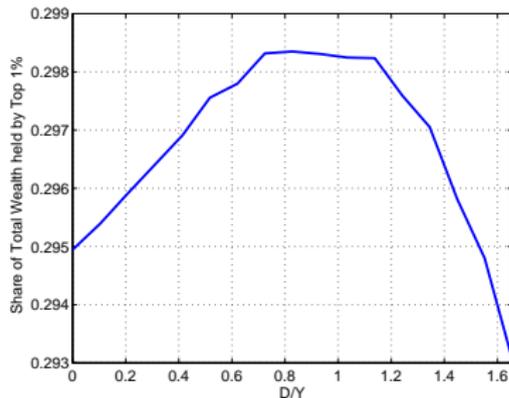
(2) tail inequality ($1/\zeta$ or share of wealth held by top 1%) is hump-shaped function of λ and hence GDP: “Kuznets curve”

Financial Development and Tail Inequality

“Zoomed out”



“Zoomed in”



Two offsetting effects give rise to hump shape:

- “leverage effect”
- “return equalization effect”

Implications for Business Cycles: Credit Crunches

Aggregate Dynamics

- Based on Buera and Moll (2013)
- Consider same model but with time varying λ

$$k_t \leq \lambda_{t-1} a_t$$

- $t - 1$ subscript due to slightly different setup in Buera-Moll: firms own and accumulate capital, issue debt, face collateral constraints
- turns out this is equivalent to previous setup: firms rent capital in rental market, face rental limit.
- Useful equivalence: previous setup is easier to solve (one less state variable)

Aggregate Dynamics

Proposition

Aggregate quantities satisfy

$$Y_t = Z_t K_t^\alpha L^{1-\alpha}$$

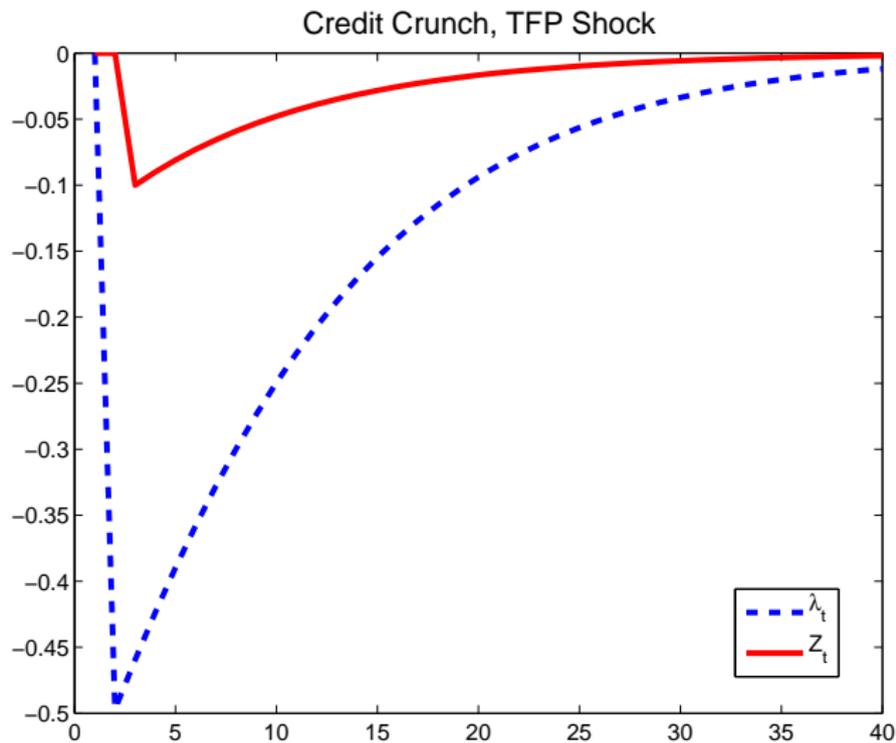
$$K_{t+1} = \beta [\alpha Y_t + (1 - \delta) K_t]$$

where

$$Z_t = \left(\frac{\int_{\underline{z}_t}^{\infty} z \psi(z) dz}{1 - \Psi(\underline{z}_t)} \right)^\alpha = \mathbb{E}[z | z \geq \underline{z}_t]^\alpha$$

is measured TFP. The cutoff is defined by

$$\lambda_{t-1} (1 - \Psi(\underline{z}_t)) = 1.$$

Credit Crunch \Leftrightarrow TFP Shock

Chari, Kehoe & McGrattan (2007) on Great Depression

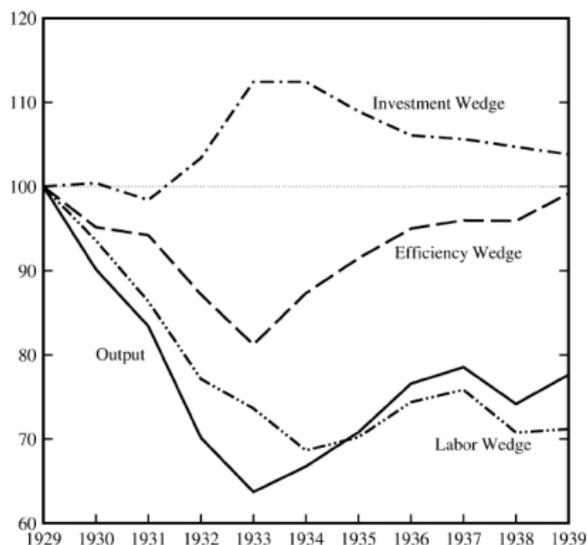


FIGURE 1.—U.S. output and three measured wedges (annually; normalized to equal 100 in 1929).

“Note that this investment wedge pattern does not square with models of business cycles in which financial frictions increase in downturns and decrease in recoveries.” (p.801)

Conclusion (1)

- Size of productivity losses from financial frictions depends on stochastic process of productivity shocks.
- Self-financing as counteracting force to capital misallocation.

Conclusion (2)

- Financial frictions do not necessarily show up as an “investment wedge.”
- **undistorted** Euler equation for aggregate of firm owners.
- Should be obvious: with different form of heterogeneity, financial frictions can show up **anywhere**, e.g.
 - as investment wedge if investment costs are heterogenous.
 - as labor wedge if recruitment costs are heterogenous.
- Trying to learn about the sources of aggregate fluctuations using a rep. agent framework and aggregate data, may seem appealing...
... but once you start thinking about heterogeneity, this goes out the window.

Heterogeneous Agent Models in Continuous Time: General Approach

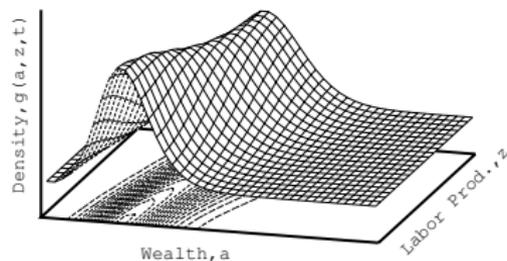
Heterogeneous Agent Models in Continuous Time

- Very preliminary draft here:

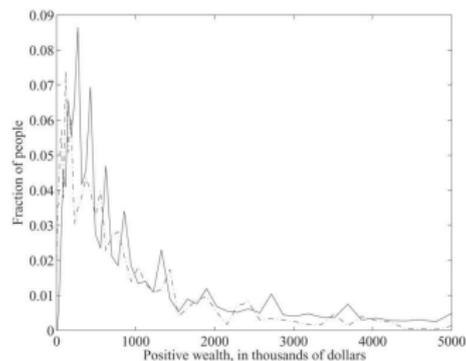
<http://www.princeton.edu/~moll/HACT.pdf>

- Large literature on heterogeneous agent models. But little is known theoretically and models often difficult to compute
- **Our goal:** develop continuous time methods so as to reap technical advantages of cont. time
- Results so far:
 - (1) Cont. time version of Aiyagari (1994) and efficient computational algorithm for transition dynamics
 - (2) Show how to handle borrowing constraints in cont. time.
 - (3) Theoretical result: tight characterization of stationary r :
 $r^* < \rho$ and gap (“overaccumulation”) pinned down by number of borrowing constrained.

Contrast: Typical Wealth Dist. in Het. Agent Model



(a) Wealth distribution in our model



(b) Cagetti and DeNardi (2006)

Movie: <http://www.princeton.edu/~moll/aiyagari.mov>

Aiyagari Model: Households

- are heterogeneous in their wealth a and work ability z (joint distribution $g(a, z, t)$), solve

$$\begin{aligned} \max_{c_t} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt \quad \text{s.t.} \\ da_t = [w_t z_t + r_t a_t - c_t] dt \\ dz_t = \mu_z(z_t) dt + \sigma_z(z_t) dW_t \\ a_t \geq \underline{a} \end{aligned}$$

- c_t : consumption
- u : utility function, $u' > 0$, $u'' < 0$.
- ρ : discount rate
- w_t : wage
- r_t : interest rate
- W_t : standard Brownian motion, independent across households
- \underline{a} : borrowing limit, e.g. if $\underline{a} = 0$, can only save

Equilibrium

Rep. firm with production function $F(K, L)$

$$r(t) = \partial_K F(K(t), 1) - \delta, \quad w(t) = \partial_L F(K(t), 1) \quad (\text{P})$$

$$K(t) = \int ag(a, z, t)dadz,$$

$$\begin{aligned} \rho v(a, z, t) = \max_c & u(c) + \partial_a v(a, z, t)[w(t)z + r(t)a - c] \quad (\text{HJB}) \\ & + \partial_z v(a, z, t)\mu_z(z) + \frac{1}{2}\partial_{zz}v(a, z, t)\sigma_z^2(z) + \partial_t v(a, z, t), \end{aligned}$$

$$\begin{aligned} \partial_t g(a, z, t) = & -\partial_a[\mu_a(a, z, t)g(a, z, t)] - \partial_z[\mu_z(z)g(a, z, t)] \quad (\text{KFE}) \\ & + \frac{1}{2}\partial_{zz}[\sigma_z^2(z)g(a, z, t)], \end{aligned}$$

$$\mu_a(a, z, t) = w(t)z + r(t)a - c(a, z, t), \quad c(a, z, t) = (u')^{-1}(\partial_a v(a, z, t))$$

Given initial condition $g_0(a, z)$, the two PDEs (HJB) and (KFE) together with (P) fully characterize equilibrium.

Borrowing Constraints?

- Q: where is borrowing constraint $a \geq \underline{a}$ in (HJB)?
- A: “in” boundary condition
- Borrowing constraint forms a “state constraint.”
- Appropriate solution for (HJB): “viscosity solution” (tells you which boundary condition to pick)
- In practice, much simpler: penalty function.
 - enlarge domain $a_{\min} < \underline{a}$
 - penalty function for $a < \underline{a}$: replace $u(c)$ in (HJB) by

$$u(c) - \xi|a - \underline{a}|, \quad \xi = 10,000, \text{ say}$$

- Can show: as $\xi \rightarrow \infty$, solution \rightarrow “viscosity solution.”

Computation

- Can solve system of PDEs (HJB) and (KFE) quite efficiently
- Current implementation in C++, planning to have MATLAB version

Overaccumulation and Borrowing Constraints

Proposition

In a stationary equilibrium:

- there is a Dirac point mass of individuals at the borrowing constraint $\underline{a} = 0$*
- There is a cutoff \hat{z} such that the z -type-specific Dirac mass satisfies $M(z) > 0$ for $z < \hat{z}$ and $M(z) = 0$ for $z \geq \hat{z}$*
- Denoting by $p(z)$ the probability that type z runs into the borrowing constraint, the stationary interest rate r^* satisfies*

$$(\rho - r^*) \int_0^\infty \int_{\underline{z}}^{\bar{z}} u'(c(a, z)) g(a, z) dz da = \int_{\underline{z}}^{\bar{z}} u'(c(0, z)) p(z) dz > 0.$$

- $p(z) \equiv -\lim_{\varepsilon \rightarrow 0} \mu_a(\varepsilon, z) g(\varepsilon, z) = \partial_z(\mu_z(z) M(z)) - \frac{1}{2} \partial_{zz}(\sigma^2(z) M(z)) \geq 0.$
- Note: also know $c(0, z) = w^* z$ (constr. \Rightarrow eat everything)

Overaccumulation and Borrowing Constraints

- As in discrete time, $r^* < \rho$.
- Gap $r^* - \rho$ important for judging amount of capital “overaccumulation”
- In contrast to conventional discrete time formulation: tight link between **overaccumulation** and **borrowing constraints**
- Intuition:
 - Euler equation holds for everyone **except** those at **borrowing constraint**

$$\frac{d\mathbb{E}[u'(c_t(a, z))]}{dt} = (\rho - r^*)u'(c_t(a, z)), \quad a > 0$$

$$\frac{d\mathbb{E}[u'(c_t(0, z))]}{dt} < (\rho - r^*)u'(c_t(0, z))$$

- In stationary equilibrium: average of LHS = 0 $\Rightarrow r^* < \rho$.

Conclusion

- **Tried to convince you:** continuous time methods extremely useful for building better heterogeneous agent models
- **Working on:**
 - existence and uniqueness (?) in Aiyagari
 - other environments, e.g. heterogeneous firms and credit constraints \Rightarrow misallocation
 - Aggregate shocks, i.e. Krusell-Smith

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Proof of Lemma (Discrete Time) [Return](#)

- Problem of an entrepreneur in recursive form:

$$v(a, z) = \max_{a'} \log[A(z)a - a'] + \beta \mathbb{E}v(a', z')$$

$$A(z) \equiv \lambda \max\{z\pi - r - \delta, 0\} + 1 + r$$

- Guess and verify. Guess

$$v(a, z) = V(z) + B \log a \quad \Rightarrow \quad \mathbb{E}v(a', z') = B \log a' + \mathbb{E}V(z')$$

- \Rightarrow FOC:

$$\frac{1}{A(z)a - a'} = \frac{\beta B}{a'} \quad \Rightarrow \quad a' = \frac{\beta B}{1 + \beta B} A(z)a, \quad c = \frac{1}{1 + \beta B} A(z)a.$$

- Substitute everything into Bellman:

$$A(z) + B \log a = \log \left[\frac{1}{1 + \beta B} A(z)a \right] + \beta \left[\mathbb{E}V(z') + B \log \frac{\beta B}{1 + \beta B} A(z)a \right]$$

- Collect terms involving $\log a$:

$$B = 1/(1 - \beta), \quad a' = \beta A(z)a. \square$$

Proof of Lemma (Continuous Time) [▶ Return](#)

- Problem of an entrepreneur in recursive form:

$$\rho v(a, z) = \max_c \left\{ \log c + \frac{1}{dt} \mathbb{E}[dv(a, z)] \quad \text{s.t.} \quad da = [A(z) - c]dt \right\}.$$

$$A(z) = \lambda \max\{z\pi - r - \delta, 0\} + r$$

- Guess and verify. Guess

$$v(a, z) = B \log a + V(z) \quad \Rightarrow \quad \mathbb{E}[dv(a, z)] = \frac{B}{a} da + \mathbb{E}[dV(z)]$$

- Substitute into HJB equation:

$$\rho V(z) + \rho B \log a = \max_c \log c + \frac{B}{a} [A(z)a - c] + \frac{1}{dt} \mathbb{E}[dV(z)].$$

- FOC: $c = a/B$. Substitute back in,

$$\rho V(z) + \rho B \log a = \log a - \log B + A(z)B - 1 + \frac{1}{dt} \mathbb{E}[dV(z)].$$

- Collect terms involving $\log a$:

$$B = 1/\rho \quad \Rightarrow \quad c = \rho a, \quad \dot{a} = [A(z) - \rho]a. \square$$

Capitalists and Workers [Return](#)

- Well-known result for neoclassical growth model:
log-utility + full depreciation \Rightarrow closed form solution.
- Show here: slightly different setup with “capitalists” and “workers” also gives closed form for $\delta < 1$; also useful as building block for other more complicated models.
- Trick: individ. constant returns but agg. decreasing returns.
- Capitalists:

$$\max_{c_t, \ell_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \log c_t \quad \text{s.t.}$$

$$k_{t+1} = Ak_t^\alpha \ell_t^{1-\alpha} - w_t \ell_t + (1 - \delta)k_t - c_t$$

- Hand-to-mouth workers: inelastically supply one unit of labor.
- Labor market clearing:

$$\ell_t = 1$$

Capitalists and Workers [Return](#)

- Profits linear in k_t after maximizing out over ℓ_t :

$$\Pi_t = \max_{\ell_t} \{ A k_t^\alpha \ell_t^{1-\alpha} - w_t \ell_t \}$$

$$\ell_t = \left(\frac{1-\alpha}{w_t} \right)^{1/\alpha} A^{1/\alpha} k_t$$

$$\Pi_t = A^{1/\alpha} \pi_t k_t, \quad \pi_t \equiv \alpha \left(\frac{1-\alpha}{w_t} \right)^{(1-\alpha)/\alpha}$$

- Problem of capitalists becomes:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \log c_t \quad \text{s.t.}$$

$$k_{t+1} = A^{1/\alpha} \pi_t k_t + (1-\delta)k_t - c_t$$

Capitalists and Workers [Return](#)

- Log-utility + CRS:

$$k_{t+1} = \beta[A^{1/\alpha}\pi_t k_t + (1 - \delta)k_t]$$

- Labor market clearing:

$$1 = \ell_t = \left(\frac{\pi_t}{\alpha}\right)^{\frac{1}{1-\alpha}} A^{\frac{1}{\alpha}} k_t \quad \Leftrightarrow \quad \pi_t = \alpha A^{\frac{\alpha-1}{\alpha}} k_t^{\alpha-1}$$

- Law of motion for aggregate capital stock is:

$$k_{t+1} = \beta[\alpha A k_t^\alpha + (1 - \delta)k_t]$$

- Steady state:

$$1 = \beta[\alpha A k^{\alpha-1} + 1 - \delta]$$

- Comparison with neoclassical growth model:

- Same steady state...
- ... but much simpler characterization of transition.